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Some topological properties of double Cesàro-Orlicz sequence spaces

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Abstract: The object of this paper is to introduce the double Cesàro-Orlicz sequence space $Ces_M^{(2)}$ using a Orlicz function M. Necessary and sufficient conditions under which the double Cesàro-Orlicz sequence space $Ces_M^{(2)}$ is nontrivial are presented. It is proved that double Cesàro-Orlicz sequence spaces $Ces_M^{(2)}$ are complete. Finally, it is obtained that if $\phi \in \Delta_2(0)$ then the space $Ces_M^{(2)}$ is separable and order continuous.

Keywords: Double sequence, double Cesàro-Orlicz sequence space, Luxemburg norm, Fatou property, order continuity.

1 Introduction

As usual, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of positive integers, real numbers and nonnegative real numbers, respectively. A double sequence on a normed linear space X is a function x from $\mathbb{N}\times\mathbb{N}$ into X and briefly denoted by $x=(x(i,j))$. Throughout this work , w and w^2 denote the spaces of all single real sequences and double real sequences, respectively.

First of all, let us recall preliminary definitions and notations.

Definition 1. *If for every $\varepsilon>0$ there exists $n_\varepsilon\in\mathbb{N}$ such that $\left\|x_{k,l}-a\right\|_X<\varepsilon$ whenever $k,l>n_\varepsilon$ then a double sequence $\left\{x_{k,l}\right\}$ is said to be converge (in terms of Pringsheim) to $a\in X$* [12].

A double sequence $\left\{x_{k,l}\right\}$ is called a Cauchy sequence if and only if for every $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)$ such that $\left|x_{k,l}-x_{p,q}\right|<\varepsilon$ for all $k,l,p,q\geq n_0$.

A double series is infinity sum $\sum_{k,l=1}^{\infty}x_{k,l}$ and its convergence implies the convergence by $\|.\|_X$ of partial sums sequence $\{S_{n,m}\}$, where $S_{n,m}=\sum_{k=1}^{n}\sum_{l=1}^{m}x_{k,l}$(see [1], [6]).

Definition 2. *If each double Cauchy sequence in X converges an element of X according to norm of X, then X is said to be a double complete space. A normed double complete space is said to be a double Banach space* [1].

Definition 3. *A Banach space $(X,\|.\|)$ which is a subspace of $w^{(2)}$ is said to be double Köthe sequence space if:*

(i) *for any $x\in w^{(2)}$ and $y\in X$ such that $|x(i,j)|\leq|y(i,j)|$ for all $i,j\in\mathbb{N}$, we have $x\in X$ and $\|x\|\leq\|y\|$,*
(ii) *there is $x\in X$ with $x(i,j)\neq 0$ for all $i,j\in\mathbb{N}$.*

An element x from a double Köthe sequence space X is called order continuous if for any sequence (x_n) in X_+ (the positive cone of X) such that $x_n\leq|x|$ for all $n\in\mathbb{N}$ and $x_n\to 0$ coordinatewise, we have $\|x_n\|\to 0$.

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A double Köthe sequence space X is said to be order continuous if any $x \in X$ order continuous. It is easy to see that X is order continuous if and only if $\|x^{*}\| \to 0$ as $n,m \to \infty$, where

$$x^{*}(i,j) = \begin{cases} x(i,j), & if \;\; i \geq n+1 \; and \; j \geq m+1 \\ 0, & others \end{cases}$$

for any $x \in X$.

A double Köthe sequence space X has the Fatou property if for any sequence (x_n) in X_+, and any $x \in w^{(2)}$ such that $x_n \to x$ coordinatewise and $\sup_n \|x_n\| < \infty$, we have that $x \in X$ and $\|x_n\| \to \|x\|$ [2].

It is known that for any Köthe sequence (function) space the Fatou property implies its completeness [10] .

Definition 4. *A function $\rho : X \to [0,\infty)$, where X is real vector space is called a modular if it is satisfies the following conditions:*

(i) $\rho(x) = 0$ *if and only if* $x = 0$;
(ii) $\rho(\alpha x) = \rho(x)$ *for all scalar* α *with* $|\alpha| = 1$;
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *for all* $x,y \in X$ *and all* $\alpha,\beta \geq 0$ *with* $\alpha + \beta = 1$.

The modular ρ is called convex if
(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ *for all* $x,y \in X$ *and all* $\alpha,\beta \geq 0$ *with* $\alpha + \beta = 1$.

For any modular ρ on X, the space

$$X_\rho = \left\{ x \in X : \rho(\lambda x) \to 0 \; as \; \lambda \to 0^{+} \right\}$$

is called the modular space. If ρ is a convex modular, the function

$$\|x\| = \inf\left\{ \lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1 \right\}$$

is norm on X_ρ, which is called the Luxemburg norm [11].

Definition 5. *An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.*

An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x \eta(t)\,dt$, where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$ for $t > 0$, η is nondecreasing and $\eta(t) \to \infty$ as $t \to \infty$.

An Orlicz function M is said to be satisfied the Δ_2-condition at zero ($M \in \Delta_2(0)$ for shortly) if there are $T > 0$ and $a > 0$ such that $M(a) > 0$ and $M(2u) \leq TM(u)$ for all $u \in [0,a]$ [4], [9], [11]. For $1 \leq p < \infty$, the Cesàro sequence space is defined by

$$Ces_p = \left\{ x \in w : \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p} < \infty \right\},$$

equipped with norm

$$\|x\| = \left(\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^{p} \right)^{\frac{1}{p}}.$$

This space was first introduced by Shiue [14] It is very useful in the theory of matrix operators and others.

The arithmetic mean map σ is defined on w by the formula

$$\sigma x = (\sigma x(i))_{i=1}^{\infty} \ \text{, where } \sigma x(i) = \frac{1}{i}\sum_{j=1}^{i}|x(j)|$$

for any $i \in \mathbb{N}$ and $x \in w$. Given any Orlicz function M, the space

$$Ces_M = \{x \in w : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where

$$\rho_M(\lambda x) = \sum_{i=1}^{\infty} M\left(\frac{\lambda}{i}\sum_{j=1}^{i}|x(j)|\right)$$

which is called the Cesàro-Orlicz sequence space. This space equipped with the Luxemburg norm

$$\|x\|_{Ces_M} = \inf\left\{\lambda > 0 : \rho_M\left(\frac{x}{\lambda}\right) \leq 1\right\}$$

([2]). In the case, when $M(u) = |u|^p$ $,1 \leq p < \infty$, we get the Cesaro sequence spaces Ces_p. The topological and geometric properties of Cesàro-Orlicz sequence spaces and their generalizations have been studied in [2], [3], [5], [7], [13], [14].

In this paper, for double sequences, we introduce sequence space $Ces_M^{(2)}$ using a Orlicz function M and obtain its some topological properties. The double sequence spaces $Ces_M^{(2)}$ is defined by follows;

Given any Orlicz function M, we define on $w^{(2)}$ the following two modulars;

$$I_M^{(2)}(x) = \sum_{n,m=1}^{\infty} M(|x(n,m)|) \ \text{ and } \ \rho_M^{(2)}(x) = I_M^{(2)}(\sigma^{(2)}x)$$

where

$$\sigma^{(2)}x = \left(\sigma^{(2)}x(n,m)\right), \ \sigma^{(2)}x(n,m) = \frac{1}{nm}\sum_{i,j=1}^{n,m}|x(i,j)|.$$

Let M be an Orlicz function. The double Cesàro-Orlicz sequence space $Ces_M^{(2)}$ is defined by

$$Ces_M^{(2)} = \left\{x \in w^{(2)} : \rho_M^{(2)}(\lambda x) < \infty \text{ for some } \lambda > 0\right\}$$

where $\rho_M^{(2)}$ is convex modular defined as above. This double sequence space is a normed space equipped with Luxemburg norm

$$\|x\|_M = \inf\left\{\lambda > 0 : \rho_M^{(2)}\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

2 Conclusion

Theorem 1. *The following conditions are equivalent:*

(1) $Ces_M^{(2)} \neq \{0\}$,

(2) $\exists n_1, m_1 \ \sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{1}{n.m}\right) < \infty$,

(3) $\forall k > 0, \ \exists n_k, \ m_k \ \sum_{n=n_k}^{\infty}\sum_{m=m_k}^{\infty} M\left(\frac{k}{n.m}\right) < \infty$.

Proof. $(1) \Rightarrow (2)$. Let $Ces_M^{(2)} \neq \{0\}$. Then there is $z \in Ces_M^{(2)}$ such that $z \neq 0$. Since $z \neq 0$, there exists $l_1,\ l_2 \in \mathbb{N}$ such that $z(l_1, l_2) \neq 0$. Therefore $y = (y(i,j)) \in Ces_M^{(2)}$, where

$$y(i,j) = \begin{cases} z(i,j), & if\ i = l_1, j = l_2 \\ 0, & others \end{cases}$$

and consequently $x = (x(i,j)) \in Ces_M^{(2)}$, where

$$x(i,j) = \begin{cases} 1, & if\ i = l_1, j = l_2 \\ 0, & others \end{cases} .$$

Hence there exists $k > 0$ such that

$$\rho_M^{(2)}(kx) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M\left(\frac{k}{n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\right) = \sum_{n=l_1}^{\infty}\sum_{m=l_2}^{\infty} M\left(\frac{k}{n.m}\right) < \infty.$$

We consider two different cases for $k > 0$.

(i) $k > 1$. Then we have $\frac{1}{n.m} < \frac{k}{n.m}$ for all $n, m \in \mathbb{N}$. Since the Orlicz function M is nondecreasing, we get $M\left(\frac{1}{n.m}\right) < M\left(\frac{k}{n.m}\right)$ for all $n, m \in \mathbb{N}$. Hence

$$\sum_{n=l_1}^{\infty}\sum_{m=l_2}^{\infty} M\left(\frac{1}{n.m}\right) < \sum_{n=l_1}^{\infty}\sum_{m=l_2}^{\infty} M\left(\frac{k}{n.m}\right) < \infty.$$

So if we take $n_1 = l_1$ and $m_1 = l_2$, the condition (2) is satisfied for $k > 1$.

(ii) $0 < k < 1$. Then there exists $s \in \mathbb{N}$ such that $\frac{1}{s^2} \leq k$ and so we have $\frac{1}{s^2.n.m} \leq \frac{k}{n.m}$ for all $n, m \in \mathbb{N}$. Since the Orlicz function M is nondecreasing, we get

$$\sum_{n=l_1}^{\infty}\sum_{m=l_2}^{\infty} M\left(\frac{1}{s^2.n.m}\right) \leq \sum_{n=l_1}^{\infty}\sum_{m=l_2}^{\infty} M\left(\frac{k}{n.m}\right).$$

Thus,

$$\begin{aligned}
\sum_{n=sl_1}^{\infty}\sum_{m=sl_2}^{\infty} M\left(\frac{1}{n.m}\right) &= \sum_{n=sl_1}^{\infty}\left\{M\left(\frac{1}{n.(sl_2)}\right) + M\left(\frac{1}{n.(sl_2+1)}\right) + M\left(\frac{1}{n.(sl_2+2)}\right) + \cdots + M\left(\frac{1}{n.(sl_2+(s-1))}\right)\right.\\
&\quad \left. + M\left(\frac{1}{n.(s(l_2+1))}\right) + M\left(\frac{1}{n.(s(l_2+1)+1)}\right) + \ldots + M\left(\frac{1}{n.(s(l_2+1)+(s-1))}\right) + \ldots\right\}\\
&\leq \sum_{n=sl_1}^{\infty}\left\{M\left(\frac{1}{n.(sl_2)}\right) + M\left(\frac{1}{n.(sl_2)}\right) + M\left(\frac{1}{n.(sl_2)}\right) + \ldots + M\left(\frac{1}{n.(sl_2)}\right)\right.\\
&\quad \left. + M\left(\frac{1}{n.(s(l_2+1))}\right) + M\left(\frac{1}{n.(s(l_2+1))}\right) + \ldots + M\left(\frac{1}{n.(s(l_2+1))}\right) + \ldots\right\}\\
&= \sum_{n=sl_1}^{\infty}\left\{s.M\left(\frac{1}{n.(sl_2)}\right) + s.M\left(\frac{1}{n.(s(l_2+1))}\right) + \ldots\right\}\\
&= \sum_{n=sl_1}^{\infty}\left\{s.\sum_{m=l_2}^{\infty} M\left(\frac{1}{n.(s.m)}\right)\right\} = s.\sum_{m=l_2}^{\infty}\left\{\sum_{n=sl_1}^{\infty} M\left(\frac{1}{m.(s.n)}\right)\right\}\\
&\leq s.\sum_{m=l_2}^{\infty}\left\{s.\sum_{n=l_1}^{\infty} M\left(\frac{1}{m.(s^2.n)}\right)\right\} = s^2.\sum_{m=l_2}^{\infty}\sum_{n=l_1}^{\infty} M\left(\frac{1}{s^2.m.n}\right)\\
&< \infty.
\end{aligned}$$

If we take $n_1 = s.l_1$ and $m_1 = s.l_2$, the proof is completed.

$(2) \Rightarrow (3)$. Let the condition (2) be satisfied. Then there exist $n_1, m_1 \in \mathbb{N}$ such that $\sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{1}{n.m}\right) < \infty$. We consider two different cases for $k > 0$.

(i) $0 < k < 1$. Then we have $\frac{k}{n.m} < \frac{1}{n.m}$ and

$$\sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{k}{n.m}\right) < \sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{1}{n.m}\right) < \infty.$$

Taking $n_k := n_1$ and $m_k := m_1$ we get $\sum_{n=n_k}^{\infty}\sum_{m=m_k}^{\infty} M\left(\frac{k}{n.m}\right) < \infty$.

(ii) $k > 1$. Then there exists $s \in \mathbb{N}$ such that $k \leq s$. Let define $n_k := n_1.s$ and $m_k := m_1.s$. Hence we have

$$\begin{aligned}
\sum_{n=n_k}^{\infty}\sum_{m=m_k}^{\infty} M\left(\frac{k}{n.m}\right) &\leq \sum_{n=n_k}^{\infty}\sum_{m=m_k}^{\infty} M\left(\frac{s}{n.m}\right) = \sum_{n=n_1.s}^{\infty}\left\{\sum_{m=m_1.s}^{\infty} M\left(\frac{s}{n.m}\right)\right\} \\
&= \sum_{n=n_1.s}^{\infty}\left\{M\left(\frac{s}{n.(s.m_1)}\right) + M\left(\frac{s}{n.(s.m_1+1)}\right) + \ldots + M\left(\frac{s}{n.(s.m_1+(s-1))}\right)\right. \\
&\quad + M\left(\frac{s}{n.(s.(m_1+1))}\right) + M\left(\frac{s}{n.(s.(m_1+1)+1)}\right) + \ldots \\
&\quad \left. + M\left(\frac{s}{n.(s.(m_1+1)+(s-1))}\right) + \ldots\right\} \\
&\leq \sum_{n=n_1.s}^{\infty}\left\{s.M\left(\frac{1}{n.m_1)}\right) + s.M\left(\frac{1}{n.(m_1+1)}\right) + \ldots\right\} \\
&= \sum_{n=n_1.s}^{\infty}\left\{s.\sum_{m=m_1}^{\infty} M\left(\frac{1}{n.m)}\right)\right\} = s.\sum_{m=m_1}^{\infty}\left\{\sum_{n=n_1.s}^{\infty} M\left(\frac{1}{n.m)}\right)\right\} \\
&\leq s^2.\sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{1}{n.m}\right) < \infty.
\end{aligned}$$

$(3) \Rightarrow (1)$. Let the condition (3) holds. By assumption, there exist $n_1, m_1 \in \mathbb{N}$ such that $\sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{1}{n.m}\right) < \infty$ for $k = 1$. Define $x = (x(i,j))$ such that

$$x(i,j) = \begin{cases} 1, \; if \; i = n_1, j = m_1 \\ 0, \quad others \end{cases}.$$

Clearly, $x \in w^{(2)}$ and

$$\rho_M^{(2)}(kx) = \rho_M^{(2)}(x) = \sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{1}{n.m}\right) < \infty.$$

Hence $x \in Ces_M^{(2)}$, which implies $Ces_M^{(2)} \neq \{0\}$.

Theorem 2. *Let M be Orlicz function. For the conditions:*

(1) $\liminf_{t\to 0} \frac{t.M'(t)}{M(t)} > 1$,
(2) $\exists \varepsilon > 0, \exists A > 0, \exists u_0, \forall 0 \leq u \leq u_0 \;\; M(u) \leq A.u^{1+\varepsilon}$,
(3) $\exists n_1, m_1 \; \sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{1}{n.m}\right) < \infty$.

we have the implications $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. $(1) \Rightarrow (2)$. (see [2]).

$(2) \Rightarrow (3)$. Let the condition (2) holds. Since $\frac{1}{n.m} \to 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{n.m} \leq u_0$ for all $n,m \geq N$. Hence, we get

$$\begin{aligned}\sum_{n=N}^{\infty}\sum_{m=N}^{\infty} M\left(\frac{1}{n.m}\right) &\leq \sum_{n=N}^{\infty}\sum_{m=N}^{\infty} A.\left(\frac{1}{n.m}\right)^{1+\varepsilon} \\ &\leq A.\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{n.m}\right)^{1+\varepsilon} \\ &< \infty.\end{aligned}$$

This completes the proof.

Theorem 3. *Let M_1 and M_2 be Orlicz functions. If there exist $b > 0$, $t_0 > 0$ such that $M_2(t_0) > 0$ and $M_2(t) \leq M_1(b.t)$ for all $t \in [0,t_0]$ then $Ces_{M_1}^{(2)} \subset Ces_{M_2}^{(2)}$.*

Proof. We may assume that $b \geq 1$ and defining $u = b.t$ we get

$$M_2\left(\frac{u}{b}\right) \leq M_1(u) \tag{1}$$

for all $u \in [0, b.t_0]$. Let $x \in Ces_{M_1}^{(2)}$. Then there exists $\lambda > 0$ such that $\rho_{M_1}^{(2)}(\lambda.x) < \infty$. Define

$$A_x = \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{\lambda}{n.m}\sum_{i=1}^{n}\sum_{j=1}^{m}|x(i,j)| > b.t_0\right\}$$

. The set A_x is finite, because otherwise we have

$$\begin{aligned}\rho_{M_1}^{(2)}(\lambda.x) &\geq \sum_{(n,m)\in A_x} M_1\left(\frac{\lambda}{n.m}\sum_{i=1}^{n}\sum_{j=1}^{m}|x(i,j)|\right) \\ &> \sum_{(n,m)\in A_x} M_1(b.t_0) > \sum_{(n,m)\in A_x} M_2(t_0) = \infty\end{aligned}$$

by (1) and this gives a conradiction. Let take $\lambda^* = \frac{c}{b}$ for c enough, we get

$$\rho_{M_2}^{(2)}(\lambda^*.x) \leq \rho_{M_1}^{(2)}(c.x) \leq \rho_{M_1}^{(2)}(\lambda.x) < \infty$$

which implies $x \in Ces_{M_2}^{(2)}$.

Theorem 4. *If $x \in w^{(2)}$, $\{x_n\} \subset Ces_M^{(2)}$, $\sup_n \|x_n\| < \infty$ and $0 \leq x_n \uparrow x$ coordinatewise, then $x \in Ces_M^{(2)}$ and $\|x_n\| \to \|x\|$.*

Proof. Assume that $x_n \in Ces_M^{(2)}$, $\sup_n \|x_n\| < \infty$ for all $n \in \mathbb{N}$ and $0 \leq x_n(i,j) \uparrow x(i,j)$ for each $i,j \in \mathbb{N}$. Denote $A = \sup_n \|x_n\|$. It is known that $\|x_n\| \leq A < \infty$ for all $n \in \mathbb{N}$ and so we have $0 \leq \frac{x_n}{A} \leq \frac{x_n}{\|x_n\|}$. Hence $\rho_M^{(2)}\left(\frac{x_n}{A}\right) \leq 1$ and since the modular $\rho_M^{(2)}$ is monotone, we get $\rho_M^{(2)}\left(\frac{x_n}{A}\right) \leq \rho_M^{(2)}\left(\frac{x_n}{\|x_n\|}\right) \leq 1$.

Since $x_n(i,j) \uparrow x(i,j)$ for each $i,j \in \mathbb{N}$, we have $\frac{x_n(i,j)}{A} \to \frac{x(i,j)}{A}$ for each $i,j \in \mathbb{N}$. By the Beppo Levi Theorem we get

$$\rho_M^{(2)}\left(\frac{x}{A}\right) = \lim_{n\to\infty}\rho_M^{(2)}\left(\frac{x_n}{A}\right) = \sup_n \rho_M^{(2)}\left(\frac{x_n}{A}\right) \leq 1$$

which means that $x \in Ces_M^{(2)}$ and $\|x\| \leq A$. Since $x_n \uparrow x$ coordinatewise and monotonicity of the norm, we get $\sup_n \|x_n\| \leq \|x\|$ and so $\|x\| = \sup_n \|x_n\| = \lim_{n\to\infty}\|x_n\|$.

By the above theorem, we have that the space $Ces_M^{(2)}$ has Fatou property. Consequently, the double Cesàro-Orlicz sequence space $Ces_M^{(2)}$ is a Banach space.

Theorem 5. *Define*

$$A_M^{(2)} = \left\{ x \in Ces_M^{(2)} : \forall k > 0, \exists n_k, m_k \sum_{n=n_k}^{\infty} \sum_{m=m_k}^{\infty} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)| \right) < \infty \right\}.$$

Then following assertions are true:

(1) $A_M^{(2)}$ *is a closed separable subspace of* $Ces_M^{(2)}$*,*
(2) $A_M^{(2)} = cl\left\{ x \in Ces_M^{(2)} : x(i,j) \neq 0 \text{ for only finite number of } i,j \in \mathbb{N} \right\}$*,*
(3) $A_M^{(2)}$ *is the subspace of all order continuous elements of* $Ces_M^{(2)}$*.*

Proof. (1) It is easy to see that $A_M^{(2)}$ is a subspace of $Ces_M^{(2)}$. We will show that $A_M^{(2)}$ is a closed subspace of $Ces_M^{(2)}$. Let take $\{x_s\} \subset A_M^{(2)}$ such that $x_s \to x$, $x \in Ces_M^{(2)}$. We must show that $x \in A_M^{(2)}$. Take any $k > 0$. Since $\rho_M^{(2)}(k(x - x_s)) \to 0$ for all $k > 0$, there exists $s \in \mathbb{N}$ such that $\rho_M^{(2)}(2k(x - x_s)) < 1$. Since $x_s \in A_M^{(2)}$ for all $s \in \mathbb{N}$, there exist $n_s, m_s \in \mathbb{N}$ such that

$$\sum_{n=n_s}^{\infty} \sum_{m=m_s}^{\infty} M\left(\frac{2k}{n.m} \sum_{i,j=1}^{n,m} |x_s(i,j)| \right) < \infty.$$

We can take $n_k := n_s$, $m_k := m_s$. Since Orlicz function M is convex, we have

$$\begin{aligned}
\sum_{n=n_s}^{\infty} \sum_{m=m_s}^{\infty} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)| \right) &= \sum_{n=n_s}^{\infty} \sum_{m=m_s}^{\infty} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} \left| \frac{2\left(x(i,j) - x_s(i,j)\right)}{2} + \frac{2x_s(i,j)}{2} \right| \right) \\
&= \sum_{n=n_s}^{\infty} \sum_{m=m_s}^{\infty} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} \left| \frac{2\left(x(i,j) - x_s(i,j)\right)}{2} + \frac{2x_s(i,j)}{2} \right| \right) \\
&\leq \sum_{n=n_s}^{\infty} \sum_{m=m_s}^{\infty} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} \left(\left| \frac{2\left(x(i,j) - x_s(i,j)\right)}{2} \right| + \left| \frac{2x_s(i,j)}{2} \right| \right) \right) \\
&\leq \sum_{n=n_s}^{\infty} \sum_{m=m_s}^{\infty} \left\{ \frac{1}{2} M\left(\frac{2k}{n.m} \sum_{i,j=1}^{n,m} |x(i,j) - x_s(i,j)| \right) + \frac{1}{2} M\left(\frac{2k}{n.m} \sum_{i,j=1}^{n,m} |x_s(i,j)| \right) \right\} \\
&\leq \frac{1}{2} \rho_M^{(2)}(2k(x - x_s)) + \frac{1}{2} \sum_{n=n_s}^{\infty} \sum_{m=m_s}^{\infty} M\left(\frac{2k}{n.m} \sum_{i,j=1}^{n,m} |x_s(i,j)| \right) \\
&< \infty.
\end{aligned}$$

Since $k > 0$ is arbitrary, we get $x \in A_M^{(2)}$. This shows that $A_M^{(2)}$ is a closed subspace of $Ces_M^{(2)}$.

(2) Let us define $B_M^{(2)} = \left\{ x \in Ces_M^{(2)} : x(i,j) = 0 \text{ for a.e. } i,j \in \mathbb{N} \right\}$. We will prove that $A_M^{(2)}$ is equal to $clB_M^{(2)}$. If $B_M^{(2)} = \varnothing$, then $clB_M^{(2)} \subset A_M^{(2)}$. Let $B_M^{(2)} \neq \varnothing$. Then there exists $x = (x(i,j)) \in B_M^{(2)}$ such that

$$x(i,j) = \begin{cases} 1, \text{ if } & (i,j) = (l_1, l_2) \\ 0, & \text{others} \end{cases}.$$

Take $k > 0$. By the Theorem 1, there exist $\exists n_k, m_k \in \mathbb{N}$ such that

$$\sum_{n=n_k}^{\infty} \sum_{m=m_k}^{\infty} M\left(\frac{k}{n.m} \right) < \infty.$$

We assume that $n_k \geq l_1$ and $m_k \geq l_2$. By the fact that $A_M^{(2)}$ is a linear subspace of $Ces_M^{(2)}$, we get $x \in A_M^{(2)}$ and so $clB_M^{(2)} \subset A_M^{(2)}$.

For the inclusion $A_M^{(2)} \subset clB_M^{(2)}$, let us take $x = (x(i,j)) \in A_M^{(2)}$ and define $x^{k,l} = \left(x^{k,l}(i,j)\right)$ such that

$$x^{k,l}(i,j) = \begin{cases} x(i,j), \text{ if } & i \leq k \text{ and } j \leq l \\ 0, & \text{others} \end{cases}.$$

for any $k,l \in \mathbb{N}$. It is obvious that $x^{k,l} \in B_M^{(2)}$. Take any $\lambda > 0$ and $\varepsilon > 0$. Since $x = (x(i,j)) \in A_M^{(2)}$, there exist $k_0, l_0 \in \mathbb{N}$ such that

$$\begin{aligned} R_{k_0,l_0}(x) = & \sum_{n=1}^{k_0} \sum_{m=l_0+1}^{\infty} M\left(\frac{\lambda}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)|\right) + \sum_{n=k_0+1}^{\infty} \sum_{m=1}^{l_0} M\left(\frac{\lambda}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)|\right) + \sum_{n=k_0+1}^{\infty} \sum_{m=l_0+1}^{\infty} M\left(\frac{\lambda}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)|\right) \\ & < \varepsilon. \end{aligned}$$

Then for any $k \geq k_0, l \geq l_0$, we get

$$\begin{aligned} \rho_M^{(2)}(\lambda(x - x^{k,l})) \leq \rho_M^{(2)}(\lambda(x - x^{k_0,l_0})) \\ & \leq \sum_{n=1}^{k_0} \sum_{m=l_0+1}^{\infty} M\left(\frac{\lambda}{n.m} \sum_{i=1}^{n} \sum_{j=l_0+1}^{m} |x(i,j)|\right) + \sum_{n=k_0+1}^{\infty} \sum_{m=1}^{l_0} M\left(\frac{\lambda}{n.m} \sum_{i=k_0+1}^{n} \sum_{j=1}^{m} |x(i,j)|\right) \\ & + \sum_{n=k_0+1}^{\infty} \sum_{m=l_0+1}^{\infty} M\left(\frac{\lambda}{n.m} \sum_{i=k_0+1}^{n} \sum_{j=l_0+1}^{m} |x(i,j)|\right) \\ & \leq \sum_{n=1}^{k_0} \sum_{m=l_0+1}^{\infty} M\left(\frac{\lambda}{n.m} \sum_{i=1}^{n} \sum_{j=1}^{m} |x(i,j)|\right) + \sum_{n=k_0+1}^{\infty} \sum_{m=1}^{l_0} M\left(\frac{\lambda}{n.m} \sum_{i=1}^{n} \sum_{j=1}^{m} |x(i,j)|\right) \\ & + \sum_{n=k_0+1}^{\infty} \sum_{m=l_0+1}^{\infty} M\left(\frac{\lambda}{n.m} \sum_{i=1}^{n} \sum_{j=1}^{m} |x(i,j)|\right) = R_{k_0,l_0}(x) < \varepsilon. \end{aligned}$$

This implies $x^{k,l} \to x$. Then $x \in clB_M^{(2)}$ and so $A_M^{(2)} \subset clB_M^{(2)}$.

(3) Let $x \in A_M^{(2)}$. We will show that x is order continuous. Take any $k > 0$ and $s > 0$. Then there exist $n_k, m_k \in \mathbb{N}$ such that

$$\begin{aligned} R_{n_k,m_k}(x) = & \sum_{n=1}^{n_k} \sum_{m=m_k+1}^{\infty} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)|\right) + \sum_{n=n_k+1}^{\infty} \sum_{m=1}^{m_k} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)|\right) + \sum_{n=n_k+1}^{\infty} \sum_{m=m_k+1}^{\infty} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)|\right) \\ & < \frac{\varepsilon}{2}. \end{aligned}$$

Assume that $x_s \downarrow 0$ coordinatewise and $x_s \leq |x|$ for all $s \in \mathbb{N}$. Let us denote

$$\eta(n,m) = M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} |x(i,j)|\right)$$

and

$$\eta_s(n,m) = M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} |x_s(i,j)|\right).$$

Since $x_s \downarrow 0$ coordinatewise, we have $\eta_s(n,m) \to 0$ as $s \to \infty$ for all $n,m \in \mathbb{N}$. Thus, there exists $s_\varepsilon \in \mathbb{N}$ such that $\sum_{n=1}^{n_k-1} \sum_{m=1}^{m_k-1} \eta_s(n,m) < \frac{\varepsilon}{2}$ for all $s \geq s_\varepsilon$. Moreover,

$$\begin{aligned}\sum_{n=1}^{n_k} \sum_{m=m_k+1}^{\infty} \eta_s(n,m) + \sum_{n=n_k+1}^{\infty} \sum_{m=1}^{m_k} \eta_s(n,m) + \sum_{n=n_k+1}^{\infty} \sum_{m=m_k+1}^{\infty} \eta_s(n,m) &< \sum_{n=1}^{n_k} \sum_{m=m_k+1}^{\infty} \eta(n,m) + \sum_{n=n_k+1}^{\infty} \sum_{m=1}^{m_k} \eta(n,m) \\ &+ \sum_{n=n_k+1}^{\infty} \sum_{m=m_k+1}^{\infty} \eta(n,m) \\ &= R_{n_k,m_k}(x) < \frac{\varepsilon}{2}\end{aligned}$$

for all $n \geq n_k, m \geq m_k$ and $s \in \mathbb{N}$. Consequently, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} M\left(\frac{k}{n.m} \sum_{i,j=1}^{n,m} |x_s(i,j)|\right) < \varepsilon$$

for all $s \geq s_\varepsilon$, which implies $\rho_M^{(2)}(kx_s) \to 0$ as $s \to \infty$. Since k is arbitrary, we get $\|x_s\| \to 0$.

Let $x \in Ces_M^{(2)}$ be an order continuous element. Since $\|x^*\| \to 0$, where

$$x^*(i,j) = \begin{cases} x(i,j), \ if \ \ i \geq n+1 \ and \ j \geq m+1 \\ 0, \qquad\qquad others \end{cases}$$

as $n,m \to \infty$, so it is easy to see that $x \in cl\left\{x \in Ces_M^{(2)} : x(i,j) = 0 \text{ for a.e. } i,j \in \mathbb{N}\right\}$.

Finally, we show that $A_M^{(2)}$ is separable. Define the set

$$C_M^{(2)} = \left\{x \in Ces_M^{(2)} : x(i,j) = 0 \text{ for a.e. } i,j \in \mathbb{N} \text{ and } x(i,j) \in \mathbb{Q}\right\}.$$

Then, the set $C_M^{(2)}$ is countable and it is obvious that $clC_M^{(2)} \subset clB_M^{(2)}$. For the converse inclusion, take $x = (x(i,j)) \in clB_M^{(2)}$, where

$$x(i,j) = \begin{cases} x(i,j), \ \ if \ \ i \leq k \ and \ j \leq l \\ 0, \qquad\qquad others \end{cases}$$

and $x_s = (x_s(i,j)) \in C_M^{(2)}$, where

$$x_s(i,j) = \begin{cases} x_s(i,j), \ \ if \ \ i \leq k \ and \ j \leq l \\ 0, \qquad\qquad others \end{cases}$$

such that $x_s(i,j) \to x(i,j)$ as $s \to \infty$. We will show that $\|x_s - x\| \to 0$. Take any $\lambda > 0$. We have

$$\lambda \sum_{i,j=1}^{k,l} |x_s(i,j) - x(i,j)| \leq 1$$

for s large enough. Thus, by convexity of M,

$$\begin{aligned}\rho_M^{(2)}(\lambda(x_s-x)) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M\left(\frac{\lambda}{n.m}\sum_{i,j=1}^{n,m}|x_s(i,j)-x(i,j)|\right)\\ &\leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M\left(\frac{\lambda}{n.m}\sum_{i,j=1}^{k,l}|x_s(i,j)-x(i,j)|\right)\\ &\leq \lambda.\sum_{i,j=1}^{k,l}|x_s(i,j)-x(i,j)|.\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M\left(\frac{1}{n.m}\right).\end{aligned}$$

Since $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M\left(\frac{1}{n.m}\right) < \infty$, we get $\rho_M^{(2)}(\lambda(x_s-x)) \to 0$ as $s\to\infty$. By the arbitrariness of λ, we have $\|x_s-x\|\to 0$ as $s\to\infty$. This implies that $x\in clC_M^{(2)}$. Consequently, $clC_M^{(2)} = clB_M^{(2)}$. Since $A_M^{(2)} = clB_M^{(2)} = clC_M^{(2)}$ and the space $C_M^{(2)}$ is countable, we get $A_M^{(2)}$ is separable space.

Theorem 6. *If $M\in\triangle_2(0)$, then $A_M^{(2)} = Ces_M^{(2)}$.*

Proof. Let $x\in Ces_M^{(2)}$. Thus, there exists $\alpha>0$ such that $\rho_M^{(2)}(\alpha x)<\infty$. We will show that for any $\lambda>0$ there exist $n_\lambda, m_\lambda\in\mathbb{N}$ such that $\sum_{n=n_\lambda}^{\infty}\sum_{m=m_\lambda}^{\infty} M\left(\frac{\lambda}{n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\right)<\infty$.

If $\lambda<\alpha$, by the monotonicity of M

$$\sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{\lambda}{n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\right) < \sum_{n=n_1}^{\infty}\sum_{m=m_1}^{\infty} M\left(\frac{\alpha}{n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\right)<\infty.$$

Let $\lambda>\alpha$. Since $M\in\triangle_2(0)$, we have $M\in\triangle_l(0)$ for any $l>1$, whence for $l:=\frac{\lambda}{\alpha}$ there exist $k>0, u_0>0$ such that $M(l.u)\leq k.M(u)$ for all $u\leq u_0$. By $\rho_M^{(2)}(\alpha x)<\infty$, there exists s_λ such that $\frac{\alpha}{n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\leq u_0$ for all $n,m\geq s_\lambda$. Then, we get

$$\begin{aligned}\sum_{n=s_\lambda}^{\infty}\sum_{m=s_\lambda}^{\infty} M\left(\frac{\lambda}{n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\right) &= \sum_{n=s_\lambda}^{\infty}\sum_{m=s_\lambda}^{\infty} M\left(\frac{\alpha.\lambda}{\alpha.n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\right)\\ &= \sum_{n=s_\lambda}^{\infty}\sum_{m=s_\lambda}^{\infty} M\left(l.\frac{\alpha}{n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\right)\\ &\leq k.\sum_{n=s_\lambda}^{\infty}\sum_{m=s_\lambda}^{\infty} M\left(\frac{\alpha}{n.m}\sum_{i,j=1}^{n,m}|x(i,j)|\right)\\ &<\infty.\end{aligned}$$

This implies that $x\in A_M^{(2)}$. Hence we get $A_M^{(2)} = Ces_M^{(2)}$.

Corollary 1.*If $M\in\triangle_2(0)$, then*

(i) *the double Cesàro-Orlicz sequence space $Ces_M^{(2)}$ is separable,*
(ii) *the double Cesàro-Orlicz sequence space $Ces_M^{(2)}$ is order continuous.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] Başar F., Summability Theory and Its Applications, Bentham Science Publishers, e-books, Monographs, İstanbul, 2012.
[2] Cui Y., Hudzik H., Petrot N., Suantai S., Szkymaszkiewicz A., Basic Topological and Geometric Properties of Cesàro-Orlicz Spaces, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 4, 2005.
[3] Kubiak D., A Note on Cesàro-Orlicz Sequence Spaces, J. Math. Anal. Appl. 349 (2009) 291-296.
[4] Krasnoselskii M. A., Rutickii Ya. B., Convex Functions and Orlicz Spaces, (Gooningen: P. Nordhoff Ltd.), 1961.
[5] Lee P. Y., Cesàro Sequence Spaces, Math. Chronicle, New Zeland 13 (1984) 29-45.
[6] Limaye B. V., Zelstser M., On the Pringsheim convergence of double series, Proc. Eston. Aca. Sci. 58,2 (2009), 108-121.
[7] Lim S. K., Lee P. Y., An Orlicz extension of Cesàro Sequence Spaces, Comment. Math. Prace Mat. 28 (1988) 117-128.
[8] Lindenstrauss I. J., Tzafiri L., Classical Banach Spaces I, Lecture Notes in Math.(New York, Tokyo: Springer-Verlag, Berlin, Heidelberg), !983.
[9] Lindenstrauss I. J., L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., 10(1971), 379-390.
[10] Luxemburg W. A., Banach Function Spaces, Thesis (Delft) 1955.
[11] Musielak J., Orlicz Spaces and Modular Spaces, Lecture Notes in Math. 1034 (Springer-Verlag) (1983).
[12] Pringsheim A., Zur Theorie de Zweifach Unendlichen Zahlenfolgen, Math. Ann., 53 (1900), 289-321.
[13] Saejung S., Another Look at Cesàro Sequence Spaces, J. Math. Anal. Appl. 366 (2010) 530-537.
[14] Shue J. S., On the Cesàro Sequence Spaces, Tamkang J. Math. 1 (1970), 143-150.