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On bounded second Riesz $p(\cdot)$-variable variation

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Abstract: In this paper we define the concept of functions of bounded second $p(\cdot)$-variable variation. Further we obtain some relation between a function of bounded second $p(\cdot)$-variable variation on an interval $[a,b]$ with the Luxemburg norm on the variable $L^{p(x)}[a,b]$ space and the norm of the usual $L_p[a,b]$

Keywords: Bounded variation, bounded p-variation, bounded second p-variation, Mulsielak Orlicz Space, Lebesgue space, variable Lebesgue space, Luxemburg norm.

1 Introduction

The concept of functions of bounded variation first appeared in a paper by Camile Jordan [5] in 1881 while dealing with the convergence of Fourier series. He established the relationship between the functions of bounded variation and the monotonic functions. Since then this topic has attracted researchers studying mathematical analysis all across the world and a number of extensions has been given in many directions. The first generalization is the functions of bounded p-variation given by Wiener [14] which nowadays are widely being accepted as the definition given in the Wiener sense. L.Young [15] in 1937 further generalized this to the functions of Φ-variation on a closed interval $[a,b]$. De La Valle Poussin [3] took this concept in a very different direction. He introduce the notion of bounded second variation on a closed interval and obtained relationship between the functions of second bounded variation with the convex functions. A.M. Russel and C.J.F. Upton [12] obtained the functions of bounded second p-variation $(1 \leq p < \infty)$ in the sense of Wiener. It is important to mention here that all of the above generalization of the function of bounded variation were motivated by their applications in several different areas of mathematics few of which are Fourier analysis [10], [13], operator theory [1],[4] calculus of variation, geometric measure theory and mathematical physics.

For decades, modular space has been an active area of research primarily due, to its very rich in structure besides being a Banach space. Recently a new modular space call the variable Lebesgue space is being developed which generalizes both Orlicz and Musielak Orlicz space. Although it has its origin from 1931 in the work of Orlicz. However serious work on this space began after the fundamental paper by Kovacik and Rakosnk [7] in 1991.

Recently Rene Erlin Castillo, Nelson Merentes and Humberto Rafeiro [2] further extended the notion of function of bounded p-variation to function of bounded $p(\cdot)$-variable variation. Further in 2016 George Kakochasvili and Shalva Zviadadze [6] studied functions of bounded Riesz $p(\cdot)$-variation. In this paper we have extended the concept of bounded second p-variation defined by Merentes [8] to the variable $p(\cdot)$-variation and using certain concepts from the variable Lebesgue spaces we have partially generalized certain results proved for p-variation and bounded second p-variation.

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2 Premilinaries

We shall begin this section with some basics and definitions of variable Lebesgue space, function of bounded variation and there generalizations which will be used through this paper.

First we begin with a brief introduction to the variable Lebesgue space. Let $p:[a,b]\to[1,\infty)$ be a measurable function such that

$$p^* = esssup_{x\in[a,b]}p(x) \quad \text{and} \quad p_* = essinf_{x\in[a,b]}p(x) \tag{1}$$

Denote by $L^{p(x)}[a.b]$, the collection of all measurable functions $f:[a,b]\to\mathbb{R}$ such that for some $\lambda=\lambda(f)>0$, $I_{p(\cdot)}^{[a,b]}\left(\frac{f}{\lambda}\right)<\infty$, where $I_{p(\cdot)}^{[a,b]}$ is the modular function defined by

$$I_{p(\cdot)}^{[a,b]}(f)=\int_a^b |f(x)|^{p(x)}dx \tag{2}$$

This is a linear space and is a Banach Spaces with respect to the Luxemburg norm defined by

$$\|f\|_{L_{p(\cdot)}[a,b]}=\inf\left\{\lambda>0: I_{p(\cdot)}^{[a,b]}\left(\frac{f}{\lambda}\right)\le 1\right\} \tag{3}$$

For a given $p:[a,b]\to[1,\infty)$, we define the conjugate exponent function by

$$\frac{1}{p(x)}+\frac{1}{q(x)}=1,\;\; x\in[a,b] \tag{4}$$

Note that here we do not allow either $p(x)$ or $q(x)$ to tend to infinity and so we also exclude the tendency of either $p(x)$ or $q(x)$ to 1. Thus we always assume that

$$1<p^*\le p(x)\le p_*<\infty \quad \text{and} \quad 1<q^*\le q(x)\le q_*<\infty$$

Now for any partition of $[a,b]$ of the form

$$\pi^*: a=x_o<y_1\le z_1<x_1<y_2\le z_2<x_2<\cdots<y_n\le z_n<x_n=b \;\text{ with }\; Q_k=(x_k,x_{k+1}). \tag{5}$$

Define

$$\frac{1}{\tilde{p}_{Q_k}}=\frac{1}{|Q_k|}\int_{Q_k}\frac{1}{p(x)}dx \;\text{ and }\; \frac{1}{\tilde{q}_{Q_k}}=\frac{1}{|Q_k|}\int_{Q_k}\frac{1}{q(x)}dx \tag{6}$$

where $p(\cdot)$ and $q(\cdot)$ are conjugate functions. We further define the discrete variable Lebesgue space denoted by $l^{p(\cdot),\mathscr{Q}}$(see [6]) by

$$l^{p(\cdot),\mathscr{Q}}=\left\{\{x_{Q_k}\}_{Q_k\in\mathscr{Q}}: \sum_{Q_k\in\mathscr{Q}}|x_{Q_k}|^{\tilde{p}_{Q_k}}<+\infty\right\} \tag{7}$$

and is equipped with the Luxemburg's norm

$$\|x\|_{l^{p(\cdot),\mathscr{Q}}}=\inf\left\{\lambda>0: \sum_{Q_k\in\mathscr{Q}}\left|\frac{x_{Q_k}}{\lambda}\right|^{\tilde{p}_{Q_k}}\le 1\right\},$$

where we have denoted $\mathscr{Q}=\{Q_k\}$ to be a partition of the form (5). Note that $\{e_Q\}_{Q\in\mathscr{Q}}$ is the canonical basis of the discrete variable Lebesgue space where e_Q has 1 at the index Q and 0 otherwise.

Also for a function $p:[a,b]\to[1,\infty)$ and its conjugate function $q:[a,b]\to[1,\infty)$, we have the Holder's inequality as

$$\sum_{Q_k\in\mathcal{Q}} |x_{Q_k}\cdot y_{Q_k}| \le C\,\|x\|_{l^{p(\cdot),\mathcal{Q}}}\,\|y\|_{l^{q(\cdot),\mathcal{Q}}}\,,\quad x\in l^{p(\cdot),\mathcal{Q}}\,,\ y\in l^{q(\cdot),\mathcal{Q}} \tag{8}$$

Now before we begin our main section, we shall go through some definition of functions of bounded variations

Definition 1. *Let $f:[a,b]\to\mathbb{R}$ be a function. Then for a partition*

$$\pi : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

of $[a,b]$, define

$$V(f:[a,b]) = \sup_{\pi}\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|\,, \tag{9}$$

where supremum is taken over all partitions π of the interval $[a,b]$. If $V(f:[a,b])<\infty$, then f is said to be a function of bounded variation on $[a,b]$. We denote the collection of all functions of bounded variation on $[a,b]$ by $BV(f:[a,b])$.

In 1973 N. Wiener [14] further introduced functions of bounded p-variations $(1<p<\infty)$ on an interval $[a,b]$.

Definition 2.[14] *A function $f:[a,b]\to\mathbb{R}$ is said to be a function of bounded p-variation$(1<p<\infty)$ in the sense of Wiener iff*

$$V_p^w(f:[a,b]) = \sup_{\pi}\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|^p < \infty, \tag{10}$$

where supremum is taken over all partition π given by

$$\pi : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

of $[a,b]$. We denote the collection of all functions of bounded p-variation$(1<p<\infty)$ in Wiener sense over $[a,b]$ by $BV_p^w(f:[a,b])$

Definition 3.[11] *Let $f:[a,b]\to\mathbb{R}$ be a function. Then for a given partition*

$$\pi = a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

of $[a,b]$, define

$$V_p^r(f:[a,b]) = \sup_{\pi}\sum_{k=0}^{n-1} \frac{|f(x_{k+1}) - f(x_k)|^p}{|x_{k+1}-x_k|^{p-1}} \quad (1<p<\infty), \tag{11}$$

where supremum is taken over all partition π of $[a,b]$. Now, if $V_p^r(f:[a,b])<\infty$, then we say that f is a function of bounded Riesz p-variation on $[a,b]$ and we take $BV_p^r(f:[a,b])$ to be the collection of all function of bounded Riesz p-variation $(1<p<\infty)$ on $[a,b]$.

George Kakochashvili,Shalva Zviadadze [6] recently in 2016 introduce the concept of functions of bounded Riesz $p(\cdot)$-variable variation on $[a,b]$.

Definition 4.[6] *Let $p:[a,b]\to[1,\infty)$ be a measurable function. Then a function $f:[a,b]\to\mathbb{R}$ is said to be a function of bounded Riesz $p(\cdot)$-variable variation on $[a,b]$, if*

$$D(f) = \sup_{\pi}\sum_{k=0}^{n-1} \frac{|f(x_{k+1}) - f(x_k)|^{\tilde{p}_{Q_k}}}{|x_{k+1}-x_k|^{\tilde{p}_{Q_k}-1}} < \infty, \tag{12}$$

where supremum is taken over all partition of $[a,b]$ *given by*

$$\pi = a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

of $[a,b]$ *with* $Q_k = (x_{k+1} - x_k)$.

We denote the collection of all function of bounded Riesz $p(\cdot)$-variable variation on $[a,b]$ by $BV_{p(\cdot)}[a,b]$.

$$BV_{p(\cdot)}(f:[a,b]) = \{f:[a,b] \to \mathbb{R} : D(f) < \infty\}.$$

De la Valle Poussin [3] studied the class of all functions of bounded second variation as follows.

Definition 5.[3] *For a partition* π *of the form* $\pi : a = x_o < z_1 \leq y_1 < x_2 < \cdots < x_{n-1} \leq z_n \leq y_n < x_n = b$ *of* $[a,b]$. *A function* $f:[a,b] \to \mathbb{R}$ *is said to be of bounded second variation on* $[a,b]$, *if*

$$V^2(f:[a,b]) = \sup_{\pi} \sum_{k=0}^{n-1} \left| \frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right| < \infty. \tag{13}$$

We denote the collection of all such function by $BV^2[a,b]$

$$i.e., \;\; BV^2(f:[a,b]) = \left\{ f:[a,b] \to \mathbb{R} : V^2(f:[a,b]) < \infty \right\}.$$

Merentes in his paper [8] studied functions of bounded Riesz $(p,2)$-variation on $[a,b]$ and proved that if $f:[a,b] \to \mathbb{R}$ is a function of bounded $(p,2)$-variation$(1 < p < \infty)$, then $V_p^2(f:[a,b]) = \|f''\|^p_{L_p[a,b]}$ where $\|\cdot\|^p_{L_p[a,b]}$ is the L_p norm of the function f and $V_p^2(f:[a,b])$ is defined as follows.

Definition 6.[8] *Let* $f:[a,b] \to \mathbb{R}$ *be a function with* $1 < p < \infty$. *For a given* π *partition of the form*

$$\pi : a = x_o < z_1 \leq y_1 < x_2 < \cdots < x_{n-1} \leq z_n \leq y_n < x_n = b, \tag{14}$$

define

$$V_p^2(f:[a,b]) = \sup_{\pi} \sum_{k=0}^{n-1} \left| \frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right|^p \left(\frac{1}{x_{k+1} - x_k} \right)^{p-1}, \tag{15}$$

where supremum is taken over all partition π *of the form (14) over the interval* $[a,b]$. *We say that* f *is a function of bounded Riesz* $(p,2)$*-variation on* $[a,b]$ *if* $V_p^2(f:[a,b]) < \infty$. *We denote by* $BV_p^2(f:[a,b])$ *the collection of all functions such that* $V_p^2(f:[a,b]) < \infty$.

3 Main result

In this section, we shall begin with our definition of bounded second $p(\cdot)$-variable variation on $[a,b]$ and prove some results.

Definition 7. *Let* $p:[a,b] \to [1,\infty)$ *be a measurable function such that* $p^* < \infty$. *For a partition* π^* *of* $[a,b]$ *given by*

$$\pi^* : a = x_o < y_1 \leq z_1 < x_1 < y_2 \leq z_2 < x_2 < \cdots < y_n \leq z_n < x_n = b \;\; with \;\; Q_k = (x_k, x_{k+1}). \tag{16}$$

We define the class of all functions $f:[a,b] \to \mathbb{R}$ *which are of bounded second* $p(\cdot)$*-variable variation as follows.*
Let

$$\rho^2_{p(\cdot)}[f:\pi^*] = \sum_{k=0}^{n-1} \left| \frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right|^{\tilde{p}_{Q_k}} \frac{1}{|Q_k|^{\tilde{p}_{Q_k}-1}}$$

and

$$D^2(f) = \sup_{\pi^*} \rho^2_{p(\cdot)}[f:\pi^*],$$

where supremum is taken over all the partitions π^* of the form (16). We say that f is a function of bounded second $p(\cdot)$-variable variation on $[a,b]$. We denote the set $BD^2_{p(\cdot)}(f:[a,b])$ to be the collection of all those functions for which $D^2(f) < \infty$.

$$i.e \quad BD^2_{p(\cdot)}(f:[a,b]) = \left\{f:[a,b]\to\mathbb{R} : D^2(f)<\infty\right\}$$

Remark. Every constant function $f:[a,b]\to\mathbb{R}$ are elements of $BD^2_{p(\cdot)}(f:[a,b])$

Example 1. Let $f:[a,b]\to\mathbb{R}$ be a real valued function given by $f(x)=\alpha x+\beta$ for some fixed reals α and β. Then clearly $D^2(f)<\infty$. Thus $f\in BD^2_{p(\cdot)}(f:[a,b])$

Remark. If $p(x)=p, \forall \quad x\in[a.b]$, then the class of all functions $BD^2_{p(\cdot)}(f:[a,b])$ coincides with the class of functions of Riez-$(p,2)$ variation on $[a,b]$ which we have denoted by $BD^2_p(f:[a,b])$

Remark. If $p(x)=1, \forall \quad x\in[a.b]$, then the class of all functions $BD^2_{p(\cdot)}(f:[a,b])$ coincides with the class of functions of bounded second variation defined by De la valle Poussin [3] which we have denoted by $BV^2(f:[a,b])$.

Theorem 1. *Let $p:[a,b]\to[1,\infty)$ be a measurable function such that $0<p_*\le p^*<\infty$. If $D^2(f)<\infty$, then the function f is of bounded second variation on $[a,b]$, and*

$$V^2(f:[a,b]) \le \left(D^2(f)\right)^{\frac{1}{p_*}} |b-a|^{q^*} \tag{17}$$

Proof. Consider a partition π^* of $[a,b]$ as

$$\pi^*: a=x_o<y_1\le z_1<x_1<y_2\le z_2<x_2<\cdots<y_n\le z_n<x_n=b$$

with $Q_k=(x_k,x_{k+1})$. Then by Holder's inequality, we have

$$\sum_{k=0}^{n-1}\left|\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right| = \sum_{k=0}^{n-1}\left|\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right|^{\tilde{p}_{Q_k}}\frac{|Q_k|^{1-\frac{1}{\tilde{p}_{Q_k}}}}{|Q_k|^{1-\frac{1}{\tilde{p}_{Q_k}}}}$$

$$\le C\left\|\left\{\left(\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right)\frac{1}{|Q_k|^{1-\frac{1}{\tilde{p}_{Q_k}}}}\right\}_{Q_k\in\mathscr{Q}}\right\|_{l^{p(\cdot)},\mathscr{Q}}\left\|\left\{|Q_k|^{\frac{1}{\tilde{q}_{Q_k}}}\right\}_{Q_k\in\mathscr{Q}}\right\|_{l^{q(\cdot)},\mathscr{Q}},$$

where $q:[a,b]\to[1,\infty)$ is the conjugate exponent.
Since $D^2(f)<\infty$,without loss of generality we may assume that $D^2(f)\ge 1$, so that

$$1\ge\sum_{k=0}^{n-1}\left|\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right|^{\tilde{p}_{Q_k}}\frac{1}{|Q_k|^{\tilde{p}_{Q_k}-1}}\frac{1}{D^2(f)}$$

$$\ge\left[\sum_{k=0}^{n-1}\left|\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right|\frac{1}{|Q_k|^{1-\frac{1}{\tilde{p}_{Q_k}}}}\left(\frac{1}{D^2(f)}\right)^{\frac{1}{\tilde{p}_{Q_k}}}\right]^{\tilde{p}_{Q_k}}$$

$$\ge\left[\sum_{k=0}^{n-1}\left|\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right|\frac{1}{|Q_k|^{1-\frac{1}{\tilde{p}_{Q_k}}}}\left(\frac{1}{D^2(f)}\right)^{\frac{1}{p_*}}\right]^{\tilde{p}_{Q_k}}$$

So that

$$\left\| \left\{ \left(\frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right) \frac{1}{|Q_k|^{1-\frac{1}{\tilde{p}_{Q_k}}}} \right\}_{Q_k \in \mathcal{Q}} \right\|_{l^{p(\cdot)},\mathcal{Q}} \leq \left(D^2(f)\right)^{\frac{1}{p_*}} \tag{18}$$

Further we see that

$$\left\| \left\{ |Q_k|^{\frac{1}{\tilde{q}_{Q_k}}} \right\}_{Q_k \in \mathcal{Q}} \right\|_{l^{q(\cdot)},\mathcal{Q}} \leq |b-a|^{\frac{1}{q^*}} \tag{19}$$

Thus from equation (18) and (19), we get

$$\sum_{k=0}^{n-1} \left| \frac{f(x_{k+1}) - f(z_{k+1})}{x_{k+1} - z_{k+1}} - \frac{f(y_{k+1}) - f(x_k)}{y_{k+1} - x_k} \right| \leq \left(D^2(f)\right)^{\frac{1}{p_*}} |b-a|^{\frac{1}{q^*}} \tag{20}$$

Taking supremum over all such partitions of $[a,b]$, we get

$$V^2(f:[a,b]) \leq \left(D^2(f)\right)^{\frac{1}{p_*}} |b-a|^{\frac{1}{q^*}} \tag{21}$$

Now using Theorem 1 with Theorem 1.1 [3], we have the following corollary

Corollary 1. *Let $p:[a,b] \to [1,\infty)$ be a measurable function such that $1 < p_* < \infty$ and $1 < q^* < \infty$. If $D^2(f) < \infty$, then f is absolutely continuous on $[a,b]$ and f can also be expressed as a difference of two convex function.*

Theorem 2. *Let $p:[a,b] \to [1,\infty)$ such that $1 < p_* \leq p^* < \infty$. If $D^2(f) < \infty$, then first derivative of f exists for every $x \in (a,b)$.*

Proof. Since $D^2(f) < \infty$ implies that $V^2(f:[a,b]) < \infty$ by Theorem 1. Then the existence of the right-hand derivative and left hand derivative is evident from the fact that f can be express as a difference of two convex function. The further proof can be done in a similar way as was done for fixed $1 < p < \infty$.(see [8])

Theorem 3. *Let $p:[a,b] \to [1,\infty)$ be a measurable function such that $1 < p_* \leq p^* < \infty$. If $D^2(f) < \infty$ and $f'' \in L_{p^*}[a,b]$, then $f' \in BV_{p(\cdot)}[a,b]$ and f' is absolutely continuous. Moreover*

$$\|f''\|^{p_*}_{L_{p(\cdot)}[a,b]} \leq D^2(f) \leq r_p \|f''\|^{p^*}_{L_{p^*}[a,b]} \tag{22}$$

Proof. Let us consider the partition

$$\pi^* : a = x_o < y_1 \leq z_1 < x_1 < y_2 \leq z_2 < x_2 < \cdots < y_n \leq z_n < x_n = b$$

with $Q_k = (x_k, x_{k+1})$.
We take small $h > 0$ such that $h \leq \min\left\{ \frac{|Q_0|}{2}, \frac{|Q_1|}{2}, \cdots \frac{|Q_{n-1}|}{2} \right\}$. Then by the definition of $D^2(f) < \infty$, we have

$$\sum_{k=0}^{n-1} \left| \frac{f(x_{k+1}) - f(x_{k+1} - h)}{h} - \frac{f(x_k + h) - f(x_k)}{h} \right|^{\tilde{p}_{Q_k}} \frac{1}{|Q_k|^{\tilde{p}_{Q_k}-1}} \leq D^2(f) \tag{23}$$

Also since $D^2(f) < \infty$ and so by Theorem 2, we see that derivative of f exists at each point $x \in (a,b)$. Thus taking limit $h \to 0$ in equation (23), we get

$$\sum_{k=0}^{n-1} \left| f'(x_{k+1}) - f'(x_k) \right|^{\tilde{p}_{Q_k}} \frac{1}{|Q_k|^{\tilde{p}_{Q_k}-1}} \leq D^2(f) \tag{24}$$

This shows that f' is absolutely continuous and now taking supremum on RHS of equation (24) over all such partitions of $[a,b]$, we have

$$\sup_{\pi^*}\sum_{k=0}^{n-1}\left|f'(x_{k+1})-f'(x_k)\right|^{\tilde{p}_{Q_k}}\frac{1}{|Q_k|^{\tilde{p}_{Q_k}-1}}\leq D^2(f)$$

$$i.e \quad D(f')\leq D^2(f)<\infty \tag{25}$$

Thus we find that $f'\in BV_{p(\cdot)}[a,b]$. Further by embedding in the variable lebesgue space we have $f''\in L_{p(\cdot)}[a,b]$ and

$$\int_b^a |f''(x)|^{p(x)}dx\leq D(f')$$

see [6, Theorem 3.1].
So that

$$\left(\int_b^a\left|\frac{f''(x)}{D(f')^{\frac{1}{p_*}}}\right|^{p(x)}dx\right)\leq\left(\int_b^a\left|\frac{f''(x)}{D(f')^{\frac{1}{p(x)}}}\right|^{p(x)}dx\right)=\frac{1}{D(f')}\left(\int_b^a|f''(x)|^{p(x)}dx\right)\leq 1.$$

Thus

$$\left(\int_b^a\left|\frac{f''}{D(f')^{\frac{1}{p_*}}}\right|^{p(x)}dx\right)\leq 1.$$

Thus by the definition of Luxemburg's norm on variable $L_{p(\cdot)}[a,b]$ space, we find that

$$\|f''\|_{L_{p(\cdot)}[a,b]}\leq D(f')^{\frac{1}{p_*}}$$

So that

$$\|f''\|^{p_*}_{L_{p(\cdot)}[a,b]}\leq D(f') \tag{26}$$

From equation (25) and (26), we find that

$$\|f''\|^{p_*}_{L_{p(\cdot)}[a,b]}\leq D^2(f). \tag{27}$$

Next since f' is absolutely continuous and hence continuous. So by mean value theorem, we can always find some t_k and s_k such that $x_k<t_k<y_{k+1}$ and $z_{k+1}<s_k<x_{k+1}$ for $k=0,1,2,3\cdots(n-1)$ and

$$\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}=f'(t_k)\ \text{ and }\ \frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}=f'(s_k) \tag{28}$$

So by Holder's Inequality, we have

$$\begin{aligned}\left|\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right|^{\tilde{p}_{Q_k}}&\leq r_p\left(\left\{\int_{x_k}^{x_{k+1}}|f''(\zeta)|^{\tilde{p}_{Q_k}}d\zeta\right\}^{\frac{1}{\tilde{p}_{Q_k}}}(x_{k+1}-x_k)^{1-\frac{1}{\tilde{p}_{Q_k}}}\right)^{\tilde{p}_{Q_k}}\\&\leq r_p\left\{\int_{x_k}^{x_{k+1}}|f''(\zeta)|^{\tilde{p}_{Q_k}}d\zeta\right\}(x_{k+1}-x_k)^{(1-\frac{1}{\tilde{p}_{Q_k}})\tilde{p}_{Q_k}}\\&\leq r_p\left\{\int_{x_k}^{x_{k+1}}|f''(\zeta)|^{\tilde{p}_{Q_k}}d\zeta\right\}(x_{k+1}-x_k)^{\tilde{p}_{Q_k}-1}\end{aligned}$$

So that
$$\left|\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right|^{\tilde{p}_{Q_k}}\frac{1}{(x_{k+1}-x_k)^{\tilde{p}_{Q_k}-1}}\leq r_p\left\{\int_{x_k}^{x_{k+1}}\left|f''(\zeta)\right|^{\tilde{p}_{Q_k}}d\zeta\right\} \tag{29}$$
Now taking summation on both the sides of equation (29), we get

$$\begin{aligned}\sum_{k=0}^{n-1}\left|\frac{f(x_{k+1})-f(z_{k+1})}{x_{k+1}-z_{k+1}}-\frac{f(y_{k+1})-f(x_k)}{y_{k+1}-x_k}\right|^{\tilde{p}_{Q_k}}\frac{1}{(x_{k+1}-x_k)^{\tilde{p}_{Q_k}-1}} &\leq r_p\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}\left|f''(\zeta)\right|^{\tilde{p}_{Q_k}}d\zeta\\ &\leq r_p\int_a^b\left|f''(\zeta)\right|^{p^*}d\zeta\\ &= r_p\left\|f''\right\|^{p^*}_{L_{p^*}[a,b]}\end{aligned}$$

$$D^2(f)\leq r_p\left\|f''\right\|^{p^*}_{L_{p^*}[a,b]} \tag{30}$$
Hence from equation (27) and (30), we finally get

$$\left\|f''\right\|^{p_*}_{L_{p(\cdot)}[a,b]}\leq D^2(f)\leq r_p\left\|f''\right\|^{p^*}_{L_{p^*}[a,b]}$$

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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