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The right Rieaman-Liouville fractional Hermite-Hadamard type inequalities for quasi-convex functions

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Abstract: Recently, in [5], with a new approach, the authors obtained a new fractional Hermite-Hadamard type inequality for convex functions by using only the right Riemann-Liouville fractional integral. They also had new equalities to have new fractional trapezoid and midpoint type inequalities for convex functions, In this papers, we will use the same equalities to have new fractional trapezoid and midpoint type inequalities for quasi-convex functions. Our results generalise the study [3].

Keywords: Quasi-convex functions, Hermite-Hadamard inequality, Right Rieaman-Liouville fractional integral, Trapezoid type inequalities, Midpoint type inequalities.

1 Introduction

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b \in I$ with $a<b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality [1,2].

Definition 1. *A function $f:[a,b] \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if*

$$f(tx+(1-t)y) \leq \sup\{f(x), f(y)\}.$$

for each $x,y \in [a,b]$, $t \in [0,1]$

In [3], Ion used the following equality to obtain trapezoid type inequalities for quasi-convex functions

Lemma 1. *Let $a,b \in I$ with $a<b$ and $f: I^\circ \to \mathbb{R}$ is a differentiable mapping (I° is the interior of I). If $f' \in L[a,b]$, then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du = \frac{b-a}{2}\int_0^1 (1-2t)\, f'(ta+(1-t)b)\,dt. \tag{2}$$

Definition 2. *Let $a,b \in \mathbb{R}$ with $a<b$ and $f \in L[a,b]$. The left and right Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x>a \ \text{ and } \ J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function (see [6, page 69] and [7, page 4]).

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In [5], Kunt et al. proved the following Hermite-Hadamard type fractional integral inequality:

Theorem 1. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f\in L[a,b]$, then the following inequality for the right Riemann-Liouville fractional integral holds:*

$$f\left(\frac{a+\alpha b}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\leq\frac{f(a)+\alpha f(b)}{\alpha+1} \tag{3}$$

with $\alpha>0$.

Proof. See [5, Theorem 3]

Lemma 2. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L[a,b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:*

$$\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)=\frac{b-a}{\alpha+1}\int_0^1\left[(\alpha+1)(1-t)^{\alpha}-1\right]f'(ta+(1-t)b)\,dt \tag{4}$$

with $\alpha>0$.

Proof. See [5, Lemma 3]

Lemma 3. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L[a,b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)-f\left(\frac{a+\alpha b}{\alpha+1}\right)=(b-a)\left[\int_0^{\frac{\alpha}{\alpha+1}}-t^{\alpha}f'(tb+(1-t)a)\,dt+\int_{\frac{\alpha}{\alpha+1}}^1(1-t^{\alpha})f'(tb+(1-t)a)\,dt\right] \tag{5}$$

with $\alpha>0$.

Proof. See [5, Lemma 4]

2 The right fractional trapezoid and midpoint type inequalities for quasi-convex function

In this section we will obtain some new right Riemann-Liouville fractional trapezoid and midpoint type inequalities for quasi-convex function by using Lemma 2 and Lemma 3.

Theorem 2. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|$ is quasi-convex on $[a,b]$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right|\leq\frac{b-a}{\alpha+1}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\frac{2\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}} \tag{6}$$

with $\alpha>0$.

Proof. Using Lemma 2 and the quasi-convexity of $|f'|$, we have

$$\begin{aligned}\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right| &\leq \frac{b-a}{\alpha+1}\int_0^1\left|(\alpha+1)(1-t)^{\alpha}-1\right|\left|f'(ta+(1-t)b)\right|dt\\ &=\frac{b-a}{\alpha+1}\int_0^1\left|(\alpha+1)t^{\alpha}-1\right|\left|f'(tb+(1-t)a)\right|dt\\ &\leq \frac{b-a}{\alpha+1}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\left[\begin{array}{c}\int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}(1-(\alpha+1)t^{\alpha})dt\\ +\int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1((\alpha+1)t^{\alpha}-1)dt\end{array}\right]\\ &\leq \frac{b-a}{\alpha+1}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\frac{2\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}.\end{aligned}$$

This completes the proof.

Remark. In Theorem 2, if one takes $\alpha=1$, one has the inequality proved in [3, Theorem 1].

Theorem 3. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is quasi-convex on $[a,b]$ for $q\geq 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right|\leq\frac{b-a}{\alpha+1}\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\frac{2\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}\tag{7}$$

with $\alpha>0$.

Proof. Using Lemma 2, power mean inequality and the quasi-convexity of $|f'|^q$, we have

$$\begin{aligned}\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right| &\leq \frac{b-a}{\alpha+1}\int_0^1\left|(\alpha+1)(1-t)^{\alpha}-1\right|\left|f'(ta+(1-t)b)\right|dt\\ &=\frac{b-a}{\alpha+1}\int_0^1\left|(\alpha+1)t^{\alpha}-1\right|\left|f'(tb+(1-t)a)\right|dt\\ &\leq \frac{b-a}{\alpha+1}\left[\begin{array}{c}\left(\int_0^1\left|(\alpha+1)t^{\alpha}-1\right|dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^1\left|(\alpha+1)t^{\alpha}-1\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]\\ &\leq \frac{b-a}{\alpha+1}\left[\begin{array}{c}\left(\int_0^1\left|1-(\alpha+1)t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^1\left|1-(\alpha+1)t^{\alpha}\right|\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}}\end{array}\right]\\ &\leq \frac{b-a}{\alpha+1}\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\int_0^1\left|1-(\alpha+1)t^{\alpha}\right|dt\\ &\leq \frac{b-a}{\alpha+1}\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\frac{2\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}.\end{aligned}$$

This completes the proof.

Corollary 1. *In Theorem 3, if one takes $\alpha=1$, one has the following trapezoid type inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{4}\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}.$$

Theorem 4. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is quasi-convex on $[a,b]$ for $q>1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right|\le\frac{b-a}{\alpha+1}\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(R_1(\alpha,p)+R_2(\alpha,p)\right)^{\frac{1}{p}} \tag{8}$$

where

$$R_1(\alpha,p)=\int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}\left(1-(\alpha+1)t^{\alpha}\right)^p dt \quad \text{and} \quad R_2(\alpha,p)=\int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^{1}\left((\alpha+1)t^{\alpha}-1\right)^p dt,$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.

Proof. Using Lemma 2, Hölder inequality and the quasi-convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right| &\le \frac{b-a}{\alpha+1}\int_0^1\left|(\alpha+1)(1-t)^{\alpha}-1\right|\left|f'(ta+(1-t)b)\right|dt\\
&=\frac{b-a}{\alpha+1}\int_0^1\left|(\alpha+1)t^{\alpha}-1\right|\left|f'(tb+(1-t)a)\right|dt\\
&\le\frac{b-a}{\alpha+1}\left[\left(\int_0^1\left|(\alpha+1)t^{\alpha}-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}}\right]\\
&\le\frac{b-a}{\alpha+1}\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left[\left(\begin{array}{c}\int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}\left(1-(\alpha+1)t^{\alpha}\right)^p dt\\+\int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1\left((\alpha+1)t^{\alpha}-1\right)^p dt\end{array}\right)^{\frac{1}{p}}\right]\\
&\le\frac{b-a}{\alpha+1}\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(R_1(\alpha,p)+R_2(\alpha,p)\right)^{\frac{1}{p}}.
\end{aligned}$$

This completes the proof.

Remark. In Theorem 4, if one takes $\alpha=1$, one has the inequality proved in [3, Theorem 2].

Theorem 5. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|$ is quasi-convex on $[a,b]$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)-f\left(\frac{a+\alpha b}{\alpha+1}\right)\right|\le(b-a)\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\frac{2\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}} \tag{9}$$

with $\alpha>0$.

Proof. Using Lemma 3 and the quasi-convexity of $|f'|$, we have

$$\begin{aligned}
\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)-f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| &\le(b-a)\left[\begin{array}{c}\int_0^{\frac{\alpha}{\alpha+1}}t^{\alpha}\left|f'(tb+(1-t)a)\right|dt\\+\int_{\frac{\alpha}{\alpha+1}}^1\left(1-t^{\alpha}\right)\left|f'(tb+(1-t)a)\right|dt\end{array}\right]\\
&\le(b-a)\left[\begin{array}{c}\int_0^{\frac{\alpha}{\alpha+1}}t^{\alpha}\left[\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right]dt\\+\int_{\frac{\alpha}{\alpha+1}}^1\left(1-t^{\alpha}\right)\left[\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right]dt\end{array}\right]\\
&\le(b-a)\left[\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right]\left[\int_0^{\frac{\alpha}{\alpha+1}}t^{\alpha}dt+\int_{\frac{\alpha}{\alpha+1}}^1\left(1-t^{\alpha}\right)dt\right]\\
&\le(b-a)\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\frac{2\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}.
\end{aligned}$$

This completes the proof.

Corollary 2. *In Theorem 5, if one takes $\alpha = 1$, one has the following midpoint type inequality.*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.$$

Theorem 6. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is quasi-convex on $[a,b]$ for $q\ge 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{b-}f(a) - f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| \le (b-a)\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\frac{2\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}. \tag{10}$$

with $\alpha>0$.

Proof. Using Lemma 3, power mean inequality and the quasi-convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{b-}f(a) - f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| &\le (b-a)\left[\begin{array}{c}\int_0^{\frac{\alpha}{\alpha+1}} t^\alpha\left|f'(tb+(1-t)a)\right|dt \\ +\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^\alpha)\left|f'(tb+(1-t)a)\right|dt\end{array}\right] \\
&\le (b-a)\left[\begin{array}{c}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^\alpha\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^\alpha)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^\alpha)\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right] \\
&\le (b-a)\left[\begin{array}{c}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^\alpha\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^\alpha)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^\alpha)\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}}\end{array}\right] \\
&\le (b-a)\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left[\int_0^{\frac{\alpha}{\alpha+1}} t^\alpha dt + \int_{\frac{\alpha}{\alpha+1}}^1 (1-t^\alpha)dt\right] \\
&\le (b-a)\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\frac{2\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}.
\end{aligned}$$

This completes the proof.

Corollary 3. *In Theorem 6, if one takes $\alpha=1$, one has the following midpoint type inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,du - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}.$$

Theorem 7. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is convex on $[a,b]$ for $q>1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{b-}f(a) - f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| \le (b-a)\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left[\begin{array}{c}R_3^{\frac{1}{p}}(\alpha,p)\left(\frac{\alpha}{\alpha+1}\right)^{\frac{1}{q}} \\ +R_4^{\frac{1}{p}}(\alpha,p)\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\end{array}\right], \tag{11}$$

where

$$R_3(\alpha,p)=\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt \quad and \quad R_4(\alpha,p)=\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^\alpha)^p\,dt,$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.

Proof. Using Lemma 3, Hölder inequality and the quasi-convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)-f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| &\leq (b-a)\left[\begin{array}{c}\int_0^{\frac{\alpha}{\alpha+1}}t^{\alpha}\left|f'(tb+(1-t)a)\right|dt\\ +\int_{\frac{\alpha}{\alpha+1}}^1(1-t^{\alpha})\left|f'(tb+(1-t)a)\right|dt\end{array}\right]\\
&\leq (b-a)\left[\begin{array}{c}\left(\int_0^{\frac{\alpha}{\alpha+1}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha}{\alpha+1}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha}{\alpha+1}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]\\
&\leq (b-a)\left[\begin{array}{c}\left(\int_0^{\frac{\alpha}{\alpha+1}}t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha}{\alpha+1}}\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right]dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha}{\alpha+1}}^1\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}^q\right]dt\right)^{\frac{1}{q}}\end{array}\right]\\
&\leq (b-a)\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left[\begin{array}{c}R_3^{\frac{1}{p}}(\alpha,p)\left(\frac{\alpha}{\alpha+1}\right)^{\frac{1}{q}}\\ +R_4^{\frac{1}{p}}(\alpha,p)\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

This completes the proof.

Remark. In Theorem 7, if one takes $\alpha=1$, one has the following midpoint type inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq (b-a)\left[\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right]^{\frac{1}{q}}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}.$$

3 Conclusions

In this paper, we proved an equality for differentiable functions. By using this equality, we have some new trapezoid and midpoint type inequalities for functions whose derivatives in absolute values at certain powers are quasi-convex.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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