The relation between quasi valuation and valuation ring and filtered ring

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Abstract: In this paper we show the relation between filtered ring and quasi valuation and valuation ring. We show if $R$ is a filtered ring then we can define a quasi valuation. And if $R$ is some kind of filtered ring then we can define a valuation. Then we prove some properties and relations for $R$.

Keywords: Filtered ring, Quasi valuation ring, Valuation ring, Strongly filtered ring.

1. Introduction

In algebra valuation ring and filtered ring are two most important structure [5],[6],[7]. We know that filtered ring is also the most important structure since filtered ring is a base for graded ring especially associated graded ring and completion and some similar results, on the Andreadakis–Johnson filtration of the automorphism group of a free group [1], on the depth of the associated graded ring of a filtration [2],[3]. So, as these important structures, the relation between these structure is useful for finding some new structures, and if $R$ is a discrete valuation ring then $R$ has many properties that have many usage for example Decidability of the theory of modules over commutative valuation domains [7], Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices [6].

In this article we investigate the relation between filtered ring and valuation and quasi valuation ring. We prove that if we have filtered ring then we can find a quasi valuation on it. Continuously we show that if $R$ be a strongly filtered then exist a valuation, Similarly we prove it for PID. At the end we explain some properties for them.

2. Preliminaries

Definition 2.1 A filtered ring $R$ is a ring together with a family $\{R_n\}_{n \geq 0}$ of subgroups of $R$ satisfying in the following conditions:

i. $R_0 = R$;
ii. $R_{n+1} \subseteq R_n$ for all $n \geq 0$;
iii. $R_nR_m \subseteq R_{n+m}$ for all $n, m \geq 0$.

Definition 2.2. Let $R$ be a ring together with a family $\{R_n\}_{n \geq 0}$ of subgroups of $R$ satisfying the following conditions:

i. $R_0 = R$;
ii. $R_{n+1} \subseteq R_n$ for all $n \geq 0$;
iii. $R_nR_m = R_{n+m}$ for all $n, m \geq 0$.

Then we say $R$ has a strong filtration.
Definition 2.3. Let \( R \) be a ring and \( I \) an ideal of \( R \). Then \( R_n = I^n \) is called \( I \)-adic filtration.

Definition 2.4. A map \( f : M \to N \) is called a homomorphism of filtered modules if: (i) \( f \) is \( R \)-module an homomorphism and (ii) \( f(M_n) \subseteq N_n \) for all \( n \geq 0 \).

Definition 2.5. A subring \( R \) of a filed \( K \) is called a valuation ring of \( K \) if for every \( \alpha \in K, \alpha \neq 0 \), either \( \alpha \in R \) or \( \alpha^{-1} \in R \).

Definition 2.6. Let \( \Delta \) be a totally ordered abelian group. A valuation \( v \) on \( R \) with values in \( \Delta \) is a mapping \( v : R^* \to \Delta \) satisfying:

i. \( v(ab) = v(a) + v(b) \);

ii. \( v(a + b) \geq \min\{v(a), v(b)\} \).

Definition 2.7. Let \( \Delta \) be a totally ordered abelian group. A quasi valuation \( v \) on \( R \) with values in \( \Delta \) is a mapping \( v : R^* \to \Delta \) satisfying:

i. \( v(ab) \geq v(a) + v(b) \);

ii. \( v(a + b) \geq \min\{v(a), v(b)\} \).

Remark 2.1. \( R \) is said to be vaulted ring: \( R_v = \{x \in R : v(x) \geq 0\} \) and \( v^{-1}(\infty) = \{x \in R : v(x) = \infty\} \).

Definition 2.8. Let \( K \) be a filed. A discrete valuation on \( K \) is a valuation \( v : K^* \to \mathbb{Z} \) which is surjective.

Theorem 2.1. If \( R \) is a UFD then \( R \) is a PID (see [2]).

Proposition 2.1. Any discrete valuation ring is a Euclidean domain(see[3]).

Remark 2.2. If \( R \) is a ring, we will denote by \( Z(R) \) the set of zero-divisors of \( R \) and by \( T(R) \) the total ring of fractions of \( R \).

Definition 2.9. A ring \( R \) is said to be a Manis valuation ring (or simply a Manis ring) if there exist a valuation \( v \) on its total fractions \( T(R) \), such that \( R = R_v \).

Definition 2.10. A ring \( R \) is said to be a Prüfer ring if each overring of \( R \) is integrally closed in \( T(R) \).

Definition 2.11. A Manis ring \( R_v \) is said to be \( v \)-closed if \( R_v/v^{-1}(\infty) \) is a valuation domain (see Theorem 2 of [8]).

3. Quasi Valuation and Valuation derived from Filtered ring

Let \( R \) be a ring with unit and \( R \) a filtered ring with filtration \( \{R_n\}_{n \geq 0} \).

Lemma 3.1. Let \( R \) be a filtered ring with filtration \( \{R_n\}_{n \geq 0} \). Now we define \( v : R \to \mathbb{Z} \) such that for every \( \alpha \in R \) and \( v(\alpha) = \min\{i \mid \alpha \in R_i \setminus R_{i+1}\} \).

Then we have \( v(\alpha \beta) \geq v(\alpha) + v(\beta) \).

Proof. For any \( \alpha, \beta \in R \) with \( v(\alpha) = i \) and \( v(\beta) = j \), \( \alpha \beta \in R_i R_j \subseteq R_{i+j} \).

Now let \( v(\alpha \beta) = k \) then we have \( \alpha \beta \in R_k \setminus R_{k+1} \).

We show that \( k \geq i + j \).

Let \( k < i + j \) so we have \( k + 1 \leq i + j \) hence \( R_{k+1} \supseteq R_{i+j} \) then \( \alpha \beta \in R_{i+j} \subseteq R_{k+1} \) it is contradiction. So \( k \geq i + j \).

Now we have \( v(\alpha \beta) \geq v(\alpha) + v(\beta) \).
Lemma 3.2. Let $R$ be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i \mid \alpha \in R_i \setminus R_{i+1}\}$.

Then $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$

Proof. For any $\alpha, \beta \in R$ such that $v(\alpha) = i$ and $v(\beta) = j$ and $v(\alpha + \beta) = k$ so we have $\alpha + \beta \in R_k \setminus R_{k+1}$. Without losing the generality, let $i < j$ so $R_i \subset R_j$ hence $\beta \in R_i$. Now if $k < i$ then $k + 1 \leq i$ and $R_i \subset R_{k+1}$ so $\alpha + \beta \in R_i \subset R_{k+1}$ it is contradiction. Hence $k \geq i$ and so we have $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$.

Theorem 3.1. Let $R$ be a filtered ring. Then there exist a quasi valuation on $R$.

Proof. Let $R$ be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i \mid \alpha \in R_i \setminus R_{i+1}\}$.

Then

i) By lemma (3.1) we have $v(\alpha \beta) \geq v(\alpha) + v(\beta)$.

ii) By Lemma(3.2) we have $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$. So by Definition 2.7 $R$ is quasi valuation ring.

Proposition 3.1. Let $R$ be a strongly filtered ring. Then there exists a valuation on $R$.

Proof. By theorem (3.1) we have $v(\alpha \beta) \geq v(\alpha) + v(\beta)$ and $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$. Now we show $v(\alpha \beta) = v(\alpha) + v(\beta)$. Let $v(\alpha \beta) > v(\alpha) + v(\beta)$ so $k > i + j$ and it is contradiction. So $v(\alpha \beta) = v(\alpha) + v(\beta)$, then there is a valuation on $R$.

Corollary 3.1. Let $R$ be a strongly filtered ring, then $R$ is a Euclidean domain.

Proof. By proposition (3.1) $R$ is a discrete valuation and so by proposition (2.1) $R$ is a Euclidean domain.

Proposition 3.2. Let $P$ be a prime ideal of $R$ and $\{P^n\}_{n \geq 0}$ be $P$-adic filtration. Then there exists a valuation on $R$.

Proof. By theorem (3.1) we have $v(\alpha \beta) \geq v(\alpha) + v(\beta)$ and $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$. Now we show $v(\alpha \beta) = v(\alpha) + v(\beta)$. Let $v(\alpha \beta) > v(\alpha) + v(\beta)$ so $k > i + j$ then $\alpha \beta \in P^k \subset P^{i+j}$ and $k \geq i + j + 1$, since $P$ is a prime ideal hence $\alpha \in P^{i+1}$ or $\beta \in P^{j+1}$ and it is contradiction. So $v(\alpha \beta) = v(\alpha) + v(\beta)$, then there is a valuation on $R$.

Proposition 3.3. Let $R$ be a PID then there is a valuation on $R$.

Proof. By theorem (3.1) and proposition (3.2) there is a valuation on $R$.

Corollary 3.2. If $R$ is an UFD then there exists a valuation on $R$, then $R$ is a Euclidean domain.

Corollary 3.3. Let $R$ be a ring and $P$ is a prime ideal of $R$. If $R$ has a $P$-adic filtration and $R = \bigcup_{i=0}^{\infty} P^i$, then $R$ is a Euclidean domain.

Proof. By proposition (3.2) $R$ is a discrete valuation and so by proposition (2.1) $R$ is a Euclidean domain.

Corollary 3.4. Let $R$ be a PID then $R$ is a Euclidean domain.

Proof. By proposition (3.3) and proposition (2.1) we have $R$ is a Euclidean domain.

Corollary 3.5. Let $R$ be a UFD then $R$ is a Euclidean domain.

Corollary 3.6. Let $R$ be a strongly filtered ring. Then $R$ is Manis ring.

Corollary 3.7. Let $P$ be a prime ideal of $R$ and $\{P^n\}_{n \geq 0}$ be $P$-adic filtration. Then $R$ is Manis ring.

Proposition 3.4. Let $R_v$ be a Manis ring. If $R_v$ is $v$-closed, then $R_v$ is Prüfer.
Proof. See proposition 1 of [9]

**Proposition 3.5.** Let \( R \) be a strongly filtered ring. Then \( R \) is \( v \)-closed.

**Proof.** By proposition (3.1) and definition (2.9) we have \( R \) is Manis ring and \( R = R_v \).

Now let \( \alpha, \beta \in R \) and
\[
v(\alpha) = i \text{ and } v(\beta) = j
\]

Consequently if
\[
(\alpha + v^{-1}(\infty))(\beta + v^{-1}(\infty)) \in v^{-1}(\infty)
\]

Then \( i + j \geq \infty \) so \( \alpha \in v^{-1}(\infty) \) or \( \beta \in v^{-1}(\infty) \). Hence by definition (2.11) \( R \) is \( v \)-closed.

**Corollary 3.8.** Let \( R \) be a strongly filtered ring. Then \( R \) is Prüfer.

**Proof.** By proposition (3.6) \( R \) is \( v \)-closed so by proposition (3.4) \( R \) is Prüfer.

**Proposition 3.6.** Let \( P \) is a prime ideal of \( R \) and \( \{P^n\}_{n \geq 0} \) be \( P \)-adic filtration. Then \( R \) is \( v \)-closed.

**Proof.** By proposition (3.2) and definition (2.9) we have \( R \) is Manis ring and \( R = R_v \).

Now let \( \alpha, \beta \in R \) and
\[
v(\alpha) = i \text{ and } v(\beta) = j
\]

Consequently if
\[
(\alpha + v^{-1}(\infty))(\beta + v^{-1}(\infty)) \in v^{-1}(\infty)
\]

Then \( i + j \geq \infty \) so \( \alpha \in v^{-1}(\infty) \) or \( \beta \in v^{-1}(\infty) \). Hence by definition (2.11) \( R \) is \( v \)-closed.

**Corollary 3.9.** Let \( P \) is a prime ideal of \( R \) and \( \{P^n\}_{n \geq 0} \) be \( P \)-adic filtration. Then \( R \) is Prüfer.

**Proof.** By proposition (3.6) \( R \) is \( v \)-closed so by proposition (3.4) \( R \) is Prüfer.

**References**


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