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A numerical algorithm with residual error estimation for solution of high-order Pantograph-type functional differential equations using Fibonacci polynomials

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Abstract: In this article a functional differential equation known as the high-order delay pantograph-type equation, which contains a linear functional argument, is considered and a new matrix method based on the Fibonacci polynomials and collocation points is presented to find the approximate solution of the pantograph equations under the initial conditions. Also, the numerical examples are given demonstrate the applicability of the technique. In addition, an error analysis technique based on residual function is developed and applied to some problems to demonstrate the validity of the method.

Keywords: Fibonacci polynomials, Pantograph equations; Matrix method; Collocation method; Residual error analysis.

1 Introduction

Functional-differential equations with proportional delays are usually referred to as pantograph equations or generalized equations. The name pantograph originated from the work of Ockendon and Tayler [1] on the collection of current by the pantograph head of an electric locomotive. These equations arise in many applications such as nonlinear dynamical systems, electrodynamics, number theory, astrophysics, quantum mechanics and cell growth, among others [2-4]. In recent years, there has been a growing interest in the numerical treatment of the pantograph equations of the retarded and the advanced type. A special feature of this type is the existence of compactly supported solutions [5]. This issue was studied in [6] and has direct applications to approximation theory and to wavelets [7].

Pantograph equations are characterized by the presence of a linear functional argument and play an important role in explaining many ODEs- based model fail. These equations arise in industrial applications [1, 8] and in studies based on biology, economy, control and electrodynamics [9, 10].

The different numerical methods have been used to find the approximate solutions of multi-pantograph and generalized pantograph equations [11-20]. In addition, the Fibonacci method has been used to find the approximate solutions of differential, integral, integro-differential, difference equations [21-23]. The basic motivation of this work is to apply the Fibonacci method to the nonhomogenous and the homogenous generalized pantograph equations with variable coefficients, which are extented of the multi-pantograph equations, in the above mentioned studies. Our aim in this study

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is to develop and to apply the Fibonacci collocation method to the pantograph equation

$$y^{(m)}(t) = \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t)y^{(k)}(\lambda_{jk}t+\mu_{jk}) + g(t), 0 \leq t \leq b \tag{1}$$

with the initial conditions

$$\sum_{k=0}^{m-1} c_{ik}y^{(k)}(0) = \lambda_i, i = 0,1,...,m-1. \tag{2}$$

Here $P_{jk}(t)$ and $g(t)$ are continuous functions defined in the interval $0 \leq t \leq b; c_{ik}, \lambda_i, \lambda_{jk}$, and μ_{jk} are real or complex constants. Our purpose is to find an approximate solution of (1) expressed in the truncated Fibonacci series form

$$y(t) = \sum_{n=1}^{N} a_n F_n(t), 0 \leq t \leq b \tag{3}$$

where $a_n, n = 1,2,3,...,N$ are the unknown Fibonacci coefficients. Here N is chosen any positive integer such that $N \geq m$ and $F_n(t), n = 1,2,3,...,N$ are the Fibonacci polynomials defined by

$$F_n(t) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} t^{n-2j-1},$$

$$[(n-1)/2] = \begin{cases} (n-2)/2, & n \text{ even} \\ (n-1)/2, & n \text{ odd}. \end{cases}$$

2 Fundamental matrix relations

First, we can write the Fibonacci polynomials $F_n(t)$ in the matrix form as follows,

$$F^T(t) = CT^T(t) \Leftrightarrow F(t) = T(t)C^T \tag{4}$$

where $F(t) = [F_1(t)F_2(t)...F_N(t)]$ if

$$T(t) = [1t\ldots t^{N-1}],$$

N is even,

$$C = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{1} & 0 & \binom{2}{0} & 0 & \cdots & 0 \\ 0 & \binom{2}{1} & 0 & \binom{3}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{(n-2)/2}{(n-2)/2} & 0 & \binom{n/2}{(n-4)/2} & 0 & \cdots & 0 \\ 0 & \binom{n/2}{(n-2)/2} & 0 & \binom{(n+2)/2}{(n-4)/2} & \cdots & \binom{n-1}{0} \end{bmatrix}_{NxN}$$

if N is odd,

$$C=\begin{bmatrix}\binom{0}{0} & 0 & 0 & 0 & \cdots & 0\\ 0 & \binom{1}{0} & 0 & 0 & \cdots & 0\\ \binom{1}{1} & 0 & \binom{2}{0} & 0 & \cdots & 0\\ 0 & \binom{2}{1} & 0 & \binom{3}{0} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \binom{(n-1)/2}{(n-3)/2} & 0 & \binom{(n+1)/2}{(n-5)/2} & \cdots & 0\\ \binom{(n-1)/2}{(n-1)/2} & 0 & \binom{(n+1)/2}{(n-3)/2} & \cdots & \cdots & \binom{n-1}{0}\end{bmatrix}_{NxN}.$$

We now consider the solution $y(t)$ of Eq. (1) defined by the truncated Fibonacci series (3). Then the finite series (3) can be written in the matrix form

$$y(t)=F(t)A,\ A=\begin{bmatrix}a_1\ a_2\ \cdots\ a_N\end{bmatrix}^T,$$

or from Eq. (4)

$$y(t)=T(t)C^TA. \tag{5}$$

On the other hand, it is clearly seen from [16] that the relation between the matrix $T(t)$ and its derivative $T^{(1)}(t)$ is

$$T^{(0)}(t)=T(t),T^{(1)}(t)=T(t)B^T \tag{6}$$

where

$$B^T=\begin{bmatrix}0 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 2 & \cdots & 0 & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & N-1\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}.$$

It follows from (6) and (4) that

$$\begin{aligned}T^{(0)}(t)&=T(t)\\ T^{(1)}(t)&=T(t)B^T\\ T^{(2)}(t)&=T^{(1)}(t)B^T=T(t)(B^T)^2\\ &\vdots\\ T^{(k)}(t)&=T^{(k-1)}(t)(B^T)^{k-1}=T(t)(B^T)^k\end{aligned} \tag{7}$$

and thus

$$F^{(k)}(t) = T^{(k)}(t)C^T = T(t)(B^T)^k C^T. \tag{8}$$

From the relations (5), (7) and (8), we have recurrence relations

$$\begin{aligned} y^{(k)}(t) &= F^{(k)}(t)A \\ &= T^{(k)}(t)C^T A \\ &= T(t)(B^T)^k C^T A, k = 0,1,2,...,m. \end{aligned} \tag{9}$$

Similarly, we obtain the matrix relations as

$$\begin{aligned} &T(\lambda_{jk}t + \mu_{jk}) = T(t)B(\lambda_{jk}, \mu_{jk}) \\ &y(\lambda_{jk}t + \mu_{jk}) = T(\lambda_{jk}t + \mu_{jk})C^T A \\ &y^{(k)}(\lambda_{jk}t + \mu_{jk}) = T(t)B(\lambda_{jk}, \mu_{jk})(B^T)^k C^T A \end{aligned} \tag{10}$$

where for $\lambda_{jk} \neq 0$ and $\mu_{jk} \neq 0$,

$$B(\lambda_{jk}, \mu_{jk}) = \begin{bmatrix} \binom{0}{0}(\lambda_{jk})^0(\mu_{jk})^0 & \binom{1}{0}(\lambda_{jk})^0(\mu_{jk})^1 & \dots & \binom{N-1}{0}(\lambda_{jk})^0(\mu_{jk})^{N-1} \\ 0 & \binom{1}{1}(\lambda_{jk})^1(\mu_{jk})^0 & \dots & \binom{N-1}{1}(\lambda_{jk})^1(\mu_{jk})^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \binom{N-1}{N-1}(\lambda_{jk})^{N-1}(\mu_{jk})^0 \end{bmatrix}$$

and for $\lambda_{jk} \neq 0$ and $\mu_{jk} = 0$,

$$B(\lambda_{jk}, 0) = \begin{bmatrix} (\lambda_{jk})^0 & 0 & \dots & 0 \\ 0 & (\lambda_{jk})^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (\lambda_{jk})^{N-1} \end{bmatrix}.$$

3 Method of solution

We can construct the fundamental matrix equation for Eq.(1) now. For this aim, we substitute the matrix relations (9) and (10) into Eq. (1) and obtain the matrix equation

$$T(t)(B^T)^m C^T A = \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t)T(t)B(\lambda_{jk}, \mu_{jk})(B^T)^k C^T A + g(t). \tag{11}$$

The collocation points t_i are defined as

$$t_i = \frac{b}{N-1}(i-1), i = 1,2,...,N. \tag{12}$$

By substituting (12) in Eq. (11), we obtain the system of the matrix equations

$$T(t_i)(B^T)^m C^T A = \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(t_i)T(t_i)B(\lambda_{jk},\mu_{jk})(B^T)^k C^T A + g(t_i), i = 1,2,...,N$$

or shortly the fundamental matrix equation

$$\left\{T(B^T)^m C^T - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}TB(\lambda_{jk},\mu_{jk})(B^T)^k C^T\right\}A = G \tag{13}$$

where $P_{jk} = \begin{bmatrix} P_{jk}(t_1) & 0 & \cdots & 0 \\ 0 & P_{jk}(t_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & P_{jk}(t_N) \end{bmatrix}$, $G = \begin{bmatrix} g(t_1) \\ g(t_2) \\ \vdots \\ g(t_N) \end{bmatrix}$, $T = \begin{bmatrix} T(t_1) \\ T(t_2) \\ M \\ T(t_N) \end{bmatrix} = \begin{bmatrix} 1 & t_1 & \cdots & t_1^{N-1} \\ 1 & t_2 & \cdots & t_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & \cdots & t_N^{N-1} \end{bmatrix}$.

Hence, the fundamental matrix equation (13) for Eq. (1) can be written in the form

$$WA = G \text{ or } [W;G] \tag{14}$$

where

$$W = T(B^T)^m C^T - \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}TB(\lambda_{jk},\mu_{jk})(B^T)^k C^T,$$

$$W = [w_{ij}]\, i,j = 1,2,...,N.$$

Here, Eq. (14) corresponds to a system of N linear algebraic equations with unknown Fibonacci coefficients $a_1,a_2,...,a_N$. By means of the relation (9), for the conditions (2), we can obtain the matrix forms

$$\sum_{k=0}^{m-1} c_{ik}T(0)(B^T)^k C^T A = [\lambda_i], i = 1,2,3,...,m.$$

On the other hand, we can write the matrix form for the conditions as

$$U_iA = [\lambda_i] \text{ or } [U_i;\lambda_i], i = 1,2,3,...,m \tag{15}$$

where

$$U_i = \sum_{k=0}^{m-1} c_{ik}T(0)(B^T)^k C^T = [u_{i1}u_{i2}u_{i3}...u_{iN}], i = 1,2,3,...,m.$$

To obtain the solution of Eq. (1) under conditions (2) by replacing the row matrices (15) by the last $m$rows of the matrix (14), we have the new augmented matrix [15, 16],

$$[\tilde{W};\tilde{G}]=\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1N} & ; & g(t_1) \\ w_{21} & w_{22} & \cdots & w_{2N} & ; & g(t_2) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-m)1} & w_{(N-m)2} & \cdots & w_{(N-m)N} & ; & g(t_{N-m}) \\ u_{11} & u_{12} & \cdots & u_{1N} & ; & \lambda_1 \\ u_{21} & u_{22} & \cdots & u_{2N} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mN} & ; & \lambda_m \end{bmatrix}. \tag{16}$$

If rank$\tilde{W}=rank[\tilde{W};\tilde{G}]=N$, we can write$A=(\tilde{W})^{-1}\tilde{G}$. Thus, we uniquely determine the matrix A (thereby the coefficients$a_1,a_2,...,a_N$). Therefore, Eq. (1) with conditions (2) has a unique solution which is given by Fibonacci series solution (3). On the other hand, when$\left|\tilde{W}\right|=0$, that is if rank $\tilde{W}=rank[\tilde{W};\tilde{G}]\leq N$, we can find a particular solution. Otherwise if rank$\tilde{W}\neq rank[\tilde{W};\tilde{G}]\leq N$, then there is no solution.

4 Error analysis based on residual function

In this section, we will give an efficient error estimation for the Fibonacci polynomial approximation and also a technique to obtain the corrected solution of the problem (1) and (2) by using the residual correction method [24-28]. For our purpose, we define the residual function for the present method as

$$R_N(t)=y_N{}^{(m)}(t)-\sum_{j=0}^{J}\sum_{k=0}^{m-1}P_{jk}(t)y_N{}^{(k)}(\lambda_{jk}t+\mu_{jk})-g(t) \tag{17}$$

where $y_N(t)$ is the approximate solution of the problem (1) and (2). Thus, $y_N(t)$satisfies the problem

$$\begin{aligned} & y_N{}^{(m)}(t)-\sum_{j=0}^{J}\sum_{k=0}^{m-1}P_{jk}(t)y_N{}^{(k)}(\lambda_{jk}t+\mu_{jk})=g(t)+R_N(t) \\ & \sum_{k=0}^{m-1}c_{ik}y^{(k)}(0)=\lambda_i. \end{aligned} \tag{18}$$

Also, the error function $e_N(t)$ can be defined as

$$e_N(t)=y(t)-y_N(t) \tag{19}$$

where $y(t)$is the exact solution of the problem (1) and (2). Substituting (19) into (1) and (2) and using (17) and (18), we have the error differential equation with the homogenous conditions

$$\begin{aligned} & e_N{}^{(m)}(t)-\sum_{j=0}^{J}\sum_{k=0}^{m-1}P_{jk}(t)e_N{}^{(k)}(\lambda_{jk}t+\mu_{jk})=-R_N(t), \\ & \sum_{k=0}^{m-1}c_{ik}e_N{}^{(k)}(0)=0, i=0,1,...,m-1. \end{aligned} \tag{20}$$

Solving the problem (20) in the same way as in Section 3, we get the approximation $e_{N,M}(t)$ to$e_N(t)$, $M>N$which is the error function based on the residual function$R_N(t)$.

Consequently, by means of the Fibonacci polynomials $y_N(t)$and $e_{N,M}(t)$, we obtain the corrected exponential solution

$$y_{N,M}(t) = y_N(t) + e_{N,M}(t).$$

Also, we construct the Fibonacci error function $e_N(t) = y(t) - y_N(t)$, and the corrected Fibonacci error function

$$\begin{aligned} e_N(t) &= y(t) - y_N(t), \\ E_{N,M}(t) &= e_N(t) - e_{N,M}(t). \end{aligned}$$

5 Numerical examples

Example 1. [12] With the exact solution$y(t) = t^2$, let us first consider the pantograph equation of the second order

$$y''(t) = \frac{3}{4}y(t) + y(t/2) - t^2 + 2, y(0) = 0, y'(0) = 0, 0 \le t \le 1. \tag{21}$$

We assume that the problem has a Fibonacci polynomial solution in the form

$$y(t) = \sum_{n=1}^{N} a_n F_n(t)$$

where $N = 3, P_{00}(t) = 3/4, P_{10}(t) = 1, g(t) = -t^2 + 2, \lambda_{00} = 1, \mu_{00} = 0, \lambda_{10} = 1/2, \mu_{10} = 0.$

From Eq. (12), the collocation points for $N = 3$ are computed

$$\{t_1 = 0, t_2 = \frac{1}{2}, t_3 = 1\}$$

and from Eq. (13), the fundamental matrix equation of the problem is

$$\left\{T(B^T)^2 C^T - P_{00}TB(\lambda_{00}, \mu_{00})(B^T)^0 C^T - P_{10}TB(\lambda_{10}, \mu_{10})(B^T)^0 C^T\right\}A = G$$

where

$$P_{00} = \begin{bmatrix} 3/4 & 0 & 0 \\ 0 & 3/4 & 0 \\ 0 & 0 & 3/4 \end{bmatrix}, P_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$
$$C^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 2 \\ 7/4 \\ 1 \end{bmatrix},$$
$$B(\lambda_{00}, \mu_{00}) = B(1,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B(\lambda_{10}, \mu_{10}) = B(1/2, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is

$$[W;G] = \begin{bmatrix} -7/4 & 0 & 1/4 & ; & 2 \\ -7/4 & -5/8 & 0 & ; & 7/4 \\ -7/4 & -5/4 & -3/4 & ; & 1 \end{bmatrix}.$$

From Eq. (15), the matrix forms for the initial conditions are

$$U_jA = [\lambda_j] \text{ or } [U_j;\lambda_j]; j = 0,1$$

or clearly

$$[U_0;\lambda_0] = \begin{bmatrix} 1 \; 0 \; 1 \; ; \; 0 \end{bmatrix},$$

$$[U_1;\lambda_1] = \begin{bmatrix} 0 \; 1 \; 0 \; ; \; 0 \end{bmatrix}.$$

From system (16), the new augmented matrix based on conditions can be written as

$$\left[\tilde{W};\tilde{G}\right] = \begin{bmatrix} -7/4 & 0 & 1/4 & ; 2 \\ 1 & 0 & 1 & ; 0 \\ 0 & 1 & 0 & ; 0 \end{bmatrix}.$$

Solving this system, the Fibonacci coefficients matrix is obtained as,

$$A = \begin{bmatrix} -1 \; 0 \; 1 \end{bmatrix}^T.$$

Hence, by substituting the Fibonacci coefficients matrix into Eq. (3),

$$y(t) = \sum_{n=1}^{N} a_n F_n(t) = a_1F_1(t) + a_2F_2(t) + a_3F_3(t) = (-1).1 + 0.t + 1.(t^2+1) = t^2.$$

Thus, the solution of the problem for $N = 3$ becomes$y(t) = t^2$.

Example 2. Consider the following problem [18],

$$y'(t) = \frac{1}{2}e^{t/2}y(\frac{t}{2}) + \frac{1}{2}y(t), y(0) = 1, 0 \le t \le 1. \tag{22}$$

Note that the exact solution is $y(t) = e^t$.

From Eq. (13), the fundamental matrix equation is

$$\left\{T(B^T)C^T - P_{00}TB(\lambda_{00},\mu_{00})(B^T)^0C^T - P_{10}TB(\lambda_{10},\mu_{10})(B^T)^0C^T\right\}A = G$$

where

$$P_{00}(t) = \frac{1}{2}e^{t/2}, P_{10}(t) = \frac{1}{2}, g(t) = 0,, \lambda_{00} = \frac{1}{2}, \mu_{00} = 0, \lambda_{10} = 1, \mu_{10} = 0.$$

Therefore, we obtain the solution of the problem for $N = 13$

$$y(t) = (3.313e-9)t^{12} + (2.15461e-8)t^{11} + (2.81237e-7)t^{10} + (2.74986e-6)t^{9} + (2.48057e-5)t^{8}$$
$$+(1.98411e-4)t^{7} + (1.38889e-3)t^{6} + (8.33333e-3)t^{5} + (4.16667e-2)t^{4} + (1.66667e-1)t^{3} + (0.5)t^{2} + t + 1.$$

In order to estimate the errors for $N = 13$ we consider the following error problem of (18)

$$\begin{aligned} &e_{13}'(t) - \tfrac{1}{2}e^{t/2}e_{13}(\tfrac{t}{2}) - \tfrac{1}{2}e_{13}(t) = -R_{13}(\mathrm{t}) \\ &e_{13}(0) = 0. \end{aligned} \tag{23}$$

Here the residual function is

$$R(t) = e_{13}'(t) - \frac{1}{2}e^{t/2}e_{13}(\frac{t}{2}) - \frac{1}{2}e_{13}(t).$$

By solving the error problem (23) for $M = 14$ introduced in Section 4, the estimated error function approximation $e_{13,14}(t)$ is obtained as

$$\begin{aligned} e_{13,14}(t) = {} & (2.55591e-10)t^{13} - (1.52296e-9)t^{12} + (4.03835e-9)t^{11} - (6.28288e-9)t^{10} + (6.36498e-9)t^{9} \\ & -(4.40373e-9)t^{8} + (2.12132e-9)t^{7} + (7.11204e-10)t^{6} + (1.6278e-10)t^{5} - (2.43206e-11)t^{4} \\ & +(2.16109e-12)t^{3} - (9.00273e-14)t^{2} + (4.0e-30)t + (6.0e-38) \end{aligned}$$

Thus, we obtain the corrected exponential solution as

$$y_{13,14}(t) = y_{13}(t) + e_{13,14}(t).$$

In Figure 1, we compare the absolute error $|e_{13}(t_i)|$with the corrected absolute error$|E_{13,14}(t_i)|$. The present approximate solution converges the exact solution as the number of collocation points increases. Table 1 shows the numerical results of the absolute errors and the corrected absolute errors for $N = 12, 13$ and$M = 13, 14$. It reveals that the residual function will give more accurate results with the smaller corrected errors. Furthermore, the absolute errors of the solutions obtained by the other methods [12, 16, 19, 20] are compared with the absolute error of the solutions obtained by presented method for $N = 13$ in the Table 2.

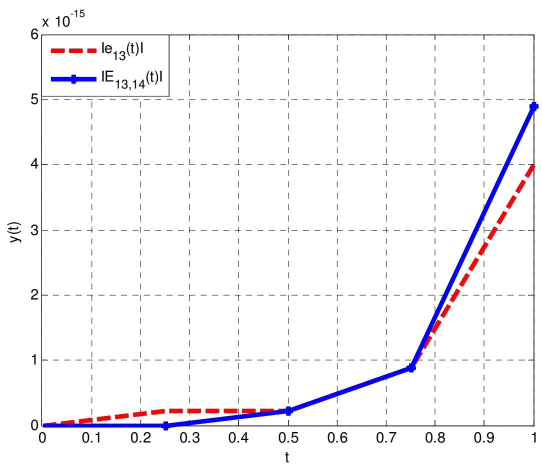

Fig. 1. Comparison of the absolute error with the corrected absolute error for Eq.(21)

Table 1. Numerical results of the error functions for $N = 12, 13$ and $M = 13, 14$.

t_i	Absolute and Corrected error			Absolute and Corrected error	
	$\lvert e_{12}(t_i)\rvert$	$\lvert E_{12,13}(t_i)\rvert$		$\lvert e_{13}(t_i)\rvert$	$\lvert E_{13,14}(t_i)\rvert$
0	0	0		0	0
0.2	2.97170e-15	6.57501e-17		7.96088e-17	1.60808e-18
0.4	3.71271e-15	3.14207e-17		9.83759e-17	6.58744e-19
0.6	4.55735e-15	5.48645e-17		1.20240e-16	1.66569e-17
0.8	5.83019e-15	2.15545e-16		1.43143e-16	5.83700e-18
1	1.46701e-13	4.50428e-15		3.85733e-15	9.54018e-17

Table 2. Comparison of the absolute errors of Eq. (21)

t_i	Spline method [20] $h = 0.001, m = 3, N = 12$	Adomain method with 13 terms[12] $N = 12$	Taylor method [16] $N = 12$	Variatonal Method [19]	Present method $N = 13$
0.2	1.37e-11	0.00	2.220e-16	2.44e-05	7.96088e-17
0.4	3.27e-11	2.22e-16	1.332e-15	2.28e-04	9.83759e-17
0.6	5.86e-11	2.22e-16	2.189e-13	9.00e-04	1.20240e-16
0.8	9.54e-11	1.33e-15	9.361e-12	2.50e-03	1.43143e-16
1	1.43e-10	4.88e-15	1.729e-10	5.71e-03	3.85733e-15

Example 3. Consider the pantograph equation of first order [16],

$$y'(t) = -y(t) + \mu_1(t)y(t/2) + \mu_2(t)y(t/4), y(0) = 1, 0 \leq t \leq 1. \tag{24}$$

with the exact solution$y(t) = e^{-t}\cos(t)$.

Here, $\mu_1(t) = -e^{-0.5t}\sin(0.5t), \mu_2(t) = -2e^{-0.75t}\cos(0.5t)\sin(0.25t)$. From Eq. (13), the fundamental matrix equation of the problem is

$$\left\{T(B^T)^1C^T - P_{00}TB(\lambda_{00},\mu_{00})(B^T)^0C^T - P_{10}TB(\lambda_{10},\mu_{10})(B^T)^0C^T - P_{20}TB(\lambda_{20},\mu_{20})(B^T)^0C^T\right\}A = G$$

Note that $P_{00}(t) = -1, P_{10}(t) = \mu_1(t), P_{20}(t) = \mu_2(t), g(t) = 0$, and

$$\lambda_{00} = 1, \mu_{00} = 0, \lambda_{10} = 1/2, \mu_{10} = 0, \lambda_{20} = 1/4, \mu_{20} = 0.$$

Hence, we find the solution of the problem for $N = 8$

$$\begin{aligned} y(t) = &-(0.52794e-3)t^7 - (0.13344e-2)t^6 + (0.34245e-1)t^5 \\ &-(0.16701)t^4 + (0.33340)t^3 - (0.59313e-5)t^2 - t + 1 \end{aligned}$$

Table 3 shows a comparison of the numerical results of the absolute errors and the corrected absolute errors for $N = 8$ and $M = 9$. In addition, Figure 2 illustrates a comparison of the absolute error $|e_8(t_i)|$ with the corrected absolute error $|E_{8,9}(t_i)|$and it indicates the significant decrease in the absolute error owing to the error correction by the residual function.

In Figure 3, the absolute error and the solution obtained by the present method are compared with the absolute error and the solution of the Taylor method given in [16]. It is seen from Figure 4 that the results obtained by the present method is very superior to that obtained by the Taylor method.

Table 3. Numerical results of the error functions for$N = 8$ and $M = 9$

t_i	Absolute error $\lvert e_8(t_i)\rvert$	Corrected absolute error $\lvert E_{8,9}(t_i)\rvert$
0	0	0
0.2	1.45223e-08	5.33333e-10
0.4	1.11496e-08	2.00920e-10
0.6	6.35221e-09	1.07101e-09
0.8	8.88920e-10	1.89453e-09
1	4.61651e-07	3.17734e-08

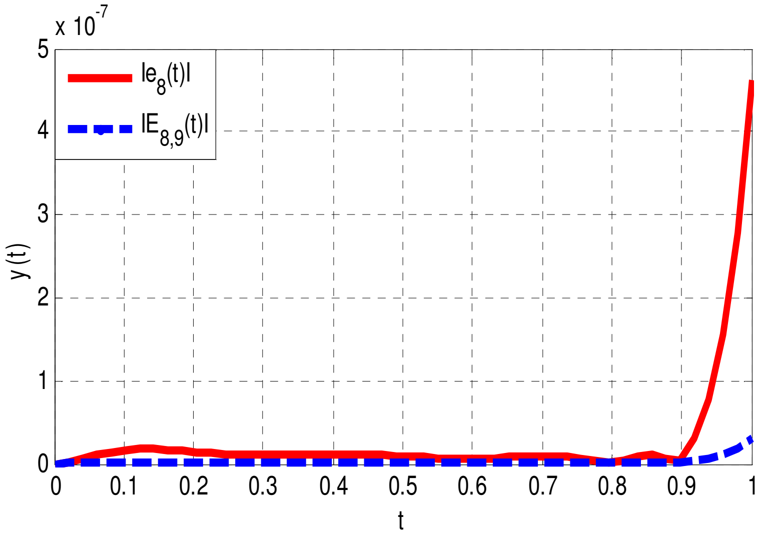

Fig. 2. Comparison of the absolute error with the corrected absolute error for Eq. (24)

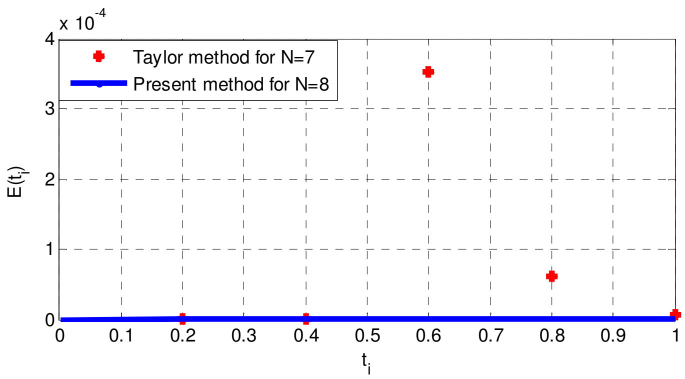

Fig. 3. Comparison of the absolute errors $E(t_i)$

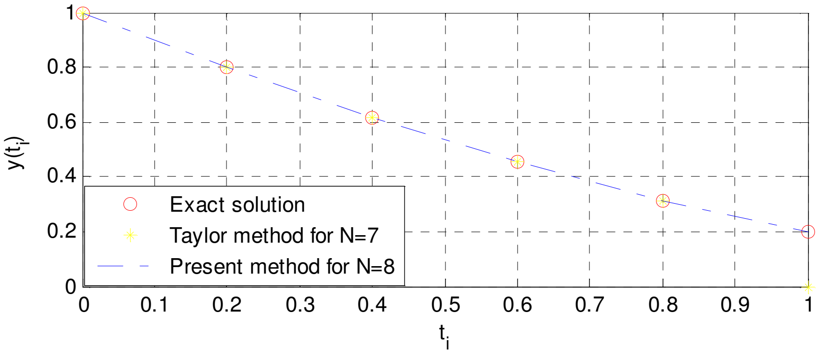

Fig. 4. Comparison of the solutions $y(t_i)$

6 Conclusions

A new approach using the Fibonacci polynomials to solve numerically the pantograph equations is presented in this study. The comparison of the results obtained by the present method with other methods reveals that the present method is more convenient, reliable and effective. An error analysis technique based on residual function is also developed and applied problems to demonstrate the validity and the applicability of this method. If the exact solution of the problem is not known, by using this technique it is possible to estimate the error function and also to reduce the error due to the residual function. It is seen that, the accuracy improves, when N is increased. Tables and figures indicate that as Nincreases, the errors decrease more rapidly. Another considerable advantage of the method is that Fibonacci coefficients of the solution are found very easily by using the computer programs.

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