Journal of Abstract and Computational Mathematics
http://www.ntmsci.com/jacm

On Saturated Semigroups

Noor Alam

Department of Mathematics, College of Preparatory Year, University of Hail, Hail-2440, K.S.A

Received: 11.7.2019, Accepted: 17.11.2019
Published online: 30.11.2019

Abstract: We show that the class of permutative semigroups (medial semigroups) satisfying the homotypical identity $[axy = axay]$ is saturated and, as a corollary, we found that an externally commutative semigroup satisfying the identity $[ax = axa]$ is saturated. Next, we show that the variety $[xy^2 = xy,\ xyz = yxz\ (xyz = xzy)]$ of left (right) commutative semigroups satisfying the identity $[xy^2 = xy]$ is saturated.

Keywords: Epimorphism, dominion, medial semigroup, externally commutative semigroup, right commutative semigroup, variety, saturated semigroup, zigzag equations.

1 Introduction

Let U, S be semigroups with $U \subseteq S$. Following Isbell [14], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \to T$, $u\beta = u\gamma$ for all $u \in U$ implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called the dominion of U in S and is denoted by $\mathrm{Dom}(U,S)$. It can be easily verified that $Dom(U,S)$ is a subsemigroup of S containing U. We say that U is *closed* in S if $Dom(U,S) = U$ and *absolutely closed* if it is closed in every containing semigroup S. We call a variety of semigroups absolutely closed if each of its member is absolutely closed. A semigroup U is called *saturated* if $Dom(U,S) \neq S$ for every properly containing semigroup S. A class (resp. variety) $\mathscr{V}$ of semigroups will be called saturated if every member of $\mathscr{V}$ is saturated.

A morphism $\alpha : A \to B$ in a category $\mathscr{C}$ is called an *epimorphism* (epi for short) if for all $C \in \mathscr{C}$ and for all morphisms $\beta, \gamma : B \to C$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. It is easy to check that a morphism $\alpha : S \to T$ is epi if and only if the inclusion $i : S\alpha \to T$ is epi. Also it can be easily seen that for any subsemigroup U of any semigroup S, the inclusion map $i : U \to S$ is epi if and only if $Dom(U,S) = S$. In such a case, we say that U is epimorphically embedded in S. We also note that every onto morphism is an epimorphism, but the converse is not true in general in the category of all semigroups.

2 Preliminaries

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 2.1([14, Theorem 2.3] or [12, Theorem VII.2.13]). Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$d = a_0y_1 = x_1a_1y_1 = x_1a_2y_2 = x_2a_3y_2 = \cdots = x_ma_{2m-1}y_m = x_ma_{2m}, \quad (1)$$

* Corresponding author e-mail: alam.nooramu@gmail.com

where $m \geq 1,\ a_i \in U\ (i = 0,1,\ldots,2m),\ x_i, y_i \in S\ (i = 1,2,\ldots,m)$, and

$$a_0 = x_1 a_1, \qquad a_{2m-1} y_m = a_{2m},$$
$$a_{2i-1} y_i = a_{2i} y_{i+1}, \qquad x_i a_{2i} = x_{i+1} a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a *zigzag* in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

We refer to the equations in Result 2.1, in whatever follows, as *the zigzag equations*.

Result 2.2 ([16, Result 3]). Let U be any subsemigroup of a semigroup S and let $d \in Dom(U,S) \setminus U$. If (1) is a zigzag of minimum length m over U with value d, then $x_i,\ y_i \in S \setminus U$ for $i = 1,\ 2, \ldots, m$.

In the following results, let U and S be any semigroups with U dense in S.

Result 2.3 ([16, Result 4]). For any $d \in S \setminus U$ and k any positive integer, if (1) is a zigzag of minimum length m over U with value d , then there exist $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $d = b_1 b_2 \cdots b_k d_k$.

We shall use standard notations and refer the readers to Clifford and Preston [4] and Howie [12] for any unexplained symbols and terminology. Further, in whatever follows, bracketed statements or notions are dual to the other statements or notions.

An effort has been made to identify the saturated classes of algebras in semigroup theory, ring theory, and elsewhere [3]. For example, B.J. Gardner [5, Theorem 2.10] had shown that any class of regular rings is saturated in the class of all rings whereas P.M. Higgins [9, Corollary 4], in contrast, had shown that not all regular semigroups are saturated. However, in [8], P.M. Higgins had shown that any class of generalized inverse semigroup is saturated. A necessary condition for any semigroup variety to be saturated is that it admits an identity, not a permutation identity, of which at least one side has no repeated variable [10, Theorem 6]. However, the determination of all saturated varieties of semigroups is still an open problem, although the question has been settled for commutative varieties (Higgins [11]; Khan [15]) and for heterotypical varieties. These varieties do not contain the variety of all semilattices [10]. However, there are semigroups of each of the following types which are not saturated; commutative cancellative semigroups (the injection of the natural numbers into the integers provides an example), subsemigroups of finite inverse semigroups [14], commutative periodic semigroups [11] and bands, as Trotter [19] has constructed a band with a properly epimorphically embedded subband.

In [1], the authors had shown that the class of all left[right] quasinormal bands was closed within the class of all left[right] quasinormal bands while, in [2], authors had generalized this result and had shown that the variety $V = [axy = axay]$ of semigroups was closed. In this paper, we extend both of the above results by showing that the varieties $[axy = axay\ (xya = xaya),\ axyz = ayxz],\ [xy2 = xy,\ xyz = xzy]$ of semigroups are saturated.

For more details and other related results, one may refer to [7], [8], [13] and [17].

3 Main Results

Definition 3.1. A *semigroup identity* $u = v$ is the formal identity of two words u and v formed by letters over an alphabet set X.

Definition 3.2. A semigroup is said to *satisfy an identity* if for every substitution of elements from S for the letters forming the word of the identity, the resulting words are equal in S.

Definition 3.3. The class of semigroups, in which a finite or an infinite collection $u_1 = v_1,\ \ u_2 = v_2, \ldots$ of identical relations is satisfied, is called the *variety of semigroups determined by these identical relations*, and the list of identical relations is called a *presentation of the variety*, denoted by $[u_1 = v_1,\ u_2 = v_2, \ldots]$.

We shall take Birkhoff's Theorem for (semigroup) varieties for granted.

Result 3.4 [5, Ch. 1 section 26, Theorem 3]. A non-empty class $\mathscr{V}$ of semigroups is a variety if and only if
(a) every subsemigroup of a semigroup in $\mathscr{V}$ is in $\mathscr{V}$;
(b) every homomorphic image of a semigroup in $\mathscr{V}$ is in $\mathscr{V}$;
(c) the direct product of a family of semigroups in $\mathscr{V}$ is in $\mathscr{V}$.

Definition 3.5. An identity

$$u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n)$$

in the variables $x_1, x_2, \ldots, x_n$ is called *homotypical* if $C(u) = C(v)$ and *heterotypical* otherwise.

Definition 3.6. A semiqroup is called *medial* if it satisfies the identity $xaby = xbay$.

Recall that a band B (a semigroup in which every element is an idempotent) is called left[right] quasi-normal if it satisfies the identity $axy = axay$ $[yxa = yaxa]$ (see [18]).

We shall use the following remark and proposition to prove the result.

Remark 3.7. Let U be any proper subsemigroup of a semigroup S such that $\mathrm{Dom}(U,S) = S$ and let $d \in Dom(U,S) \setminus U$. Take any $d \in Dom(U,S) \setminus U$. Then, by Result 2.1, we may let (1) be a zigzag in S over U with value d of minimal length m. As (1) is of minimal length, by Result 2.2, $x_i,\ y_i \in S \setminus U$ for all $i = 1, 2, \ldots, m$. Therefore, by Result 1.3,

$$x_i = z_i u_i,\ y_i = v_i t_i \tag{2}$$

for some $z_i,\ t_i \in S \setminus U$ and $u_i, v_i \in U$ $(i = 1, 2, ..., m)$.

Proposition 3.8. Let U be any proper subsemigroup of a semigroup S such that $\mathrm{Dom}(U,S) = S$. If U satisfies the identities $xyz = xyxz,\ xyzt = xzyt$, then, for any $a, b \in U$ and $x, y \in S \setminus U$, $xay = xacay$ and $xaby = xbay$, where $x = x'c$ and $y = dy'$ for any $c, d \in U$.

Proof. For $a_i, a_j \in U$ and $x_i,\ y_i \in S \setminus U$, $i = 1, 2, \ldots, m$, we have Now

$$\begin{aligned} xay &= x'cady' \quad \text{(by hypothesis)} \\ &= x'cacdy' \quad (\text{as } a, c, d \in U) \\ &= x'cacady' \quad \text{(by hypothesis))} \\ &= xacay \end{aligned}$$

and

$$\begin{aligned} xaby &= x'cabdy' \ \text{(by hypothesis)} \\ &= x'cbady' \ \text{(by hypothesis as } a,b,c,d \in U) \\ &= xbay \ \text{(by hypothesis as)}, \end{aligned}$$

as required.

Theorem 3.9. The variety $\mathscr{V} = [axy = axay,\ axyz = ayxz]$ of semigroups is saturated.

Proof. Let $U \in \mathscr{V}$ and suppose to the contrary that U is not saturated. Then, there exists a semigroup S containing U properly such that $Dom(U,S) = S$. Take any $d \in Dom(U,S) \setminus U$. Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations)} \\
&= x_1a_1u_1a_1y_1 && \text{(by Proposition 3.8, where } x_1 = x_1'u_1 \text{ for some } x_1' \in S\setminus U \text{ and } u_1 \in U) \\
&= x_1a_1u_1a_2y_2 && \text{(by zigzag equations)} \\
&= x_1a_2a_1u_1y_2 && \text{(by proposition 3.8)} \\
&= x_2a_3a_1u_1y_2 && \text{(by zigzag equations)} \\
&= x_2a_3a_1u_1a_3y_2 && \text{(by Remark 3.7)} \\
&= x_1a_2a_1u_1a_4 && \text{(by zigzag equations)} \\
&= x_1'u_1a_2a_1u_1a_4 && \text{(by Remark 3.7)} \\
&= x_1'u_1a_2a_1a_4 && \text{(by Proposition 3.8)} \\
&= x_1'u_1a_1a_2a_4 && \text{(by Proposition 3.8)} \\
&= x_1a_1a_2a_4 && \text{(by Remark 3.7)} \\
&= a_0a_2a_4 && \text{(by zigzag equations)} \\
&= (\prod_{i=0}^{1} a_{2i})(a_3y_2) \\
&\vdots \\
&= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}y_{m-1}) \\
&= a_0a_2a_4\cdots a_{2m-4}(a_{2m-2}y_m) && \text{(by zigzag equations)} \\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)} \\
&= x_1a_2a_1a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by Proposition 3.8)} \\
&= x_2a_3a_1a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)} \\
&= x_2a_4a_3a_1\cdots a_{2m-4}a_{2m-2}y_m && \text{(by Proposition 3.8)} \\
&= x_3a_5a_3a_1\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)} \\
&\vdots \\
&= x_{m-1}a_{2m-3}a_{2m-5}\cdots a_5a_3a_1a_{2m-2}y_m \\
&= x_{m-1}a_{2m-2}a_{2m-3}a_{2m-5}\cdots a_5a_3a_1y_m && \text{(by proposition 3.8)} \\
&= x_ma_{2m-1}a_{2m-3}a_{2m-5}\cdots a_5a_3a_1y_m && \text{(by zigzag equations)} \\
&= x_ma_{2m-1}a_{2m-3}a_{2m-5}\cdots a_5a_3a_1(a_{2m-1}y_m) && \text{(by proposition 3.8)} \\
&= x_{m-1}a_{2m-2}a_{2m-3}a_{2m-5}\cdots a_5a_3a_1a_{2m} && \text{(by zigzag equations)} \\
&= x_{m-1}a_{2m-3}a_{2m-2}a_{2m-5}\cdots a_5a_3a_1a_{2m} && \text{(by proposition 3.8)} \\
&= x_{m-2}a_{2m-4}a_{2m-2}a_{2m-5}\cdots a_5a_3a_1a_{2m} && \text{(by zigzag equations)} \\
&= x_{m-2}a_{2m-5}a_{2m-4}a_{2m-2}\cdots a_5a_3a_1a_{2m} && \text{(by proposition 3.8)} \\
&= x_{m-3}a_{2m-6}a_{2m-4}a_{2m-2}\cdots a_5a_3a_1a_{2m} && \text{(by zigzag equations)} \\
&\vdots \\
&= x_2a_3a_1a_4\cdots a_{2m-4}a_{2m-2}a_{2m}
\end{aligned}$$

$$\begin{aligned}
&= x_1a_2a_1a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by proposition 3.8)}\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&= \prod_{i=0}^{m} a_{2i} \in U.
\end{aligned}$$

a contradiction, as required. Following proposition is immediate.

Proposition 3.10. For a variety of semigroups $\mathscr{V} = [xy^2 = xy, xyz = xzy]$, if $U \in \mathscr{V}$ and S is any semigroup containing U properly such that $Dom(U,S) = S$. Then, for any $a,b \in U$ and $x \in S\setminus U$ such that $x = x'u$ for some $u \in U$ and $x' \in S\setminus U$,

$$xa = xa^2, \quad xab = xba.$$

Theorem 3.11. The variety $\mathscr{V} = [xyz = xzy, xy = xy^2]$ of semigroups is saturated.

Proof. Let $U \in \mathscr{V}$ and Suppose to the contrary that U is not saturated. Then, there exists a semigroup S containing U properly such that $Dom(U,S) = S$. Take any $d \in Dom(U,S)\setminus U$. Now

$$\begin{aligned}
d &= a_0y_1\\
&= x_1a_1y_1 && \text{(by zigzag equations)}\\
&= x_1a_1a_1y_1 && \text{(by proposition 3.10)}\\
&= x_1a_1a_2y_2 && \text{(by zigzag equations)}\\
&= x_1a_2a_1y_2 && \text{(by proposition 3.10)}\\
&= x_2a_3a_1y_2 && \text{(by zigzag equations)}\\
&= x_2{a_3}^2a_1y_2 && \text{(by proposition 3.10)}\\
&= x_1a_2a_3a_1y_2 && \text{(by zigzag equations)}\\
&= x_1a_1a_2a_3y_2 && \text{(by proposition 3.10)}\\
&= a_0a_2a_3y_2 && \text{(by zigzag equations)}\\
&\vdots\\
&= a_0a_2a_4\cdots a_{2m-4}(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&= x_1a_1a_2a_4\cdots a_{2m-4}(a_{2m-2}y_m) && \text{(by zigzag equations)}\\
&= x_1a_2a_1a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by proposition 3.10)}\\
&= x_2a_3a_1a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= x_2a_4a_3a_1\cdots a_{2m-4}a_{2m-2}y_m && \text{(by proposition 3.10)}\\
&= x_3a_5a_3a_1\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&\vdots\\
&= x_{m-1}a_{2m-2}a_{2m-3}a_{2m-5}\cdots a_5a_3a_1y_m\\
&= x_ma_{2m-1}a_{2m-3}a_{2m-5}\cdots a_5a_3a_1y_m && \text{(by zigzag equations)}\\
&= x_ma_{2m-1}a_{2m-1}(a_{2m-3}a_{2m-5}\cdots a_5a_3a_1)y_m && \text{(by proposition 3.10)}\\
&= x_ma_{2m-1}(a_{2m-3}a_{2m-5}\cdots a_5a_3a_1)(a_{2m-1}y_m) && \text{(by proposition 3.10)}\\
&= x_{m-1}a_{2m-2}a_{2m-3}a_{2m-5}\cdots a_5a_3a_1a_m && \text{(by zigzag equations)}\\
&= x_{m-1}a_{2m-3}a_{2m-2}a_{2m-5}\cdots a_5a_3a_1a_m && \text{(by proposition 3.10)}\\
&= x_{m-2}a_{2m-4}a_{2m-2}a_{2m-5}\cdots a_5a_3a_1a_m && \text{(by zigzag equations)}\\
&= x_{m-2}a_{2m-5}a_{2m-4}a_{2m-2}\cdots a_5a_3a_1a_m && \text{(by proposition 3.10)}\\
&= x_{m-3}a_{2m-6}a_{2m-4}a_{2m-2}\cdots a_5a_3a_1a_m && \text{(by zigzag equations)}\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&= x_2a_4a_3a_1\cdots a_{2m-4}a_{2m-2}a_{2m} \\
&= x_2a_3a_4a_1\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by proposition 3.10)} \\
&= x_1a_2a_4a_1\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)} \\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by proposition 3.10)} \\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)} \\
&= \prod_{i=0}^{m} a_{2i} \in U.
\end{aligned}$$

This is a contradiction, as required. As a corollary, we have:

Theorem 3.12. The variety of semigroups $\mathscr{V} = [xyz = zyx,\ ax = axa]$ is saturated. In otherwords, Externally commutative semigroup satisfying an identity $[ax = axa]$ is saturated.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Alam, N. and Khan, N.M.: Special Semigroup Amalgams of Quasi Unitary Subsemigroups and of Quasi Normal Bands, Asian-European Journal of Mathematics. Vol. 6, No. 1, (2013), 1350010
[2] Alam, N. and Khan, N.M.: Embedding of special semigroup amalgams, Thai Journal of Mathematics 12 (1), 133-143
[3] Burgess, W.: *The meaning of mono and epi in some familiar categories*, Canad. Math. Bull. Vol. 8, No. 6, (1965), 759-770.
[4] Clifford, A.H. and Preston, G.B.: The Algebraic Theory of Semigroups. Math. Surveys No.7, Vol.I., II. Amer. Math. Soc., Providence (1961), (1967).
[5] Gardner, B. J.: *Epimorphisms of regular rings*, Comment. Math. Univ. Carolinae. 161, (1975), 151-160.
[6] Gratzer, G.: *Universal Algebra*, 2nd. Ed. springer-verlag 1979.
[7] Hall, T.E. and Jones, P.R.: Epis are onto for finite regular semigroups, Proc. of Edinburgh Math. Soc. 26(1983), 151-162.
[8] Higgins, P.M.: Epis are on to for generalized inverse semigroups, Semigroup Forum, 23(1981), 255-259.
[9] Higgins, P. M.: *A semigroup with an epimorphically embedded subband*, Bull. Austral. Math. Soc. 27 (1983), 231-242.
[10] Higgins, P. M.: *Saturated and epimorphically closed varieties of semigroups*, J. Aust. Math. Soc. Ser A 36, No. 2 (1984), 153-175.
[11] Higgins, P. M.: *The Commutative varieties of semigroups for which epis are onto*, Proc. Edinburgh Math. Soc. Ser. A 94, No. 1-2 (1983), 1-7.
[12] Howie, J.M.: An Introduction to Semigroup Theory, London Math. Soc. Monograph, Vol.7. Academic Press, San Diego (1976).
[13] Howie, J.M. and Isbell, J.R.: Epimorphisms and dominions II, J.Algebra, 6(1967), 7-21.
[14] Isbell, J.R.: Epimorphisms and dominions, Proceedings of the conference on categorical Algebra, LaJolla, 232-246. Lange and Springer, Berlin, (1966).
[15] Khan, N.M.: Epimorphisms, dominions and varieties of semigroups, Semigroup Forum, 25 (1982) 331-337.
[16] Khan, N.M.: On saturated permutative varieties and consequences of permutation identities. J. Aust. Math. Soc. Ser.A 38, 186-197 (1985).
[17] Nagy, A.: Special classes of semigroups, Advances in Mathematics, Kluwer Academic Publishers, The Netherlands (2001).
[18] Petrich, M. and Reilly, N.: Completely Regular Semigroups, canadian Math. Soc. Monograph, John Wiley and Sons, Canada (1999).
[19] Trotter, P. G.: *A non-surjective epimorphism of bands*, Algebra Universalis, 22 (1986), 109-116.