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Bell-based Genocchi polynomials

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Abstract: In this study, we introduce Bell-based Genocchi polynomials of order α and then derive multifarious correlations and formulas including some implicit summation formulas and derivative properties.

Keywords: Genocchi polynomials, Bell polynomials, Mixed-type polynomials, Stirling numbers of the second kind.

1 Introduction

Throughout this paper, the familiar symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are referred to the set of all complex numbers, the set of all real numbers, the set of all integers, the set of all natural numbers, and the set of all non-negative integers, respectively. The Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α and the Genocchi polynomials $G_n^{(\alpha)}(x)$ of order α are defined as follows (*cf*. [1], [2], [5]-[7]):

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} \quad (|t|<2\pi) \tag{1}$$

and

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} \quad (|t|<\pi). \tag{2}$$

Letting $x=0$ in (1) and (2), we get $B_n^{(\alpha)}(0) := B_n^{(\alpha)}$ and $G_n^{(\alpha)}(0) := G_n^{(\alpha)}$ known as the Bernoulli numbers of order α and the Genocchi numbers of order α. When $\alpha=1$ in (1) and (2), not only the polynomials $B_n^{(\alpha)}(x)$ and numbers $B_n^{(\alpha)}$ reduce to the classical Bernoulli polynomials $B_n(x)$ and numbers B_n, but also the polynomials $G_n^{(\alpha)}(x)$ and numbers $G_n^{(\alpha)}$ reduce to the familiar Genocchi polynomials $G_n(x)$ and numbers G_n.

The Stirling polynomials $S_2(n,k:x)$ and numbers $S_2(n,k)$ of the second kind are given by the following exponential generating functions (*cf*. [5], [6], [8]):

$$\sum_{n=0}^{\infty} S_2(n,k:x)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}e^{tx} \text{ and } \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}. \tag{3}$$

The Stirling numbers of the second kind can also be derived by the following recurrence relation for $n\in\mathbb{N}_0$ (*cf*. [5], [6], [8]):

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{4}$$

where $(x)_n = x(x-1)(x-2)\cdots(x-(n-1))$ for $n\in\mathbb{N}$ with $(x)_0=1$.

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The bivariate Bell polynomials are defined as follows (*cf.* [5], [8], [9]):

$$\sum_{n=0}^{\infty} Bel_n(x;y)\frac{t^n}{n!} = e^{y(e^t-1)}e^{xt} \tag{5}$$

When $x=0$, $Bel_n(0;y) := Bel_n(y)$ called the classical Bell polynomials given by means of the following generating function (*cf.* [3], [5], [6], [8]-[10]):

$$\sum_{n=0}^{\infty} Bel_n(y)\frac{t^n}{n!} = e^{y(e^t-1)}. \tag{6}$$

The Bell numbers Bel_n are acquired by taking $y=1$ in (6), that is $Bel_n(0;1) = Bel_n(1) := Bel_n$ and are given by the following exponential generating function (*cf.* [4]):

$$\sum_{n=0}^{\infty} Bel_n\frac{t^n}{n!} = e^{(e^t-1)}. \tag{7}$$

The Bell polynomials have been intensely investigated and studied by several mathematicians, *cf.* [2]-[4], [9] and see also the references cited therein.

The Bell-based Stirling polynomials of the second kind are defined as follows (*cf.* [5]):

$$\sum_{n=0}^{\infty} {}_{Bel}S_2(n,k:x,y)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}e^{xt+y(e^t-1)}. \tag{8}$$

For more detailed information about the properties of Bell-based Stirling polynomials of the second kind, see the reference [5].

2 Bell-based Genocchi polynomials of order α

Recently, Duran et al. introduced the Bell-based Bernoulli polynomials of order α by the following exponential generating function (*cf.* [5]):

$$\sum_{n=0}^{\infty} {}_{Bel}B_n^{(\alpha)}(x;y)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+y(e^t-1)} \tag{9}$$

Diverse properties and relations of the Bell-based Bernoulli polynomials of order α have been examined in [5]. Also recently, Khan et al. [8] defined Bell-based Euler polynomials and investigate some of their porperties. By the same motivation, we now introduce the Bell-based Genocchi polynomials of order α as follows.

Definition 1. *The Bell-based Genocchi polynomials of order α are introduced by the following exponential generating function:*

$$\sum_{n=0}^{\infty} {}_{Bel}G_n^{(\alpha)}(x;y)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt+y(e^t-1)} \tag{10}$$

Some particular circumstances of ${}_{Bel}G_n^{(\alpha)}(x;y)$ are examined below.

Remark 1. *In the special case $x=0$ in (10), we acquire Bell-Genocchi polynomials ${}_{Bel}G_n^{(\alpha)}(y)$ of order α, which are also new extensions of the Genocchi numbers of order α in (2), as follows:*

$$\sum_{n=0}^{\infty} {}_{Bel}G_n^{(\alpha)}(y)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^{\alpha} e^{y(e^t-1)}. \tag{11}$$

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We also note that

$$ {}_{Bel}G_n^{(1)}(x;y) := {}_{Bel}G_n(x;y) $$

which we call the Bell-based Genocchi polynomials.

Theorem 1. *Each of the following summation formulae*

$$ {}_{Bel}G_n^{(\alpha)}(x;y) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(\alpha)} Bel_{n-k}(x;y) \tag{12} $$

$$ {}_{Bel}G_n^{(\alpha)}(x;y) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(\alpha)}(x) Bel_{n-k}(y) \tag{13} $$

$$ {}_{Bel}G_n^{(\alpha)}(x;y) = \sum_{k=0}^{n} \binom{n}{k} {}_{Bel}G_k^{(\alpha)}(y) x^{n-k} \tag{14} $$

hold for $n \in \mathbb{N}_0$.

Proof. By (10), we have

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_{Bel}G_n^{(\alpha)}(x;y) \frac{t^n}{n!} &= \left(\frac{2t}{e^t+1} \right)^{\alpha} e^{xt+y(e^t-1)} \\ &= \left(\sum_{n=0}^{\infty} G_n^{(\alpha)} \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} Bel_n(x;y) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} G_k^{(\alpha)} Bel_{n-k}(x;y) \right) \frac{t^n}{n!}, \end{aligned} $$

which implies the desired result (12). The others are similar to (12). So, we omit them. We give some theorems without their proofs which are similar to Theorem 1. So, we omit them.

Theorem 2. *The following relationship*

$$ {}_{Bel}G_n^{(\alpha_1+\alpha_2)}(x_1+x_2;y_1+y_2) = \sum_{k=0}^{n} \binom{n}{k} {}_{Bel}G_k^{(\alpha_1)}(x_1;y_1) \, {}_{Bel}G_{n-k}^{(\alpha_2)}(x_2;y_2) \tag{15} $$

is valid for $n \in \mathbb{N}_0$.

Theorem 3. *The difference operator formulas for the Bell-based Genocchi polynomials*

$$ \frac{\partial}{\partial x} {}_{Bel}G_n^{(\alpha)}(x;y) = n \, {}_{Bel}G_{n-1}^{(\alpha)}(x;y) \tag{16} $$

and

$$ \frac{\partial}{\partial y} {}_{Bel}G_n^{(\alpha)}(x;y) = {}_{Bel}G_n^{(\alpha)}(x+1;y) - {}_{Bel}G_n^{(\alpha)}(x;y). \tag{17} $$

hold for $n \in \mathbb{N}$.

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Theorem 4. *The following summation formula*

$$Bel_n(x;y) = \frac{{}_{Bel}G_{n+1}(x+1;y) + {}_{Bel}G_{n+1}(x;y)}{2(n+1)} \tag{18}$$

holds for $n \in \mathbb{N}_0$.

Theorem 5. *The following formula including the Bell-based Genocchi polynomials of higher-order and Stirling numbers of the second kind*

$$Bel_n(x;y) = \frac{n!}{(n+k)!} 2^{-k} \sum_{l=0}^{k} \binom{k}{l} {}_{Bel}G_{n+k}^{(k)}(x+l;y) \tag{19}$$

is valid for $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$.

Theorem 6. *The following correlation*

$${}_{Bel}G_n^{(\alpha)}(x;y) = \sum_{l=0}^{n} \sum_{k=0}^{\infty} \binom{n}{l} (x)_k S_2(l,k)\, {}_{Bel}G_{n-l}^{(\alpha)}(y) \tag{20}$$

holds for non-negative integers n.

Proof. By means of Definition 1 and, using (3) and (11), we obtain

$$\sum_{n=0}^{\infty} {}_{Bel}G_n^{(\alpha)}(x;y) \frac{t^n}{n!} = \frac{(2t)^{\alpha}}{(e^t+1)^{\alpha}} e^{y(e^t-1)} \left(e^t - 1 + 1\right)^x = \frac{(2t)^{\alpha}}{(e^t+1)^{\alpha}} e^{y(e^t-1)} \sum_{k=0}^{\infty} (x)_k \frac{(e^t-1)^k}{k!}$$

which gives the asserted result (20).

Theorem 7. *The following summation formula*

$$\frac{2^k}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^l\, {}_{Bel}G_n^{(l)}(x_1+x_2;y_1+y_2) = \frac{n!}{(n+k)!} \sum_{l=0}^{n+k} \binom{n+k}{l} {}_{Bel}G_l^{(k)}(x_2;y_2)\, {}_{Bel}S_2(n+k-l,k:x_1,y_1) \tag{21}$$

holds for non-negative integers k and n with $n \geq k$.

Proof. The proof is similar to that of Theorem 13 in reference [5]. So, we omit it.

The following series manipulation formula holds (*cf.* [5]):

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n}{n!} \frac{y^m}{m!}. \tag{22}$$

Theorem 8. *The following implicit summation formula holds:*

$${}_{Bel}G_{k+l}^{(\alpha)}(x;y) = \sum_{n,m=0}^{k,l} \binom{k}{n} \binom{l}{m} (x-z)^{n+m}\, {}_{Bel}G_{k+l-n-m}^{(\alpha)}(z;y). \tag{23}$$

Proof. Upon setting t by $t+u$ in (10), we derive

$$\left(\frac{2t}{e^{t+u}+1}\right)^{\alpha} e^{y\left(e^{t+u}-1\right)} = e^{-z(t+u)} \sum_{k,l=0}^{\infty} {}_{Bel}G_{k+l}^{(\alpha)}(z;y) \frac{t^k}{k!} \frac{u^l}{l!}.$$

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Also, changing z by x in the last equation, and using (22), we get

$$e^{-x(t+u)}\sum_{k,l=0}^{\infty}{}_{Bel}G_{k+l}^{(\alpha)}(x;y)\frac{t^k}{k!}\frac{u^l}{l!}=\left(\frac{2t}{e^{t+u}+1}\right)^{\alpha}e^{y\left(e^{t+u}-1\right)}$$

By the last two equations, we obtain

$$\sum_{k,l=0}^{\infty}{}_{Bel}G_{k+l}^{(\alpha)}(x;y)\frac{t^k}{k!}\frac{u^l}{l!}=e^{(x-z)(t+u)}\sum_{k,l=0}^{\infty}{}_{Bel}G_{k+l}^{(\alpha)}(z;y)\frac{t^k}{k!}\frac{u^l}{l!},$$

which yields

$$\begin{aligned}\sum_{k,l=0}^{\infty}{}_{Bel}G_{k+l}^{(\alpha)}(x;y)\frac{t^k}{k!}\frac{u^l}{l!}&=\sum_{n,m=0}^{\infty}(x-z)^{n+m}\frac{t^n}{n!}\frac{u^m}{m!}\sum_{k,l=0}^{\infty}{}_{Bel}G_{k+l}^{(\alpha)}(z;y)\frac{t^k}{k!}\frac{u^l}{l!}\\&=\sum_{k,l=0}^{\infty}\sum_{n,m=0}^{k,l}\frac{(x-z)^{n+m}\ {}_{Bel}G_{k+l-n-m}^{(\alpha)}(z;y)}{n!m!(k-l)!(l-m)!}t^k u^l,\end{aligned}$$

which implies the asserted result (23).

Theorem 9. *The following symmetric identity holds for $a,b\in\mathbb{R}$ and $n\geq 0$:*

$$\sum_{k=0}^{n}\binom{n}{k}{}_{Bel}G_{n-k}^{(\alpha)}(bx;y)\ {}_{Bel}G_{k}^{(\alpha)}(ax;y)\,a^{n-2k}=\sum_{k=0}^{n}\binom{n}{k}{}_{Bel}G_{k}^{(\alpha)}(bx;y)\ {}_{Bel}G_{n-k}^{(\alpha)}(ax;y)\,b^{n-2k}. \tag{24}$$

Proof. The proof is based on the expression

$$\Upsilon=\left(\frac{2^2t^2}{(e^{at}+1)(e^{bt}+1)}\right)^{\alpha}e^{2abxt+y(e^{at}-1)+y\left(e^{bt}-1\right)}.$$

and is similar to that of Theorem 15 in reference [5].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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