The rate of $\chi$-space defined by a modulus

N. Subramanian$^1$, P. Thirunavukkarasu$^2$, R. Babu$^3$

$^1$Department of Mathematics, SASTRA University, Thanjavur-613 401, India
$^2$P.G. and Research Department of Mathematics, Periyar E.V.R. College(Autonomous), Tiruchirappalli-620 023, India
$^3$Department of Mathematics, Shannugha Polytechnic College, Thanjavur-613 401, India

E-mail: nsmaths@yahoo.com, ptavinash1967@gmail.com, babunagar1968@gmail.com

Abstract: In this paper we introduce the modulus function of $\chi_\pi$. We establish some inclusion relations, topological results and we characterize the duals of the $\chi_\pi^2$ sequence spaces.

1. Introduction

A complex sequence, whose $k$th term is $x_k$ is denoted by $\{x_k\}$ or simply $x$. Let $w$ be the set of all sequences $x = (x_k)$ and $\varphi$ be the set of all finite sequences. Let $l_\infty$, $c$, $c_0$ be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of $l_\infty$, $c$, $c_0$ we have $\|x\| = \sup|x_k|$, where $x = (x_k) \in c_0 \subset c \subset l_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^\frac{1}{\pi} < \infty$. The vector space of all analytic sequences will be denoted by $\Lambda$. A sequence $x$ is called entire sequence if $\lim_{k \to \infty} |x_k|^\frac{1}{\pi}$. The vector space of all entire sequences will be denoted by $\Gamma$. $\chi$ was discussed in Kamthan [5]. Matrix transformation involving $\chi$ were characterized by Sridhar [14] and Sirajiudeen [13]. Let $\chi^2_\pi$ be the set of all those sequences $x = (x_k)$ such that $\left( k! \binom{x_k}{\pi} \right) ^{\frac{1}{\pi}} \to 0$ as $k \to \infty$. Then $\chi^2_\pi$ is a metric space with the metric

$$d(x, y) = \sup_k \left\{ k! \left[ \frac{|x_k| - |y_k|}{\pi} \right]^{\frac{1}{\pi}} \right\} ; k = 1, 2, 3, ...$$

Orlicz [11] used the idea of Orlicz function to construct the space ($L^M$). Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space $l_M$ contains a subspace isomorphic to $l_p$ ($1 \leq p < \infty$). Subsequently the different classes of sequence spaces were defined by Parashar and Choudhary [4], Mursaleen et al. [9], Bektas and Altin [1], Tripathy et al. [15], Rao and Subramanian [3] and many others.

The Orlicz sequence spaces is the special case of Orlicz space, studied in Ref [6].

Recall [6, 11] an Orlicz function is a function $M: [0, \infty] \to [0, \infty]$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of Orlicz function $M$ is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called modulus function, introduced by Nakano [10] and further discussed by Ruckle [12] and Maddox [8] and many others.

An Orlicz function $M$ is said to satisfy $\Delta_2$-condition for all values of $u$, if there exists a constant $k > 0$, such that $M(2u) \leq KM(u)$ ($u \geq 0$). The $\Delta_2$-condition is equivalent to $M(lu) \leq klM(u)$, for all values of $u$ and for $l > 1$. Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space...
\[ l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|X_k|}{\pi_k} \right) < \infty \text{ for some } \pi_k > 0 \right\} \] (1)

The space \( l_M \) with the norm

\[ \|x\| = \inf \left\{ \pi_k > 0 : \sum_{k=1}^{\infty} M \left( \frac{|X_k|}{\pi_k} \right) \leq 1 \right\} \] (2)

becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p, 1 \leq p < \infty \), the space \( l_M \) coincide with the classical sequence space \( l_p \). Given a sequence \( x = \{x_k\} \) its \( n \)th section is the sequence \( x^{(n)} = \{x_1, x_2, ..., \}, \delta^n = \{0, 0, ..., \}, \pi_k \) in the \( n \)th place and zero's elsewhere and \( S^n = \{0, 0, ..., \}, \pi_k \) in the \( n \)th place, \( \frac{-\pi_k}{k!} \) in the \( (n+1) \)st place and zero's elsewhere. An FK-space (Frechet Coordinate Space) is a Frechet Space which is made up of numerical sequences and has the property that the coordinate functionals \( P_k(x) = x_k (k = 1, 2, 3, ...) \) are continuous. We recall the following definitions (see [16]).

An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric space \((x, d)\) is said to have AK (or sectional convergence) if and only if \( d(x^{(n)}, x) \to 0 \) as \( n \to \infty \) (see [16]). The space is said to have AD (or) be an AD space if \( \varphi \) is dense in \( X \). We note that AK implies AD by [2].

If \( X \) is a sequence space, we define

1. \( X' \) is the continuous dual of \( X \);
2. \( X^a = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for each } x \in X\} \);
3. \( X^b = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in X\} \);
4. \( X^y = \{a = (a_k) : \sup |\sum_{k=1}^{n} a_k x_k| < \infty \text{ for each } x \in X\} \);
5. Let be an FK-space \( \varphi \). Then \( X^f = \left\{ f(\delta^{(n)}) : f \in X' \right\} \).

\( X^a, X^b, X^y \) are called the \( \alpha \)- (or K\( \ddot{o} \)the-Toeplitz) dual of \( X, \beta \)- (or generalized K\( \ddot{o} \)the-Toeplitz) dual of \( X, \gamma \)-dual of \( X \). Note that \( X^a \subset X^b \subset X^y \). If \( X \subset Y \) then \( Y^u \subset X^u \), for \( \mu = a, \beta \) or \( \gamma \).

**Lemma 1.1. (See[16, Theorem 7.27]).** Let \( X \) be an FK space \( \varphi \). Then (i) \( X^y \subset X' \). (ii) If \( X \) has AK, \( X^b = X^f \). (iii) If \( X \) has A.D., \( X^b = X^y \).

2. Definition and Preliminaries

Let \( w \) denote the set of all complex sequences \( x = \{x_k\}_{k=1}^{\infty} \) and \( f : [0, \infty) \to [0, \infty) \) be a modulus function.

Let

\[ X^f_f = \left\{ x \in w : \lim_{k \to \infty} \left( f \left( k! \left| \frac{x_k}{\pi_k} \right| \right) \right) = 0 \text{ for some } \pi_k > 0 \right\} \]

\[ \Gamma^f_f = \left\{ x \in w : \lim_{k \to \infty} \left( f \left( k! \left| \frac{x_k}{\pi_k} \right| \right) \right) = 0 \text{ for some } \pi_k > 0 \right\} \]

and

\[ \Lambda^f_f = \left\{ x \in w : \sup_{k} \left( f \left( k! \left| \frac{x_k - y_k}{\pi_k} \right| \right) \right) < \infty \text{ for some } \pi_k > 0 \right\} \]

The space \( X^f_f \) is a metric space with the metric

\[ d(x, y) = \inf \left\{ \pi_k > 0 : \sup_{k} \left( f \left( k! \left| \frac{x_k - y_k}{\pi_k} \right| \right) \right) \leq 1 \right\} \] (3)

The space \( \Gamma_f \) and \( \Lambda_f \) is a metric space with the metric
\[
d(x, y) = \inf \left\{ \pi_k > 0 : \sup_k \left( f \left( \frac{x_k - y_k}{\pi_k} \right)^2 \right) \leq 1 \right\}
\]

(4)

3. Main Result

Proposition 3.1.

\( \chi_f^\pi \subset \Gamma_f^\pi \) with the hypothesis that \( f \left( \frac{x_k}{\pi_k} \right)^2 \leq f \left( k! \frac{x_k}{\pi_k} \right)^2 \).

Proof. Let \( x \in \chi_f^\pi \). Then we have the following implications

\[
f \left( \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \right) \to 0 \text{ as } k \to \infty.
\]

(5)

But \( f \left( \frac{x_k}{\pi_k} \right)^\frac{1}{2} \leq f \left( \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \right) \); by our assumption, implies that

\[
\Rightarrow f \left( \frac{x_k}{\pi_k} \right)^\frac{1}{2} \to 0 \text{ as } k \to \infty \text{ by (5)}
\]

\[
\Rightarrow x \in \Gamma_f^\pi
\]

\[
\Rightarrow \chi_f^\pi \subset \Gamma_f^\pi.
\]

This completes the proof.

Proposition 3.2.

\( \chi_f^\pi \) has AK where \( f \) is a modulus function.

Proof. Let \( x = \{x_k\} \in \chi_f^\pi \), then \( \left\{ f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \right\} \in \chi_f^\pi \) and hence

\[
\sup_{k \geq n+1} f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \to 0 \text{ as } n \to \infty
\]

(6)

\[
d(x, x[n]) = \sup_{k \geq n+1} f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \to 0 \text{ as } n \to \infty \text{ by using (6)}.
\]

\[
\Rightarrow x[n] \to x \text{ as } n \to \infty,
\]

implying that \( \chi_f^\pi \) has AK. This completes the proof.

Proposition 3.3.

\( \chi_f^\pi \) is solid.

Proof. Let \( |x_k| \leq |y_k| \) and let \( y = (y_k) \in \chi_f^\pi \). \( f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \leq f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \), because \( f \) is non-decreasing. But

\[
f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \in \chi \), because \( y \in \chi_f^\pi \). That is, \( f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \to 0 \text{ as } k \to \infty \text{ and } f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \to 0 \text{ as } k \to \infty.
\]

Therefore, \( x = \{x_k\} \in \chi_f^\pi \). This completes the proof.

Proposition 3.4.

Let \( f \) be a modulus function which satisfies \( \Delta_2 \)-condition. Then \( \chi \subset \chi_f^\pi \).

Proof. Let \( x \in \chi \)

Then \( \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \leq \epsilon \) sufficiently large \( k \) and every \( \epsilon > 0 \). By taking \( \pi_k \geq \frac{1}{2} \), \( f \left( k! \frac{x_k}{\pi_k} \right)^\frac{1}{2} \leq f \left( \frac{x_k}{\pi_k} \right) \leq f(2\epsilon) \)

(because \( f \) is non-decreasing)
\[
f \left( k! \left( \frac{x_k}{\pi k} \right)^{\frac{1}{k}} \right) \leq kf(\epsilon)
\]

by \( \Delta_2 \)-condition, for some \( k \geq 0 \leq \epsilon. f \left( k! \left( \frac{x_k}{\pi k} \right)^{\frac{1}{k}} \right) \to 0 \) as \( k \to \infty \) (by defining \( f(\epsilon) < \frac{\epsilon}{k} \)). Hence \( x \in \chi^p_f \). From (7) and since

\[
x \in \chi^p_f,
\]

we get \( \chi \subseteq \chi^p_f \). This completes the proof.

**Proposition 3.5.**

If \( f \) is a modulus function, then \( \chi^p_f \) is linear space over the set of complex number \( \mathbb{C} \).

**Proof.** Let \( x, y \in \chi^p_f \) and \( \alpha, \beta \in \mathbb{C} \). In order to prove the result we need to find some \( \pi_k \) such that

\[
f \left( k! \left[ \frac{\alpha x_k + \beta y_k}{\pi_k} \right]^{\frac{1}{k}} \right) \to 0 \text{ as } k \to \infty
\]

Since \( x, y \in \chi^p_f \) such that

\[
f \left( k! \frac{\pi_k^{\frac{1}{k}}}{\pi_k} \right) \to 0 \text{ as } k \to \infty
\]

Since \( f \) is a non-decreasing modulus function, we have

\[
f \left( k! \left[ \frac{\alpha x_k + \beta y_k}{\pi_k} \right]^{\frac{1}{k}} \right) \leq f \left( k! \left[ \frac{\alpha x_k}{\pi k} \right]^{\frac{1}{k}} \right) + f \left( k! \left[ \frac{\beta y_k}{\pi_k} \right]^{\frac{1}{k}} \right) \leq f \left( k! \left[ \frac{\alpha x_k}{\pi k} \right]^{\frac{1}{k}} \right) + |\beta| \left( k! \left[ \frac{y_k}{\pi_k} \right]^{\frac{1}{k}} \right)
\]

Take \( \pi_k \) such that \( \frac{1}{\pi_k} = \min \left\{ \frac{1}{|\alpha| \pi_1}, \frac{1}{|\beta| \pi_2} \right\} \). Then

\[
f \left( k! \left[ \frac{\alpha x_k + \beta y_k}{\pi_k} \right]^{\frac{1}{k}} \right) \leq f \left( k! \left[ \frac{x_k}{\pi_k} \right]^{\frac{1}{k}} \right) + f \left( k! \left[ \frac{y_k}{\pi_k} \right]^{\frac{1}{k}} \right) \to 0 \text{ by (11)}.
\]

Hence \( f \left( k! \left[ \frac{\alpha x_k + \beta y_k}{\pi_k} \right]^{\frac{1}{k}} \right) \to 0 \text{ as } k \to \infty \). So \( (\alpha x + \beta y) \in \chi^p_f \). Therefore, \( \chi^p_f \) is linear. This completes the proof.

**Definition 3.6.**

Let \( P = (p_k) \) be any sequence of positive real numbers. Then we define \( \chi^p_f(P) = \left\{ x = (x_k): f \left( k! \left[ \frac{x_k}{\pi_k} \right]^{\frac{1}{k}} \right) \to 0 \text{ as } k \to \infty \right\} \). Suppose that \( P_k \) is a constant for all \( k \), the \( \chi^p_f(P) = \chi^p_f \).

**Proposition 3.7.**

Let \( 0 \leq p_k \leq q_k \) and let \( \left\{ \frac{q_k}{p_k} \right\} \) be bounded. Then \( \chi^p_f(q) = \chi^p_f t(p) \).

**Proof.** Let

\[
x \in \chi^p_f(q),
\]

\[
f \left( k! \left[ \frac{x_k}{\pi_k} \right]^{\frac{1}{k}} \right)^{q_k} \to 0 \text{ as } k \to \infty
\]

Let \( t_k = f \left( k! \left[ \frac{x_k}{\pi_k} \right]^{\frac{1}{k}} \right)^{q_k} \) and \( \lambda_k = \frac{p_k}{q_k} \).

Since \( p_k \leq q_k \), we have \( 0 \leq \lambda_k \leq 1 \).

Take \( 0 < \lambda < \lambda_k \). Define
\[ u_k = \begin{cases} t_k, & (t_k \geq 1) \\ 0, & (t_k < 1) \end{cases} \quad \text{and} \quad v_k = \begin{cases} 0, & (t_k \geq 1) \\ t_k, & (t_k < 1) \end{cases} \]

(14)

\[ t_k = u_k + v_k; t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}. \] Now it follows that \( u_k^{\lambda_k} \leq u_k \leq t_k \) and \( v_k^{\lambda_k} \leq v_k^{\lambda_k} \). Since \( t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \), then \( t_k^{\lambda_k} = t_k + v_k^{\lambda_k} \).

\[
\left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{p_k} \leq \left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{q_k}
\]

\[
\Rightarrow \left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{p_k/q_k} \leq \left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{q_k}
\]

But \( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \to 0 \) as \( k \to \infty \) by (13).

Therefore \( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \to 0 \) as \( k \to \infty \). Hence

\[ x \in \chi_f^p(p) \]

(15)

From (12) and (15) we get \( \chi_f^p(q) \subset \chi_f^p(p) \).

This completes the proof.

**Proposition 3.8.**

(a) Let \( 0 \leq \inf_{p_k} p_k \leq 1 \). Then \( \chi_f^p(p) \subset \chi_f^p \).

(b) Let \( 1 \leq p_k \leq \sup_{p_k} < \infty \). Then \( \chi_f^p \subset \chi_f^p(p) \).

**Proof.**

(a) Let \( x \in \chi_f^p(p) \)

\[
\left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{p_k} \to 0 \text{ as } k \to \infty
\]

(16)

Since \( 0 \leq \inf_{p_k} p_k \leq 1 \).

\[
\left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{p_k} \leq \left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{p_k}
\]

(17)

From (16) and (17) it follows that \( x \in \chi_f^p \). Thus \( \chi_f^p(p) \subset \chi_f^p \). We have thus proven (a).

(b) Let \( p_k \geq 1 \) for each \( k \) and \( \sup_{p_k} < \infty \).

Let \( x \in \chi_f^p \)

\[
\left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{p_k} \to 0 \text{ as } k \to \infty
\]

(18)

Since \( 1 \leq p_k \leq \sup_{p_k} < \infty \) we have

\[
\left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{p_k} \leq \left( f \left( k! \left( \frac{x_k}{\pi_k} \right)^{\frac{1}{x}} \right) \right)^{p_k}
\]

(19)
Let \( P_k \leq q \leq q_k < \infty \) for each \( k \). Then \( \chi_f^r(p) \subseteq \chi_f^r(q) \).

**Proof.** Let \( x \in \chi_f^r(p) \).

\[
\left( f \left( k! \frac{p_k}{p_k} \right)^{\frac{1}{q_k}} \right)^{p_k} \rightarrow 0 \text{ as } k \rightarrow \infty
\]

This implies that \( \left( f \left( k! \frac{p_k}{p_k} \right)^{\frac{1}{q_k}} \right)^{p_k} \leq 1 \) for sufficiently large \( k \).

Since \( f \) is non-decreasing, we get

\[
\left( f \left( k! \frac{p_k}{p_k} \right)^{\frac{1}{q_k}} \right)^{q_k} \leq \left( f \left( k! \frac{p_k}{p_k} \right)^{\frac{1}{p_k}} \right)^{p_k}
\]

\[
\Rightarrow \left( f \left( k! \frac{p_k}{p_k} \right)^{\frac{1}{p_k}} \right)^{q_k} \rightarrow 0 \text{ as } k \rightarrow \infty \text{ (by using (20))}
\]

\( x \in \chi_f^r(q) \)

Hence, \( \chi_f^r(p) \subseteq \chi_f^r(q) \).

This completes the proof.

**Proposition 3.10.**

\( \chi_f^r(p) \) is a \( r \)-convex for all \( r \) where \( 0 \leq r \leq \inf f \). Moreover if \( p_k = p \leq 1 \forall k \), then they are \( p \)-convex.

**Proof.** We shall prove the proposition for \( \chi_f^r(p) \). Let \( x \in \chi_f^r(p) \) and \( r \in \left( 0, \lim_{n \to \infty} p_n \right) \). Then, there exists \( k_0 \) such that \( r \leq p_k, \forall k > k_0 \). Now, define

\[
g^*(x) = \inf \left\{ \pi_k; f \left( k! \frac{p_k}{p_k} \right)^{\frac{1}{p_k}} + f \left( k! \frac{p_k}{p_k} \right)^{\frac{1}{p_k}} \right\}
\]

since, \( r \leq p_k \leq 1, \forall k > k_0 \). \( g^* \) is subadditive. Further, for \( 0 \leq |\lambda| \leq 1; |\lambda|^p \leq |\lambda|^r, \forall k > k_0 \).

\[
g^*(\lambda x) \leq |\lambda|^r g^*(x)
\]

Now, for \( 0 < \delta < 1 \),

\[
U = \{ x; g^*(x) \leq \delta \}, \text{ which is an absolutely } r \text{-convex set}
\]

for

\[
|\lambda|^r + |\mu|^r \leq 1; x, y \in U
\]

Now,

\[
g^*(\lambda x + \mu y) \leq g^*(\lambda x) + g^*(\mu y) \leq |\lambda|^r g^*(x) + |\mu|^r g^*(y) \leq |\lambda|^r \delta + |\mu|^r \delta \text{ using (23) and (24)}
\]

\[
\leq (|\lambda|^r + |\mu|^r) \delta \leq 1. \delta, \text{ by using (25) } \leq \delta
\]

If \( p_k = p \leq 1 \forall k \) then for \( 0 < r < 1 \), \( U = \{ x; g^*(x) \leq \delta \} \) is an absolutely \( p \)-convex set. This can be obtained by a similar analysis and therefore we omit the details. This completes the proof.

**Proposition 3.11.**

\[
\left( \chi_f^r \right)^\theta = \Lambda_f^\pi
\]

**Proof.**

**Step 1:** \( \chi_f^r \subset \Gamma_f^\pi \) by Proposition 3.1;

\[
\Rightarrow \left( \Gamma_f^\pi \right)^\theta \subset \left( \chi_f^r \right)^\theta \text{. But } \left( \Gamma_f^\pi \right)^\theta = \Lambda_f^\pi \text{ see (3)}.
\]
\[ \Lambda^\beta \subset (x^\beta) \]  

(26)  

**Step 2:** Let \( y \in (x^\beta) \) we have \( f(x) = \sum_{k=1}^{\infty} x_k y_k \) with \( x \in x^\beta \). We recall that \( S^{(k)} \) has \( \frac{1}{k!} \) in the \( k \)th place and zero’s elsewhere, with \( x = S^{(k)}, f\left( k! \frac{x_k}{\pi_k} \right) \subset \frac{1}{k!} \), 0, ... which converges to zero. Hence, \( S^{(k)} \in x^\beta \). Hence, \( d(S^{(k)}, 0) = 1 \). But \( |y_k| \leq ||f|| d(S^{(k)}, 0) < \infty \forall k \). Thus \( (y_k) \) is a bounded rate sequence and hence a rate analytic sequence. In other words \( y \in \Lambda^\beta \).

\[ (x^\beta)^\beta \subset \Lambda^\beta \]  

(27)  

**Step 3:** From (25) and (26) we obtain \( (x^\beta)^\beta = \Lambda^\beta \). This completes the proof.  

**Proposition 3.12.**  
\[ (x^\beta)^\beta = \Lambda \text{ for } \mu = \alpha, \beta, \gamma, f. \]  

**Proof.**  

**Step 1:** \( x_f \) has AK by Proposition 3.2. Hence, by Lemma 1.1 (ii).  

We get \( (x^\beta)^\beta = (x^\beta)^\gamma \). But \( (x^\beta)^\beta = \Lambda^\beta \).  

Hence  

\[ (x^\beta)^\beta = \Lambda^\beta \]  

(28)  

**Step 2:** Since AK \( \Rightarrow \) AD. Hence by Lemma 1.1 (iii).  

We get \( (x^\beta)^\beta = (x^\beta)^\gamma \). Therefore  

\[ (x^\beta)^\gamma = \Lambda^\beta \]  

(29)  

**Step 3:** \( x^\beta \) is normal by Proposition 3.3. Hence by Proposition ?? and (12), we get  

\[ (x^\beta)^\alpha = (x^\beta)^\gamma = \Lambda^\beta \]  

(30)  

From (28) and (30) we have \( (x^\beta)^\alpha = (x^\beta)^\beta = (x^\beta)^\gamma = (x^\beta)^\beta = \Lambda^\beta \).  

**Proposition 3.13.**  

The dual space of \( x^\beta \) is \( \Lambda \). In other words \( x_f = \Lambda \).  

**Proof.**  

We recall that \( S^{(k)} \) has \( \frac{\pi_k}{k!} \) in the \( k \)th place and zero’s elsewhere with  

\[ x = S^{(k)}, f(k! \frac{x_k}{\pi_k})^{\beta} = \left\{ 0, 0, ..., f\left( \frac{(1)!}{\pi_k} \right), 0, ... \right\} \]  

Hence, \( S^{(k)} \in x^\beta \). We have \( f(x) = \sum_{k=1}^{\infty} x_k y_k \) with \( x \in x^\beta \) and \( f \in (x^\beta)^\alpha \) where \( x^\beta \) is the dual space of \( x^\beta \). Take \( x = S^{(k)} \in x^\beta \). Then  

\[ |y_k| \leq ||f|| d(S^{(k)}, 0) < \infty \text{ for all } k. \]  

(31)  

Thus \( (y_k) \) is a bounded rate sequence and hence a rate analytic sequence. In other words, \( y \in \Lambda \). Therefore \( x_f = \Lambda \). This completes the proof.  

**Lemma 3.14 ([16, Theorem 8.6.1]).**  

\( Y \supset X \Leftrightarrow Y^f \subset X^f \) where \( X \) is an AD-space and \( Y \) on FK-space.  

**Proposition 3.15.**  

Let \( Y \) be any FK-space \( \supset \varphi \). Then \( Y \supset x_f^\beta \) if and only if the sequence \( S^{(k)} \) is weakly analytic.  

**Proof.** The following implications establish the result
\[ Y \ni \chi_f^\alpha \; \iff \; Y' \subset \chi_f^\beta \text{ since } \chi_f \text{ has AD by Lemma 3.14} \]
\[ \iff \text{ for each } f \in Y', \text{ the topological dual of } Y. \]
\[ \iff f(S^{(k)}) \text{ is rate of analytic.} \]
\[ \iff S^{(k)} \text{ is weakly rate of analytic.} \]

This completes the proof.

**Proposition 3.16.**

\( \chi_f^\beta \) is a complete metric space under the metric

\[ d(x, y) = \sup_k \left\{ f \left( k! \left\| \frac{x_k - y_k}{\pi_k} \right\|^\frac{1}{p} \right) : k = 1, 2, 3, \ldots \right\} \]

Where \( x = (x_k) \in \chi_f^\beta \) and \( y = (y_k) \in \chi_f^\beta \).

**Proof.** Let \( \{x^{(n)}\} \) be Cauchy sequence in \( \chi_f^\beta \). Then given any \( \epsilon > 0 \) there exists a positive integer \( N \) depending on \( \epsilon \) such that \( d(x^{(n)}, x^{(m)}) < \epsilon \) for all \( n \geq N \) and for \( m \geq N \). Hence, \( \sup_k \left\{ f \left( k! \left\| \frac{x_k^{(n)} - x_k^{(m)}}{\pi_k} \right\|^\frac{1}{p} \right) \right\} < \epsilon \) for all \( n \geq N \) and for \( m \geq N \).

Consequently \( f \left( k! \left| \frac{x_k^{(n)}}{\pi_k} \right|^{\frac{1}{p}} \right) \) is a Cauchy sequence in the metric space \( C \) of a complex numbers.

But \( C \) is complete. So,

\[ f \left( k! \left| \frac{x_k^{(n)}}{\pi_k} \right|^{\frac{1}{p}} \right) \rightarrow f \left( k! \left| \frac{x_k}{\pi_k} \right|^{\frac{1}{p}} \right) \text{ as } n \rightarrow \infty. \]

Hence there exists a positive integer no such that

\[ \sup_k \left\{ f \left( k! \left| \frac{x_k^{(n)} - x_k}{\pi_k} \right|^{\frac{1}{p}} \right) \right\} < \epsilon \text{ for all } n \leq n_0. \]

In particular, we have

\[ f \left( k! \left| \frac{x_k^{(n)} - x_k}{\pi_k} \right|^{\frac{1}{p}} \right) < \epsilon. \]

Now

\[ f \left( k! \left| \frac{x_k}{\pi_k} \right|^{\frac{1}{p}} \right) \leq f \left( k! \left| \frac{x_k^{(n)} - x_k}{\pi_k} \right|^{\frac{1}{p}} \right) + f \left( k! \left| \frac{x_k^{(n)}}{\pi_k} \right|^{\frac{1}{p}} \right) < \epsilon \rightarrow 0 \text{ as } k \rightarrow \infty. \]

Thus

\[ f \left( k! \left| \frac{x_k}{\pi_k} \right|^{\frac{1}{p}} \right) < \epsilon \rightarrow 0 \text{ as } k \rightarrow \infty. \]

That is \( x \in \chi_f^\beta \).

Therefore \( \chi_f^\beta \) is a complete metric space. This completes the proof.

**References**


