Infinitely many large energy solutions of nonlinear Schrödinger-Maxwell system

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Abstract: This paper deals with the existence of infinitely many large energy solutions for nonlinear Schrödinger-Maxwell system

\[
\begin{align*}
-\Delta u + V(x)u + \lambda \phi u &= |u|^{p-1}u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

We use the Fountain theorem under Cerami conditions 2.2 to find infinitely many large solutions for $p \in (2,6)$ and $\lambda \in \mathbb{R}^+ - \left(\frac{4}{7}, \frac{4}{3}\right)$.

Keywords: Schrödinger-Maxwell equations, variational method, Strongly indefinite functionals, Cerami conditions

1. Introduction

In this paper we are concerned with the existence of infinitely many large energy solutions for the nonlinear Schrödinger-Maxwell system

\[
\begin{align*}
-\Delta u + V(x)u + \lambda \phi u &= |u|^{p-1}u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where $\lambda \in \mathbb{R}^+ - \left(\frac{4}{7}, \frac{4}{3}\right)$ is a parameter, $V \in C(\mathbb{R}^3, \mathbb{R})$ which is satisfied in some suitable conditions and $p \in (2,6)$. In the classical model, the interaction of a charge particle with an electromagnetic field can be described by the nonlinear Schrödinger-Maxwell’s equations (see for examples [6, 9] and the references therein for more details on the physical aspects).

More precisely, we use the Fountain theorem under Cerami conditions 2.2 to find infinitely many large solutions for $p \in (2,6)$ and $\lambda \in \mathbb{R}^+ - \left(\frac{4}{7}, \frac{4}{3}\right)$ which is different from obtained results in [1,6]. If we consider $V(x) = 1$, then the system 1.1 reduced to the following system

\[
\begin{align*}
-\Delta u + u + \lambda \phi u &= |u|^{p-1}u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

which considered by Jiang et. al, [15], of course in homogeneous case. The problem of finding infinitely many large solutions is a vary classical problem. There is an extensive literature concerning the existence of infinitely many large energy solutions of a plethora of problems via the symmetric Mountain Pass theorem and Fountain theorem [4, 7, 10]. But, the existence of solutions for problem 1.1 has been discussed under different ranges of $p$, for examples [11, 3] for $p \in [3,5)$, 5 for $p \in (2,5)$ and [1, 2, 17] for $p \in (1,5)$. In particular case, with $V(x) = 1$ and $p \in (2,5)$, Ambrosetti and Ruiz have proved that the system 1.2 has infinitely many solutions for all $\lambda > 0$ [1]. Here, we will show infinitely many large energy solutions for 1.1, where $V \in C(\mathbb{R}^3, \mathbb{R})$ and $p \in (2,6)$, via the Fountain theorem under cerami condition. In recent years, for the potential $V$, many authors assumed (see for examples [19,18]).
We consider the more general case and weaken the condition of \( V^* \). We assume

\[ V_1^* \in C(\mathbb{R}^3, \mathbb{R}) \quad \text{and} \quad \exists M > 0 \quad \text{such that} \quad \Omega_M := \{ x \in \mathbb{R}^3 \mid V(x) \leq M \} \quad \text{is nonempty and has finite Lebesgue measure.} \]

In this section we give some notations and definitions on the function space endowed with the norm \( \| \cdot \|_p \). We set

\[ H^1(\mathbb{R}^3) := \{ u \in L^2(\mathbb{R}^3) \mid \nabla u \in L^2(\mathbb{R}^3) \}, \tag{3.1} \]

endowed with the norm

\[ \| u \|_{H^1} := \left( \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 \, dx \right)^{\frac{1}{2}} \tag{3.2} \]

and we consider the function space

\[ \mathbb{R} \ni t \mapsto u := \left( \int_{\mathbb{R}^3} -f \right. \quad \text{for all} \quad x \in \mathbb{R}^3, \quad u \in \mathbb{R} \quad \text{and} \quad t \in [0,1], \quad \text{where} \quad f_t(u) = \left( 1 - \frac{4}{\lambda(p+1)} \right) \int_{\mathbb{R}^3} |u|^{p+1} \, dx - \| u \|_2^2, \quad \text{where} \quad \lambda \in \mathbb{R}^+ - \left( \frac{4}{p+1} \right). \]

The assumption \( V_1^* \) implies that the potential \( V \) is not periodic and changes sign.

2. Main results

Here, we express Cerami condition which was established by G. Cerami in [12]. To approach the main result, we need the following critical point theorem.

**Definition 2.1.** Suppose that functional \( I \) is \( C^1 \) and \( c \in \mathbb{R} \), if any sequence \( \{ u_n \} \) satisfies in \( I(u_n) \rightarrow c \) and \( (1 + \| u_n \|) I'(u_n) \rightarrow 0 \) has a convergence subsequence, we say the \( I \) is said to Cerami condition at the level \( c \).

**Theorem 2.2.** (Fountain theorem under Cerami condition) Let \( X \) be a Banach space with the norm \( \| \cdot \| \) and let \( X_j \) be a sequence of subspace of \( X \) with \( \dim X_j < \infty \) for any \( j \in \mathbb{N} \). Further, \( X = \bigoplus_{j=0}^{\infty} X_j \), the closure of the direct sum of all \( X_j \).

Set \( W_k = \bigoplus_{j=k}^{\infty} X_j \), \( Z_k = \bigoplus_{j=0}^{k-1} X_j \).

Consider an even functional \( I \in C^1(X, \mathbb{R}) \), that is \( I(-u) = I(u) \) for any \( u \in X \). Suppose that for any \( k \in \mathbb{N} \), there exists \( \rho_k > r_k > 0 \) such that

\[ I_1) \quad a_k := \max_{u \in W_k, \| u \| = \rho_k} I(u) \leq 0, \]

\[ I_2) \quad b_k := \inf_{u \in Z_k, \| u \| = r_k} I(u) \rightarrow +\infty \quad \text{as} \quad k \rightarrow \infty, \]

\[ I_3) \quad \text{the Cerami condition holds at any level} \quad c > 0. \]

Then the functional \( I \) has an unbounded sequence of critical values.

Now, our main result is the following:

**Theorem 2.3.** Let \( V^*_1 \), and assumption 1.3 satisfies. Then the system 1.1 has infinitely many solutions \( \{ (u_k, \phi_k) \} \in \mathbb{N} \) in \( H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) satisfying

\[ \frac{1}{2} \int_{\mathbb{R}^3} \left( |\nabla u_k|^2 + V(x) u_k^2 \right) \, dx - \frac{\lambda}{4} \int_{\mathbb{R}^3} |\nabla \phi_k|^2 \, dx + \frac{\lambda}{2} \int_{\mathbb{R}^3} \phi_k u_k^2 \, dx - \frac{1}{p+1} |u_k|^{p+1} \, dx \rightarrow +\infty, \]

as \( k \rightarrow \infty. \)

3. Some auxiliary results and notations

In this section we give some notations and definitions on the function product space. We set

\[ \mathcal{H} := \{ u \in L^2(\mathbb{R}^3) \mid \nabla u \in L^2(\mathbb{R}^3) \}, \quad \mathcal{H} := \{ u \in L^2(\mathbb{R}^3) \mid \nabla u \in L^2(\mathbb{R}^3) \}, \tag{3.1} \]

endowed with the norm

\[ \| u \|_{H^1} := \left( \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 \, dx \right)^{\frac{1}{2}} \tag{3.2} \]

and we consider the function space
\( D^{1,2} (\mathbb{R}^3) \) := \( \{ u \in L^2 (\mathbb{R}^3) \mid \nabla u \in L^2 (\mathbb{R}^3) \} \) \hspace{1cm} (3.3)

with the norm
\[
\| u \|_{D^{1,2}} := \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \hspace{1cm} (3.4)
\]

where \( 2^* := \frac{2n}{n-2} = 6 \). Now, we consider the function space
\[
E := \left\{ u \in H^1 (\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\nabla u|^2 + u^2 \, dx < \infty \right\}.
\]

Then \( E \) is a Hilbert space \([20]\) with the inner product
\[
(u, v)_E := \int_{\mathbb{R}^3} (\nabla u \nabla v + V(\mathbf{x}) uv) \, dx \hspace{1cm} (3.5)
\]

and \( \| u \|_E := (u, v)_E^{\frac{1}{2}} \).

**Lemma 3.1.** [19] If \( V_1 \) holds. Then \( E \hookrightarrow L^p (\mathbb{R}^N, \mathbb{R}^2) \) is continuous for \( p \in [2, 2^*) \) and \( E \hookrightarrow L^p_{\text{loc}} (\mathbb{R}^N, \mathbb{R}^2) \) is compact for \( p \in [2^*, 2^*) \).

**Remark 3.2.** The system 1.1 is the Euler-Lagrange equations of the functional \( J_\lambda : E \times D^{1,2} (\mathbb{R}^3) \rightarrow \mathbb{R} \) defined by
\[
J_\lambda (u, \phi) := \frac{1}{2} \| u \|^2 + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx + \frac{\lambda}{2} \int_{\mathbb{R}^3} \phi u^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} \, dx \hspace{1cm} (3.6)
\]

The functional \( J_\lambda \in C^1 (E \times D^{1,2} (\mathbb{R}^3), \mathbb{R}) \) and its critical points are the solutions of system 1.1. It is easy to know that \( J_\lambda \) exhibits a strong indefiniteness, namely it is unbounded both from below and from above on infinitely dimensional subspaces. This indefiniteness can be removed using the reduction method described in [9]. We recall this method. For any \( u \in E \), the Lax-Milgram theorem [14] implies there exists a unique \( \phi_u \in D^{1,2} (\mathbb{R}^3) \) such that
\[-\Delta \phi_u = u^2 \]
in a weak sense. We can write an integral expression for \( \phi_u \) in the form:
\[
\phi_u = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u(y)^2}{|x-y|} \, dy, \hspace{1cm} (3.7)
\]

for any \( u \in E \).

**Lemma 3.3.** [13] For any \( u \in E \)

i. \( \| \phi_u \|_{D^{1,2}} \leq M_1 \| u \|^2_{L^2} \), where \( M_1 \) is a positive constant which does not depend on \( u \). In particular, there exists a positive constant \( M_2 \) such that
\[
\int_{\mathbb{R}^3} \phi_u u^2 \, dx \leq M_2 \| u \|^2; \hspace{1cm} (3.8)
\]

ii. \( \phi_u \geq 0 \).

According to the Lemma 3.3, we define the functional \( I_\lambda : E \rightarrow \mathbb{R} \) by
\[ I_\lambda(u) := J_\lambda(u, \phi_u). \]

**Remark 3.4.** Using the relation \(-\Delta \phi_u = u^2\) and integration by parts, we can obtain

\[ \int \nabla \phi_u^2 dx = \int \phi_u u^2 dx. \]

Then, we can consider the functional 3.6 as following

\[ I_\lambda(u) = \frac{1}{2} \| u \|_2^2 + \frac{\lambda}{4} \int \phi_u u^2 dx - \frac{1}{p+1} \int |u|^{p+1} dx. \]

(3.9)

It well-known that \( I \) is \( C^1 \)-functional with derivative given by

\[ (I'_\lambda(u), u) = \int (\nabla u \nabla v + V(x) uv + \phi_u uv - |u|^{p-1}uv) dx \]

(3.10)

Now, using the proposition 2.3 in [16] we can consider the following proposition for our functional \( J_\lambda \):

**Proposition 3.5.** The following statements are equivalent:

i) \((u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)\) is a critical point \( J_\lambda \) i.e. \((u, \phi)\) is a solution of problem 1.1;

ii) \( u \) is a critical point of \( I_\lambda \) and \( \phi_u = \phi \).

**Proof.** It follows using the remark 3.2 and theorem 2.3 in [9].

4. **Proof of main theorem**

We take an orthogonal basis \( \{e_j\} \) of product space \( X := E \) and we define \( W_k := \text{span}\{e_j\}_{j=1,...,k}, Z_k := W_k^\perp \).

**Lemma 4.1.** [13] for any \( p \in [2, 2^*) \beta_k := \sup_{u \in Z_k, \|u\| = 1} \|u\|_{L^p} \to 0 \), as \( k \to \infty \).

Now, we prove that the functional \( I_\lambda : E \to \mathbb{R} \) satisfies the Cerami condition.

**Proposition 4.2.** Under the assumption 1.3, the functional \( I_\lambda(u) \) satisfies the Cerami condition at any positive level.

**Proof.** We suppose that \( \{u_n\} \) is the Cerami sequence, that is for some \( c \in \mathbb{R}^+ \),

\[ I_\lambda(u_n) = \frac{1}{2} \| u_n \|_2^2 + \frac{\lambda}{4} \int \phi_{u_n} u_n^2 dx - \frac{1}{p+1} \int |u_n|^{p+1} dx \to c \]

(4.1)

as \( n \to \infty \) and

\[ (1 + \|u_n\|_2)I'_\lambda(u_n) \to 0 \]

(4.2)

as \( n \to \infty \). From relations 4.1 and 4.2 for \( n \) large enough,

\[ 1 + c \geq I_\lambda(u_n) - \frac{\lambda}{4} (I'_\lambda(u_n), u_n) = \frac{1}{2} \| u_n \|_2^2 + \int \phi_{u_n} u_n^2 dx - \frac{1}{p+1} \int |u_n|^{p+1} dx - \frac{\lambda}{4} \int (\nabla u_n \nabla u_n + V(x) u_n^2 + \phi_{u_n} u_n^2 - |u_n|^{p-1} u_n^2) dx. \]

Then,
\[ 1 + c \geq \left( \frac{1}{2} - \frac{\lambda}{4} \right) \| u_n \|^2_E - \left( \frac{1}{p+1} - \frac{\lambda}{4} \right) \int_{\mathbb{R}^3} |u_n|^{p+1} dx. \]  

(4.3)

We show that \( \{u_n\} \) is bounded sequence. Otherwise, there exists a subsequence of \( \{u_n\} \) satisfying \( \| u_n \|_E \to \infty \) as \( n \to \infty \). Then we consider \( \omega_n = \frac{u_n}{\| u_n \|_E} \in E \), so the sequence \( \omega_n \) is bounded. Up to a subsequence, for some \( \omega \in E \),

\[ \omega_n \to \omega \]

in \( E \).

\[ \omega_n \to \omega \text{ in } L^t(\mathbb{R}^3) \ \forall t \in [2, 2') \]

and

\[ \omega_n(x) \to \omega(x) \text{ a.e. in } \mathbb{R}^3, \]  

(4.4)

Now, we consider two cases. In first case suppose that \( \omega \neq 0 \) in \( E \). Dividing by \( \| u_n \|^2_E \) in both sides of relation 4.1 and by lemma 3.3 we can get

\[ \frac{1}{p+1} \int_{\mathbb{R}^3} \frac{|u_n|^{p+1}}{|u_n|^2} dx = 1 + \int_{\mathbb{R}^3} \phi_{\omega_n} \frac{u_n^2}{\| u_n \|^2_E} dx + O(\| u_n \|^2_E) \leq M_3 < \infty \]  

(4.5)

where \( M_3 \) is a positive constant. We consider,

\[ \Omega := \{ x \in \mathbb{R}^3 \mid \omega(x) \neq 0 \}, \]

then for all \( x \in \Omega \) and \( p \in (2, \infty) \)

\[ \frac{|u_n|^{p+1}}{|u_n|^2} \omega_n(x)^2 \to +\infty \]  

(4.6)

as \( n \to \infty \). Since \( meas(\Omega) > 0 \), using Fatou’s lemma,

\[ \frac{1}{p+1} \int_{\mathbb{R}^3} \frac{|u_n|^{p+1}}{|u_n|^2} dx \to +\infty \]  

(4.7)

as \( n \to \infty \). This is contradiction with relation 4.5. In second case, suppose that \( \omega(x) = 0 \), then we define a sequence, \( t_n \in \mathbb{R} \) as

\[ I_\lambda(t_n u_n) = \max_{t \in [0,1]} I_\lambda(t u_n). \]

For any positive \( m \), we set \( \tilde{\omega}_n = \sqrt{4m} \frac{u_n}{\| u_n \|_E} = \sqrt{4m} \omega_n \). Hence, by relation 4.5 and for \( n \) large enough,

\[ I_\lambda(t_n u_n) \geq I_\lambda(\tilde{\omega}_n) = \frac{1}{2} \| \tilde{\omega}_n \|^2_E + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{\omega_n} \tilde{\omega}_n^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |\tilde{\omega}_n|^{p+1} dx \]

\[ = 2m + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{\omega_n} \tilde{\omega}_n^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^3} |\tilde{\omega}_n|^{p+1} dx \geq m. \]  

(4.8)

Therefore, \( \lim_{n \to \infty} I_\lambda(t_n u_n) = +\infty \) by relation 4.8. Since \( I_\lambda(0) = 0 \) and \( I_\lambda(u_n) \to c \) then for \( t_n \in (0,1) \) and \( n \) large enough, we obtain that
\[
\int_{\mathbb{R}^3} (\nabla t_n u_n \nabla t_n u_n + V(x) t_n u_n t_n u_n + \phi t_n u_n t_n u_n t_n u_n - |t_n u_n|^{p-1} t_n u_n t_n u_n) \, dx = \langle I'_\lambda (t_n u_n), t_n u_n \rangle
\]

\[
= t_n \frac{d}{dt} \big|_{t=t_n} I'_\lambda (t u_n) = 0.
\]

Hence, by assumption 1.3,

\[
I_\lambda (u_n) - \frac{\lambda}{4} \langle I'_\lambda (u_n), u_n \rangle = \frac{2 - \lambda}{4} \|u_n\|^2 - \frac{4 - \lambda(p + 1)}{4(p + 1)} \int_{\mathbb{R}^3} |u_n|^{p+1} \, dx = \]

\[
\frac{1}{2} \|u_n\|^2 - \frac{1}{p + 1} \int_{\mathbb{R}^3} |u_n|^{p+1} \, dx - \frac{\lambda}{4} \|u_n\|^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} |u_n|^{p+1} \, dx = \]

\[
\frac{1}{2} \|u_n\|^2 + \frac{\lambda}{4} \left[1 - \frac{1}{\lambda(p + 1)} \int_{\mathbb{R}^3} |u_n|^{p+1} \, dx - \|u_n\|^2 \right] = \]

\[
\frac{1}{2} \|u_n\|^2 + \frac{\lambda}{4} f_\lambda (u_n) \geq \frac{1}{2\theta} \|t_n u_n\|^2 + \frac{\lambda}{4\theta} f_\lambda (t_n u_n) = \]

\[
\frac{1}{\theta} I_\lambda (t_n u_n) - \frac{\lambda}{4\theta} \langle I'_\lambda (t_n u_n), t_n u_n \rangle \to \infty,
\]

as \( n \to \infty \). This contradicts relation 4.3. Therefore, \( \{u_n\} \) is bounded sequence. Assume that \( u_n \to u \) in \( E \). By lemma 3.1 \( u_n \to u \) in \( L^p(\mathbb{R}^3) \) for any \( t \in [2, 2^*) \). By relation 3.10,

\[
\|u_n - u\|^2 \leq \langle I'_\lambda (u_n) - I'_\lambda (u), u_n - u \rangle + \int_{\mathbb{R}^3} |u_n|^{p-1} - |u|^{p-1} (u_n - u) \, dx - \int_{\mathbb{R}^3} (\phi u_n - \phi u) (u_n - u) \, dx.
\]

By the Hölder inequality, Sobolev inequality and lemma 3.3

\[
\left| \int_{\mathbb{R}^3} \phi u_n u_n (u_n - u) \, dx \right| \leq \|\phi u_n\|_{L^2} \|u_n - u\|_{L^2} \leq \|\phi u_n\|_{L^2} \|u_n\|_{L^3} \|u_n - u\|_{L^2}
\]

\[
M_4 \|\phi u_n\|_{L^3} \|u_n - u\|_{L^2} \leq M_2 M_4 \|u_n\|_{L^2} \|u_n - u\|_{L^2} \leq M_2 M_4 \|u_n\|_{L^2} \|u_n - u\|_{L^2},
\]

where \( M_4 \) is a positive constant. Again using \( u_n \to u \) in \( L^p(\mathbb{R}^3) \) for any \( t \in [2, 2^*) \), we obtain that

\[
\int_{\mathbb{R}^3} \phi u_n u_n (u_n - u) \, dx \to 0,
\]

as \( n \to \infty \). Similarly,

\[
\int_{\mathbb{R}^3} \phi u_n u (u_n - u) \, dx \to 0,
\]

as \( n \to \infty \). Hence,

\[
\int_{\mathbb{R}^3} (\phi u_n u_n - \phi u u) (u_n - u) \, dx \to 0,
\]

as \( n \to \infty \). Thus \( \|u_n - u\|_E \to 0 \). Therefore, \( I_\lambda (u) \) satisfies Cerami condition.
Proof of theorem 2.3. From proposition 4.2, \( I_2(u) \) satisfies Cerami condition. Next, we show that \( I_2(u) \) satisfies the rest conditions of theorem 2.2. First of all, we prove that \( I_2(u) \) satisfies \( I_1 \). Since \( p \in (2,6) \), so \( \lim_{{|u| \to \infty}} \frac{|u|^{p+1}}{|u|^2} = +\infty \). Then for any \( K > 0 \) there exist \( \delta > 0 \) such that for \( |u| \geq \delta \),

\[
|u|^{p+1} \geq \frac{\lambda}{4} K |u|^2.
\]  

(4.9)

Hence,

\[
I_2(u) \leq \frac{1}{2} \|u\|^2_E + \frac{\lambda M_2}{4} \|u\|_4^4 - \frac{\lambda K}{4(p+1)} \|u\|_E^2.
\]

Since, norms on finite dimension spaces \( W_k \) are equivalent,

\[
I_2(u) \leq \frac{1}{2} \|u\|^2_E + \frac{\lambda M_2}{4} \|u\|_4^4 - \frac{\lambda KM_5}{4(p+1)} \|u\|_E^2,
\]

where \( M_5 \) is a constant. Since

\[
\frac{\lambda M_2}{4} - \frac{\lambda KM_5}{4(p+1)} < 0
\]

when \( K \) is large enough, it follows that

\[
a_k := \max_{{u \in W_k, |u| = \rho_k}} I_2(u) \leq 0
\]

for some \( \rho_k > 0 \) large enough. Using the lemma 3.3 and 3.1 we show that \( I_2(u) \) satisfies in condition \( I_2 \). By definition of \( I_2 \),

\[
I_2(u) \geq \frac{1}{2} \|u\|^2_E - \frac{1}{p+1} \int \|u|^{p+1} dx \geq \frac{1}{2} \|u\|^2_E - \frac{1}{p+1} \|u\|^{p+1}_E \geq \frac{1}{2} \|u\|^2_E - \frac{\beta_k^p}{p+1} \|u\|_E^p,
\]

where \( \beta_k \) is defined in lemma 4.1. defining \( r_k := \left( \frac{\rho_k}{p+1} \right)^{\frac{1}{2-p}} \), implies that

\[
b_k := \inf_{{u \in Z_k, |u| = r_k}} I_2(u) \geq \inf_{{u \in Z_k, |u| = r_k}} \left( \frac{1}{2} \|u\|^2_E - \frac{\beta_k^p}{p+1} \|u\|_E^p \right) \geq \left( 1 - \frac{1}{p} \right) \left( \frac{p \beta_k^p}{p+1} \right)^\frac{2}{2-p} \to +\infty
\]

as \( k \to \infty \). Using 2.2 completes the proof.

References