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Taylor polynomial approach for systems of linear differential equations in normal form and residual error estimation

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Abstract: The purpose of this paper is to give a matrix method based on Taylor polynomials for solving linear differential equations system with variable coefficients in the normal form under the initial conditions by using residual error function. The presented method converts the problem into a system of algebraic equations via the matrix operations and collocation points. In order to demonstrate the accuracy of solution and efficiency of the method, two numerical examples are given with the help of computer programmes written in *Maple* and *Matlab*.

Keywords: Taylor collocation method; approximate solution; differential equations system; collocation points; residual error analysis.

1 Introduction

In many scientific problems, the differential equations systems have been encountered. Some of these differential equation systems do not have analytic solutions, so numerical methods are required. Many mathematicians have solved the systems of linear differential equations by using various methods such as Adomian decomposition method [1,2,3], Adomian-Pade technique [4], Variational iteration mathod [5], Differential transform method [6-9], Linearizability criteria [10,11], finite difference method [12], Trigonometric approximation [13] and Product summability transform method [14].

Taylor, Bessel, Berstein, Chebyshev, Legendre, Hermite, Laguerre, Exponential, Bernoulli matrix methods are used for solving differential and integral equations, integro-differential-difference equations and their systems [15-31].

In this study, we give Taylor collocation method for solving the linear differential equations system in the normal form as

$$L[y_i(x)] = {y_i}'(x) - \sum_{j=1}^{m} p_{i,j}(x) y_j(x) = g_i(x), \quad (0 \le a \le x \le b) \tag{1}$$

under the initial conditions

$$y_i(a) = c_i, \quad (i = 1,2,...,m) \tag{2}$$

where $y_i(x)$, $(i = 1,2,...,m)$ are unknown functions, $p_{i,j}(x)$ and $g_i(x)$ are the known continuous functions defined on interval $[a,b]$, and c_i are the real constants.

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In this paper, by developing the Taylor collocation method with the help of the residual error function used in [20-23,], we obtain approximate solutions of the system (1) expressed in the truncated Taylor series form

$$y_{i,N,M}(x) = y_{i,N}(x) + e_{i,N,M}(x), \quad (i = 1,2,...,m)$$

where

$$y_i \cong y_{i,N}(x) = \sum_{n=0}^{N} a_{i,n}x^n \tag{3}$$

is the Taylor polynomial solution and

$$e_{i,N,M}(x) = \sum_{n=0}^{M} a^*_{i,n}x^n\,, \quad (M > N)$$

is the estimated error function based on the residual error function. Here $a_{i,n}$ and $a^*_{i,n}$, $(n = 0,1,2,...,N)$ are the unknown Taylor coefficients.

In order to find the solutions of the system (1) with the initial conditions (2), we can use the collocation points defined by

$$x_k = a + \frac{b-a}{N}k, \quad k = 0,1,...,N, \quad 0 \le a \le x \le b. \tag{4}$$

2 Fundamental matrix relations

We can write the approximate solutions $y_{i,N}(x)$, i=1,2,...,m, given by Eq.(3) in the matrix form

$$y_{i,N}(x) = \mathrm{X}(x)\mathrm{A}_i, \quad (i = 1,2,...,m) \tag{5}$$

where

$$\mathrm{X}(x) = \begin{bmatrix} 1 \; x \; x^2 \; \cdots \; x^N \end{bmatrix}$$

and

$$\mathrm{A}_i = \begin{bmatrix} a_{i,0} \; a_{i,1} \; a_{i,2} \; \cdots \; a_{i,N} \end{bmatrix}^T.$$

From Eq.(5), we can express the approximate solutions $y_{i,N}(x)$ as

$$\mathrm{Y}(x) = \overline{\mathrm{X}}(x)\mathrm{A}. \tag{6}$$

where

$$\mathrm{Y}(x) = \begin{bmatrix} y_{1,N}(x) \\ y_{2,N}(x) \\ \vdots \\ y_{m,N}(x) \end{bmatrix}, \overline{\mathrm{X}}(x) = \begin{bmatrix} \mathrm{X}(x) & 0 & \cdots & 0 \\ 0 & \mathrm{X}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{X}(x) \end{bmatrix}, \mathrm{A} = \begin{bmatrix} \mathrm{A}_1 \\ \mathrm{A}_2 \\ \vdots \\ \mathrm{A}_m \end{bmatrix}.$$

Also, the relation between the matrix $\mathrm{X}(x)$ and its derivative $\mathrm{X}'(x)$ is

$$\mathrm{X}'(x) = \mathrm{X}(x)\,\mathrm{B} \tag{7}$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By using the relations (5) and (7), we obtain the following matrix relation

$$y'_{i,N}(x) = X(x)BA_i, \ (i = 1,2,...,m). \tag{8}$$

Hence, we can write the system (8) as

$$Y'(x) = \overline{X}(x)\overline{B}A \tag{9}$$

where

$$Y'(x) = \begin{bmatrix} y'_{1,N}(x) \\ y'_{2,N}(x) \\ \vdots \\ y'_{m,N}(x) \end{bmatrix}, \overline{B} = \begin{bmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B \end{bmatrix}.$$

3 Method for solution

We can write the system (1) in the matrix form

$$Y'(x) = P(x)Y(x) + G(x) \tag{10}$$

where

$$P(x) = \begin{bmatrix} p_{1,1}(x) & p_{1,2}(x) & \cdots & p_{1,m}(x) \\ p_{2,1}(x) & p_{2,1}(x) & \cdots & p_{2,1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{m,1}(x) & p_{m,2}(x) & \cdots & p_{m,m}(x) \end{bmatrix}, G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_m(x) \end{bmatrix}.$$

By substituting the collocation points (4) into Eq.(10), we obtain the system of matrix equations

$$Y'(x_k) = P(x_k)Y(x_k) + G(x_k)\ , (k = 0,1,...,N).$$

Briefly, the fundamental matrix equation is

$$Y' = PY + G \tag{11}$$

where

$$P = \begin{bmatrix} P(x_0) & 0 & \cdots & 0 \\ 0 & P(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P(x_N) \end{bmatrix}, Y = \begin{bmatrix} Y(x_0) \\ Y(x_1) \\ \vdots \\ Y(x_N) \end{bmatrix}, Y' = \begin{bmatrix} Y'(x_0) \\ Y'(x_1) \\ \vdots \\ Y'(x_N) \end{bmatrix}, G = \begin{bmatrix} G(x_0) \\ G(x_1) \\ \vdots \\ G(x_N) \end{bmatrix}.$$

By using the relations (6) and (9) along with the collocation points (4), we obtain

$$Y(x_k) = \overline{X}(x_k)A \text{ and } Y'(x_k) = \overline{X}(x_k)\overline{B}A,\ (k = 0,1,...,N)$$

or briefly

$$\mathrm{Y} = \mathrm{XA} \quad \text{and} \quad \mathrm{Y}' = \mathrm{X}\overline{\mathrm{B}}\mathrm{A} \tag{12}$$

where

$$\mathrm{X} = \begin{bmatrix} \overline{\mathrm{X}}(x_0) \\ \overline{\mathrm{X}}(x_1) \\ \vdots \\ \overline{\mathrm{X}}(x_N) \end{bmatrix}, \overline{\mathrm{X}}(x_s) = \begin{bmatrix} \mathrm{X}(x_k) & 0 & \cdots & 0 \\ 0 & \mathrm{X}(x_k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{X}(x_k) \end{bmatrix}.$$

By substituting the relations given by (12) into Eq.(11), we gain the fundamental matrix equation as

$$\left\{\mathrm{X}\overline{\mathrm{B}} - \mathrm{PX}\right\}\mathrm{A} = \mathrm{G}. \tag{13}$$

In Eq.(13) the full dimensions of the matrices P, X, $\overline{\mathrm{B}}$, A and G are $m(N+1)\times m(N+1)$, $m(N+1)\times m(N+1)$, $m(N+1)\times m(N+1)$, $m(N+1)\times 1$ and $m(N+1)\times 1$, respectively.

The fundamental matrix equation (13) corresponding to Eq.(1) can be written in the form

$$\mathrm{WA} = \mathrm{G} \;\; \text{or} \quad [\mathrm{W};\mathrm{G}]\,. \tag{14}$$

This is a linear system of $m(N+1)$algebraic equations in $m(N+1)$ the unknown Taylor coefficients such that

$$\mathrm{W} = \mathrm{X}\overline{\mathrm{B}} - \mathrm{PX} = [w_{p,q}], \quad p,q = 1,2,...,m(N+1).$$

By using the conditions (4) and the relations (6), the matrix form for the conditions is obtained as

$$\overline{\mathrm{X}}(a)\mathrm{A} = \mathrm{C} \tag{15}$$

where

$$\mathrm{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_m \end{bmatrix}^T.$$

Hence, the fundamental matrix form for conditions is

$$\mathrm{UA} = \mathrm{C} \;\; \text{or} \quad [\mathrm{U};\mathrm{C}] \tag{16}$$

such that

$$\mathrm{U} = \overline{\mathrm{X}}(a).$$

Consequently, we obtain the Taylor polynomial solution of the system (1) under the initial conditions (2) by replacing the row matrices (16) by last rows of the matrix (14). Then, we obtain the new augmented matrix

$$\widetilde{\mathrm{W}}\mathrm{A} = \widetilde{\mathrm{G}} \;\; \text{or} \quad \left[\widetilde{\mathrm{W}};\widetilde{\mathrm{G}}\right]. \tag{17}$$

If rank $\widetilde{\mathrm{W}}$ =rank$\left[\widetilde{\mathrm{W}};\widetilde{\mathrm{G}}\right] = m(N+1)$, then we can write

$$\mathrm{A} = \left(\widetilde{\mathrm{W}}\right)^{-1}\widetilde{\mathrm{G}}. \tag{18}$$

By solving this linear system, the unknown Taylor coefficients matrix A is determined and $a_{i,0}, a_{i,1}, ..., a_{i,N}$ $(i = 1,2,...,m)$ are substituted in Eq.(3). Thus, we find the Taylor polynomial solutions

$$y_{i,N}(x) = \sum_{n=0}^{N} a_{i,n}x^n, \quad (i = 1,2,...,m).$$

4 Residual Correction and Error Estimation

In this section, we will give an error estimation for the Taylor polynomial solutions (3) with the residual error function [20-23,24,27,29]. In addition, we will develop the Taylor polynomial solutions (3) by means of the residual error function. Firstly, we can define the residual function of the Taylor collocation method as

$$R_{i,N}(x) = L[y_{i,N}(x)] - g_i(x), \quad (i = 1,2,...,m). \tag{19}$$

Here, $y_{i,N}(x)$ represent the Taylor polynomial solutions given by (3) of the problem (1) and (2). Hence, $y_{i,N}(x)$ satisfies the problem

$$\begin{cases} y'_{i,N}(x) - \sum\limits_{j=1}^{m} p_{i,j}(x)y_{j,N}(x) = g_i(x) + R_{i,N}(x), & (i = 1,2,...,m) \\ y_{i,N}(a) = c_i, \quad (i = 1,2,...,m). \end{cases}$$

Also, the error function $e_{i,N}(x)$ can be defined as

$$e_{i,N}(x) = y_i(x) - y_{i,N}(x) \tag{20}$$

where $y_i(x)$ are the exact solutions of the problem (1) and (2). From Eqs.(1), (2), (19) and (20), we obtain a system of error differential equations

$$L[e_{i,N}(x)] = L[y_i(x)] - L[y_{i,N}(x)] = -R_{i,N}(x)$$

with the homogeneous initial conditions

$$e_{i,N}(a) = 0, \ (i = 1,2,...,m),$$

or openly, the error problem can be expressed as

$$\begin{cases} e'_{i,N}(x) - \sum\limits_{j=1}^{m} p_{i,j}(x)e_{j,N}(x) = -R_{i,N}(x), & (i = 1,2,...,m) \\ e_{i,N}(a) = 0, \quad (i = 1,2,...,m). \end{cases} \tag{21}$$

Here, the nonhomegeneous initial conditions

$$y_i(a) = c_i \text{ and } y_{i,N}(a) = c_i$$

are reduced to homogeneous initial conditions

$$e_{i,N}(a) = 0.$$

The error problem (21) can be solved by using the prosedure given in Section 3. Thus, we obtain the approximation

$$e_{i,N,M}(x) = \sum_{n=0}^{M} a^*_{i,n}x^n, \quad (M > N, \quad i = 1,2,...,m)$$

to $e_{i,N}(x)$. Consequently, the corrected Taylor polynomial solution $y_{i,N,M}(x) = y_{i,N}(x) + e_{i,N,M}(x)$ is obtained by means of the polynomials $y_{i,N}(x)$ and $e_{i,N,M}(x)$. Also, we construct the error function $e_{i,N}(x) = y_i(x) - y_{i,N}(x)$, the estimated error function $e_{i,N,M}(x)$ and the corrected error function $E_{i,N,M}(x) = e_{i,N}(x) - e_{i,N,M}(x) = y_i(x) - y_{i,N,M}(x)$.

5 Numerical Examples

In this section, two numerical examples are given to demonstrate the efficiency and applicability of the method. The computations related to the examples are calculated by using the *Maple* programme and the figures are drawn in *Matlab*. In tables and figures, we calculate the values of the Taylor polynomial solutions $y_{i,N}(x)$, the corrected Taylor polynomial solutions $y_{i,N,M}(x) = y_{i,N}(x) + e_{i,N,M}(x)$, estimated error functions $e_{i,N,M}(x)$ and corrected absolute error functions $|E_{i,N,M}(x)| = |y_i(x) - y_{i,N,M}(x)|$.

Example 1 : Consider the linear differential equations system given by

$$\begin{cases} y_1^{(1)}(x) - y_3(x) = -\cos(x) \\ y_2^{(1)}(x) - y_3(x) = -e^x \\ y_3^{(1)}(x) - y_1(x) + y_2(x) = 0 \end{cases}, 0 \le x \le 1 \tag{22}$$

with the initial conditions

$$y_1(0) = 1, y_2(0) = 0 \text{ and } y_3(0) = 2$$

which has the exact solution $y_1(x) = e^x$, $y_2(x) = \sin(x)$ and $y_3(x) = e^x + \cos(x)$ [7]. The set of the collocation points for $a = 0$, $b = 1$ and $N = 2$ is calculated as

$$\left\{x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1\right\}.$$

From Eq.(13), the fundamental matrix equation of the problem (22) is written as

$$\left\{\mathrm{X}\overline{\mathrm{B}} - \mathrm{PX}\right\}\mathrm{A} = \mathrm{G}.$$

By appliying the procedure in Section 3, the approximate solutions by means of the Taylor polynomials of the problem (22) for $N = 2$ are obtaied as

$$y_{1,2}(x) = 1 + x + 0.6706136074102369983 0x^2,$$

$$y_{2,2}(x) = x - 0.1005251013995184324x^2,$$

$$y_{3,2}(x) = 2 + x + 0.1927846772024388576 7x^2.$$

In order to calculate the corrected Taylor polynomial solutions, let us consider the error problem

$$\begin{cases} e'_{1,2}(x) - e_{3,2}(x) = -R_{1,2}(x), \\ e'_{2,2}(x) - e_{3,2}(x) = -R_{2,2}(x), \\ e'_{3,2}(x) - e_{1,2}(x) + e_{2,2}(x) = -R_{3,2}(x) \end{cases} \tag{23}$$

such that $e_{1,2}(0) = 0,\ e_{2,2}(0) = 0,\ e_{3,2}(0) = 0$ and the residual functions are

$$\begin{cases} R_{1,2}(x) = y'_{1,2}(x) - y_{3,2}(x) + \cos(x) \\ R_{2,2}(x) = y'_{2,2}(x) - y_{3,2}(x) + e^x \\ R_{3,2}(x) = y'_{3,2}(x) - y_{1,2}(x) + y_{2,2}(x). \end{cases}$$

By solving the error problem (23) for $M = 3$ with the method in Section 3, the estimated Taylor error function approximations $e_{1,2,3}(x)$, $e_{2,2,3}(x)$ and $e_{3,2,3}(x)$ are obtained as

$$e_{1,2,3}(x) = -0.198342522463333336x^2 + 0.238386300344444440x^3,$$

$$e_{2,2,3}(x) = (0.922148375866666676e-1)x^2 - 0.152417435955555569x^3,$$

$$e_{3,2,3}(x) = -0.236207314336666668x^2 + 0.290461694969999962x^3.$$

Hence, we can calculate the corrected Taylor polynomial solutions $y_{1,2,3}(x)$, $y_{2,2,3}(x)$ and $y_{3,2,3}(x)$ as

$$y_{1,2,3}(x) = 1 + x + 0.4722710849x^2 + 0.238386300344444440x^3,$$

$$y_{2,2,3}(x) = x - (0.831026381e-2)x^2 - 0.152417435955555569x^3,$$

$$y_{3,2,3}(x) = 2 + x - (0.434226371e-1)x^2 + 0.290461694969999962x^3.$$

Similarly we can calculate the corrected Taylor polynomial solutions for $N = 2$ and $M = 5$ as

$$\begin{aligned} y_{1,2,5}(x) = {} & 1 + x + 0.4997685849x^2 + 0.168225454862777624x^3 \\ & + (0.377211001762152444e-1)x^4 + (0.125314366180554194e-1)x^5, \end{aligned}$$

$$\begin{aligned} y_{2,2,5}(x) = {} & x + (0.453421e-4)x^2 - 0.166989019477499944x^3 \\ & + (0.782935853298638574e-3)x^4 + (0.764087047222199168e-2)x^5, \end{aligned}$$

$$\begin{aligned} y_{3,2,5}(x) = {} & 2 + x - (0.938985e-4)x^2 + 0.167226488756944570x^3 \\ & + (0.820919222569445051e-1)x^4 + (0.934385828125017249e-2)x^5, \end{aligned}$$

and for $N = 2$ and $M = 9$ as

$$\begin{aligned} y_{1,2,9}(x) = {} & 1 + x + 0.5000000043^2 + 0.166666599243195352x^3 + (0.416670828610961054e-1)x^4 \\ & + (0.833202202396421399e-2)x^5 + (0.139106498306773574e-2)x^6 \\ & + (0.196782559969577166e-3)x^7 + (0.247511344531048394e-4)x^8 \\ & + (0.352065501130205405e-5)x^9, \end{aligned}$$

$$\begin{aligned} y_{2,2,9}(x) = {} & x + (0.1341e-6)x^2 - (0.166668546848332878)x^3 + (0.113679634867747126e-4)x^4 \\ & + (0.829617310724284352e-2)x^5 + (0.704477670598890882e-4)x^6 \\ & - (0.275912593465932334e-3)x^7 + (0.459553932872225344e-4)x^8 \\ & - (0.863843029863531342e-5)x^9, \end{aligned}$$

$$\begin{aligned} y_{3,2,9}(x) = {} & 2 + x - (0.121e-7)x^2 + 0.166666815135405600x^3 + (0.833324772620436250e-1)x^4 \\ & + (0.833609272867619212e-2)x^5 - (0.526472155115698116e-5)x^6 \\ & + (0.204359255008057517e-3)x^7 + (0.458737358854932608e-4)x^8 \\ & + (0.379297555852531332e-5)x^9. \end{aligned}$$

Table1. Comparison of the exact solutions and the approximate solutions of the problem (22) for $N = 2$ ve $M = 3,5,9$.

	Exact Solution	Taylor Polynomial Solution	Corrected Taylor Polynomial Solutions		
x_i	$y_1(x_i)$	$y_{1,2}(x_i)$	$y_{1,2,3}(x_i)$	$y_{1,2,5}(x_i)$	$y_{1,2,9}(x_i)$
0	1	1	1	1	1
0.2	1.221402758	1.226824544	1.220797934	1.221400911	1.221402758
0.5	1.648721271	1.667653402	1.647866059	1.648719504	1.648721271
0.8	2.225540928	2.229192709	2.224307280	2.225540191	2.225540928
1.0	2.718281828	2.670613607	2.710657385	2.718246577	2.718281828
x_i	$y_2(x_i)$	$y_{2,2}(x_i)$	$y_{2,2,3}(x_i)$	$y_{2,2,5}(x_i)$	$y_{2,2,9}(x_i)$
0	0	0	0	0	0
0.2	0.198669331	0.195978996	0.198448250	0.198669599	0.198669331
0.5	0.479425539	0.474868725	0.478870255	0.479425419	0.479425539
0.8	0.717356091	0.735663935	0.716643704	0.717355092	0.717356091
1.0	0.841470985	0.899474899	0.839271300	0.841480129	0.841470980
x_i	$y_3(x_i)$	$y_{3,2}(x_i)$	$y_{3,2,3}(x_i)$	$y_{3,2,5}(x_i)$	$y_{3,2,9}(x_i)$
0	2	2	2	2	2
0.2	2.201469336	2.207711387	2.200586788	2.201468393	2.201469336
0.5	2.526303833	2.548196169	2.525452053	2.526302577	2.526303832
0.8	2.922247638	2.923382193	2.920925900	2.922246514	2.922247638
1.0	3.258584134	3.192784677	3.247039058	3.258568371	3.258584134

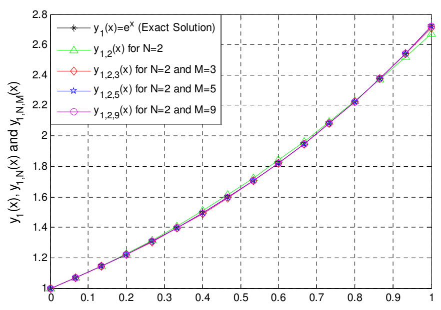

Figure 1(a). Comparison of the exact solution $y_1(x)$ and the approximate solutions

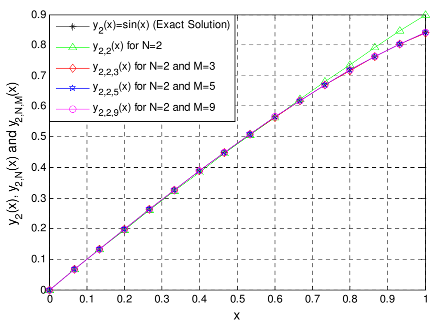

Figure 1(b). Comparison of the exact solution $y_2(x)$ and the approximate solutions

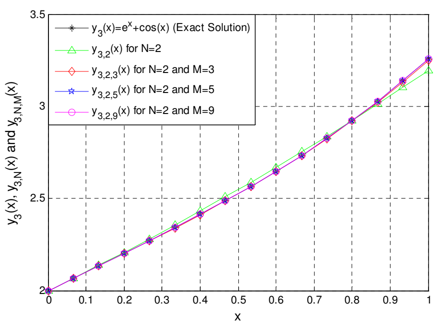

Figure 1(c). Comparison of the exact solution $y_3(x)$ and the approximate solutions

It is seen from Table 1 and Figures 1(a), 1(b), 1(c) that when the value of M is increased the accuracy of the solution increase. Now, we can compare the corrected absolute error functions in Table 2.

Table 2. Comparison of the corrected absolute errors of the problem (22) for $N = 2$ ve $M = 3,5,9$.

	Corrected absolue errors	$\lvert E_{i,N,M}(x)\rvert = \lvert y_i(x) - y_{i,N,M}(x)\rvert$	
x_i	$\lvert E_{1,2,3}(x_i)\rvert$	$\lvert E_{1,2,5}(x_i)\rvert$	$\lvert E_{1,2,9}(x_i)\rvert$
0	0	0	0
0.2	0.604824e-3	0.184731e-5	0.245769e-11
0.5	0.855212e-3	0.176646e-5	0.381529e-10
0.8	0.123365e-2	0.737484e-6	0.927858e-10
1.0	0.762444e-2	0.352519e-4	0.698288e-9
x_i	$\lvert E_{2,2,3}(x_i)\rvert$	$\lvert E_{2,2,5}(x_i)\rvert$	$\lvert E_{2,2,9}(x_i)\rvert$
0	0	0	0
0.2	0.221081e-3	0.268509e-6	0.248487e-9
0.5	0.555284e-3	0.119821e-6	0.292619e-9
0.8	0.712387e-3	0.998966e-6	0.259591e-9
1.0	0.219868e-2	0.914414e-5	0.434892e-8
x_i		$\lvert E_{3,2,5}(x_i)\rvert$	$\lvert E_{3,2,9}(x_i)\rvert$
0	0	0	0
0.2	0.882548e-3	0.942921e-6	0.528007e-10
0.5	0.851780e-3	0.125541e-5	0.990846e-10
0.8	0.132174e-2	0.112380e-5	0.135252e-9
1.0	0.115451e-1	0.157635e-4	0.561586e-10

Table 2 shows that the corrected absolute errors is close to zero as the value of M increases. Namely, the accuracy of the solution is increased.

Example 2. Consider the linear differential equations system given by

$$\begin{cases} y_1^{(1)}(x) = \cos(x)y_1(x) + y_2(x) + e^{\sin(x)} \\ y_2^{(1)}(x) = y_1(x) + \cos(x)y_2(x) + xe^{\sin(x)} \end{cases}, \quad 0 \le x \le 1 \tag{24}$$

with the initial conditions

$$y_1(0) = 2 \text{ and } y_2(0) = -2$$

which has the exact solution $y_1(x) = e^{\sin(x)}(2\cosh(x) - x)$ and $y_2(x) = e^{\sin(x)}(2\sinh(x) - 2)$. The set of the collocation points for $a = 0$, $b = 1$ and $N = 2$ is calculated as

$$\left\{x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1\right\}.$$

From Eq.(13), the fundamental matrix equation of the problem (24) is written as

$$\left\{\mathrm{X}\overline{\mathrm{B}} - \mathrm{PX}\right\}\mathrm{A} = \mathrm{G}.$$

By using the present method in Section 3, the approximate solutions by means of the Taylor polynomials for $N = 2$ are obtaied as

$$y_{1,2}(x) = 2 + x + 1.8646865593930787634x^2,$$

$$y_{2,2}(x) = -2 + 2.5859190249855302151x^2.$$

In order to calculate the corrected Taylor polynomial solutions, let us consider the error problem

$$\begin{cases} e_{1,2}^{(1)}(x)-\cos(x)e_{1,2}(x)-e_{2,2}(x)=-R_{1,2} \\ e_{2,2}^{(1)}(x)-e_{1,2}(x)-\cos(x)e_{2,2}(x)=-R_{2,2} \end{cases} \tag{25}$$

such that $e_{1,2}(0)=0,\ e_{2,2}(0)=0$ and the residual functions are

$$\begin{cases} R_{1,2}(x)=y_{1,2}^{(1)}(x)-\cos(x)y_{1,2}(x)-y_{2,2}(x)-e^{\sin(x)} \\ R_{2,2}(x)=y_{2,2}^{(1)}(x)-y_{1,2}(x)-\cos(x)y_{2,2}(x)-xe^{\sin(x)} \end{cases}$$

By solving the error problem (25) for $M=3$ by using the technique in Section 3, the estimated Taylor error function approximations $e_{1,2,3}(x)$ and $e_{2,2,3}(x)$ are obtained as

$$e_{1,2,3}(x)=-1.0553168029550909369x^2+0.9949708345094654705 1x^3,$$

$$e_{2,2,3}(x)=-1.8301377254066642699x^2+2.0381675757084778421x^3.$$

Hence, we can calculate the corrected Taylor polynomial solutions $y_{1,2,3}(x)$ and $y_{2,2,3}(x)$ as

$$y_{1,2,3}(x)=2+x+0.809369756x^2+0.9949708345094654705 1x^3$$

$$y_{2,2,3}(x)=-2+0.755781300x^2+2.0381675757084778421x^3.$$

Similarly we can calculate the corrected Taylor polynomial solutions for $N=2$ and $M=5$ as

$$y_{1,2,5}(x)=2+x+1.019087035x^2+0.38418404084877058288x^3+0.60095881149668672 09x^4 \\ -0.16172189638449954742x^5,$$

$$y_{2,2,5}(x)=-2+1.031092567x^2+1.1335297565208665254x^3+1.0728946359438083 48x^4 \\ -0.42054511631751871 06x^5,$$

and for $N=2$ and $M=9$ as

$$y_{1,2,9}(x)=2+x+0.9999897539x^2+0.500173101302650469x^3+0.332010653018222968x^4 \\ +(0.807771671168744376e-1)x^5-(0.378493987310264402e-1)x^6-(0.107759922510410888e-1)x^7 \\ -(0.415635790661426086e-1)x^8+(0.166670256823326924e-1)x^9,$$

$$y_{2,2,9}(x)=-2+1.000008919x^2+(0.561045229456268930e-15)x+1.33319973007876058x^3 \\ +0.584178089466959705x^4+(0.639409598703650772e-1)x^5-0.104116324378878744x^6 \\ -(0.642539417701755156e-1)x^7-(0.177960431765313842e-1)x^8+(0.176957626717921812e-1)x^9.$$

Table 3. Comparison of the exact solutions and the approximate solutions of the problem (22) for $N = 2$ ve $M = 3,5,9$.

	Exact Solution	Taylor Polynomial Solution	Corrected Taylor Polynomial Solutions		
x_i	$y_1(x_i)$	$y_{1,2}(x_i)$	$y_{1,2,3}(x_i)$	$y_{1,2,5}(x_i)$	$y_{1,2,9}(x_i)$
0	2	2	2	2	2
0.1	2.110534057	2.118646866	2.109088668	2.110633533	2.110534040
0.3	2.406356994	2.467821790	2.399707491	2.406565584	2.406356979
0.6	3.114745826	3.271287161	3.106286812	3.115163853	3.114745811
0.8	3.841624628	3.993399398	3.827421711	3.842077629	3.841624613
1.0	4.839428566	4.864686559	4.804340591	4.842507991	4.839428731
x_i	$y_2(x_i)$	$y_{2,2}(x_i)$	$y_{2,2,3}(x_i)$	$y_{2,2,5}(x_i)$	$y_{2,2,9}(x_i)$
0	-2	-2	-2	-2	-2
0.1	-1.988607781	-1.974140810	-1990404019	-1.988452461	-1.988607765
0.3	-1.869206372	-1.767267288	-1.876949158	-1.868927844	-1.869206358
0.6	-1.278121055	-1.069069151	-1.287674536	-1.277618692	-1.278121041
0.8	-0.458543621	-0.345011824	-0.472758169	-0.458080103	-0.458543604
1.0	0.812855337	0.585919025	0.793948876	0.816971843	0.812857152

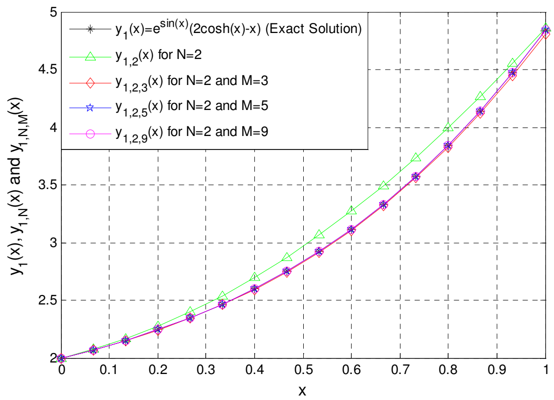

Figure 2(a). Comparison of the exact solution $y_1(x)$ and the approximate solutions

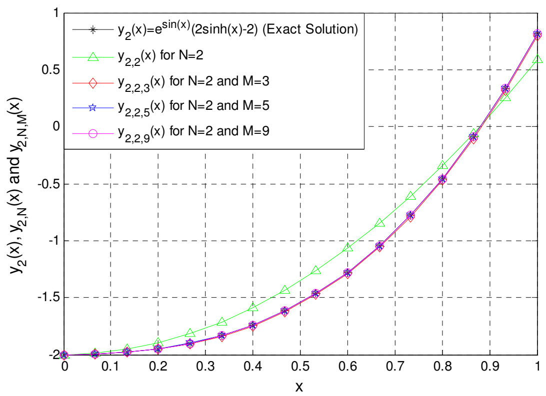

Figure 2(b). Comparison of the exact solution $y_2(x)$ and the approximate solutions

It is seen from Table 3 and Figures 2(a), 2(b) that when the value of M is increased the accuracy of the solution increase. Now, we can compare corrected absolute error functions.

Table 4. Comparison of the corrected absolute errors of the problem (24) for $N = 2$ ve $M = 3,5,9$.

Corrected absolue errors $\lvert E_{i,N,M}(x)\rvert = \lvert y_i(x) - y_{i,N,M}(x)\rvert$			
x_i	$\lvert E_{1,2,3}(x_i)\rvert$	$\lvert E_{1,2,5}(x_i)\rvert$	$\lvert E_{1,2,9}(x_i)\rvert$
0	0	0	0
0.1	0.144539e-2	0.994758e-4	0.170758e-7
0.3	0.664950e-2	0.208590e-3	0.154019e-7
0.6	0.845901e-2	0.418026e-3	0.154698e-7
0.8	0.142029e-1	0.453002e-3	0.151985e-7
1.0	0.350880e-1	0.307943e-2	0.165063e-6
x_i	$\lvert E_{2,2,3}(x_i)\rvert$	$\lvert E_{2,2,5}(x_i)\rvert$	$\lvert E_{2,2,9}(x_i)\rvert$
0	0	0	0
0.1	0.179624e-2	0.155321e-3	0.168819e-7
0.3	0.774277e-2	0.278529e-3	0.143867e-7
0.6	0.955348e-2	0.502363e-3	0.134442e-7
0.8	0.142145e-1	0.463518e-3	0.164055e-7
1.0	0.189065e-1	0.411651e-2	0.181441e-5

Table 4 shows that corrected absolute erros is close to zero as the value of M increases. Namely, the accuracy of the solution is increased.

6 Conclusions

In this study, we presented a method based on the Taylor polynomials with the aid of the residual error function for solving linear differential equations system in the normal form numerically. When the obtained results are investigated in examples, it can be seen that the presented method is very effective. Also, it can be seen from tables and fgures that the accuracy of the solution increase when the value of M is increased. In this paper, we have used the computer programmes *Maple* and *Matlab* for computations and graphics, respectively. On the other hand, the present method can be applied to the systems of ordinary and partial integro-differential equations.

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