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An efficient approach to numerical study of the coupled-bbm system with b-spline collocation method

Kamal Raslan[1], *Talaat S. EL-Danaf*[2], *Khalid Ali*[1]

[1]Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt
[2]Mathematics Department, Faculty of Science, Menoufia University, Shebein El-Koom, Egypt

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Abstract: In the present paper, a numerical method is proposed for the numerical solution of a coupled-BBM system with appropriate initial and boundary conditions by using collocation method with cubic trigonometric B-spline on the uniform mesh points. The method is shown to be unconditionally stable using von-Neumann technique. To test accuracy the error norms L_2, L_∞ are computed. Furthermore, interaction of two and three solitary waves are used to discuss the effect of the behavior of the solitary waves after the interaction. These results show that the technique introduced here is easy to apply. We make linearization for the nonlinear term.

Keywords: Collocation Method, cubic trigonometric B-splines method, coupled-BBM system.

1 Introduction

In this paper, we consider the Coupled-BBM system, which belongs to the class of Boussinesq systems, modeling two-way propagation of long waves of small amplitude on the surface of water in a channel. The system is a good candidate for modeling long waves of small to moderate amplitude. The Coupled-BBM system is given by Bona and Chen [1],

$$v_t + u_x + (vu)_x - \frac{1}{6}v_{xxt} = 0, \tag{1}$$

$$u_t + v_x + uu_x - \frac{1}{6}u_{xxt} = 0, \tag{2}$$

Where subscripts x and t denote differentiation x distance and t time, is considered, $v(x,t)$ is a dimensionless deviation of the water surface from its undisturbed position and $u(x,t)$ is the dimensionless horizontal velocity above the bottom of the channel. Boundary conditions

$$\begin{aligned} &u(a,t) = \alpha_1, u(b,t) = \alpha_2, \\ &v(a,t) = \beta_1, v(b,t) = \beta_2, \quad 0 \le t \le T. \\ &u_x(a,t) = 0, u_x(b,t) = 0, \\ &v_x(a,t) = 0, v_x(b,t) = 0, \quad 0 \le t \le T. \end{aligned} \tag{3}$$

And initial conditions.

$$u(x,0) = f(x), \quad v(x,0) = g(x), \quad a \le x \le b. \tag{4}$$

One of the advantages that (1) has over alternative Boussinesq-type systems is the easiness with which it may be integrated numerically [2]. Furthermore, it was proved in [2, 3] that the initial value problem either for $x \in \Re$ or with boundary

* Corresponding author e-mail: khalidkaram2012@yahoo.com.

conditions ($x \in [a,b]$) for (1) is well posed in certain natural function classes. The initial-boundary value problem of the form (1) posed on a bounded smooth plane domain with homogenous Dirichlet or Neumann or reflective (mixed) boundary conditions which is locally well-posed [4]. The existence and uniqueness of the system have been proved in Bona *et al.* [3]. They investigated the solution of the system as integral equation, while Chen in [5] established the existence of solitary waves for several Boussinesq types, including the Coupled-BBM system. Various numerical techniques including the finite element method have been used for the solution of Bona-Smith system of Boussinesq type in Antonopoulos *et al.* [6]. S. S. Behzadi and A. Yildirim, using Quintic B-Spline Collocation Method for Solving the Coupled-BBM System [7]. E. S. Al- Rawi and M. A. M. Sallal, using finite element method to fiend the Numerical solution of Coupled-BBM system [8]. Min Chen fined the exact traveling-wave solutions to bidirectional wave equations [9]. The numerical solutions of coupled nonlinear systems are very important in applied science, for example, the hirota-satsuma coupled KDV equation which admits soliton solution and it has many applications in communication and optical fibers; this system has been discussed numerically by Raslan *et al.* finite element methods [10]. Also, the Hirota equation has been solving by Raslan *et al.* using finite element methods [11]. A finite element algorithm based on the collocation method with trial functions taken as septic B-spline functions over the elements will be constructed. The cubic trigonometric B-spline basis together with finite element methods are shown to provide very accurate solutions in solving some partial differential equations and have been used before by several authors. In this article we are going to derive a numerical solution of the coupled BBM-system. The brief outline of this paper is as follows. In Section 2, cubic trigonometric B-spline collocation scheme is explained. In Sections 3 and 4, the method is described and applied to the coupled BBM-system. In Section 5, stability of the method is discussed. In Section 6, numerical examples are included to establish the applicability and accuracy of the proposed method computationally. Conclusion is given in Section 7 that briefly summarizes the numerical outcomes.

2 Cubic trigonometric B-spline collocation method

To construct numerical solution, consider nodal points (x_j,t_n) defined in the region $[a,b]\times[0,T]$ where

$$a = x_0 < x_1 < \ldots < x_N = b, h = x_{j+1} - x_j = \frac{b-a}{N}, \qquad j = 0,1,\ldots,N.$$

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < T, t_n = n\Delta t, n = 0,1,.....$$

The cubic trigonometric B-spline basis functions $CTB_j(x)$ at knots are given by.

$$CTB_j(x) = \frac{1}{\theta}\begin{cases} \omega^3(x_{j-2}), & x_{j-2} \le x \le x_{j-1} \\ \omega(x_{j-2})\left(\omega(x_{j-2})\phi(x_j) + \omega(x_{j-1})\phi(x_{j+1})\right) + \omega^2(x_{j-1})\phi(x_{j+2}), & x_{j-1} \le x \le x_j \\ \omega(x_{j-2})\phi^2(x_{j+1}) + \phi(x_{j+2})\left(\omega(x_{j-1})\phi(x_{j+1}) + \omega(x_j)\phi(x_{j+2})\right), & x_{j-1} \le x \le x_j \\ \phi^3(x_{j+2}), & x_{j+1} \le x \le x_{j+2} \\ 0 & otherwise \end{cases} \tag{5}$$

where $\omega(x_j) = \sin\left(\frac{x-x_j}{2}\right), \phi(x_j) = \sin\left(\frac{x_j-x}{2}\right), \theta = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right).$

Using cubic trigonometric B-spline basis function (5) the values of $CTB_j(x)$ and its derivatives at the knots points can be calculated, which are tabulated in Table 1.

3 Solution of coupled-BBM system

To apply the proposed method, we rewrite (1) and (2) as

$$\frac{\partial v(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} + \left(u(x,t)\frac{\partial v(x,t)}{\partial x} + v(x,t)\frac{\partial u(x,t)}{\partial x}\right) - \frac{1}{6}\left[\frac{\partial^3 v(x,t)}{\partial x^2 \partial t}\right] = 0,$$
$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial v(x,t)}{\partial x} + \left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) - \frac{1}{6}\left[\frac{\partial^3 u(x,t)}{\partial x^2 \partial t}\right] = 0,$$

we take the approximations $u(x,t) = U_j^n$ and $v(x,t) = V_j^n$, then from famous Cranck-Nicolson scheme and forward finite difference approximation for the derivative t, [12]. We get

$$\frac{V_j^{n+1}-V_j^n}{k} + \frac{U_{x_j}^{n+1}+U_{x_j}^n}{2} + \left[\frac{(UV_x)_j^{n+1}+(UV_x)_j^n}{2} + \frac{(VU_x)_j^{n+1}+(VU_x)_j^n}{2}\right] - \frac{1}{6}\left[\frac{(V_{xx})_j^{n+1}+(V_{xx})_j^n}{k}\right] = 0, \tag{6}$$

$$\frac{U_j^{n+1}-U_j^n}{k} + \frac{V_{x_j}^{n+1}+V_{x_j}^n}{2} + \left[\frac{(UU_x)_j^{n+1}+(UU_x)_j^n}{2}\right] - \frac{1}{6}\left[\frac{(U_{xx})_j^{n+1}+(U_{xx})_j^n}{k}\right] = 0, \tag{7}$$

where $k = \Delta t$ is the time step.

Table 1: The values of cubic trigonometric B-spline and its first and second derivatives at the knots points.

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
CTB_j	1	α_1	α_2	α_1	0
CTB'_j	0	β_1	0	β_2	0
CTB''_j	0	γ_1	γ_2	γ_1	0

where

$$\alpha_1 = \sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right), \alpha_2 = \frac{2}{1+2\cos(h)},$$
$$\beta_1 = -\frac{3}{4}\csc\left(\frac{3h}{2}\right), \beta_2 = \frac{3}{4}\csc\left(\frac{3h}{2}\right),$$
$$\gamma_1 = \frac{3\left((1+3\cos(h))\csc^2\left(\frac{h}{2}\right)\right)}{16\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}, \gamma_2 = -\frac{3\cot^2\left(\frac{h}{2}\right)}{2+4\cos(h)},$$

In the Crank-Nicolson scheme, the time stepping process is half explicit and half implicit. So the method is better than simple finite difference method.

The nonlinear terms in Eqs. (6) and (7) is linearized using the form given by Rubin and Graves [13] as: we take linearization of the nonlinear term as follows

$$\begin{aligned}(UV_x)_j^{n+1} &= U_j^n V_{x_j}^{n+1} + U_j^{n+1} V_{x_j}^n - U_j^n V_{x_j}^n,\\ (VU_x)_j^{n+1} &= V_j^n U_{x_j}^{n+1} + V_j^{n+1} U_{x_j}^n - V_j^n U_{x_j}^n,\\ (UU_x)_j^{n+1} &= U_j^n U_{x_j}^{n+1} + U_j^{n+1} U_{x_j}^n - U_j^n U_{x_j}^n\end{aligned} \tag{8}$$

Expressing $U(x,t)$ and $V(x,t)$ by using cubic trigonometric B-spline functions $CTB_j(x)$ and the time dependent parameters $c_j(t)$ and $\delta_j(t)$, for $U(x,t)$ and $V(x,t)$ respectively, the approximate solution can be written as:

$$U_N(x,t) = \sum_{j=-1}^{N+1} c_j(t)B_j(x), V_N(x,t) = \sum_{j=-1}^{N+1} \delta_j(t)B_j(x), \tag{9}$$

Using approximate function (9) and cubic trigonometric B-spline functions (5), the approximate values $U(x), V(x)$ and their derivatives up to second order are determined in terms of the time parameters $c_j(t)$ and $\delta_j(t)$, respectively, as

$$\begin{aligned}U_j &= U(x_j) = \alpha_1 c_{j-1} + \alpha_2 c_j + \alpha_1 c_{j+1},\\ U'_j &= U'(x_j) = \beta_1 c_{j-1} + \beta_2 c_{j+1},\\ U''_j &= U''(x_j) = \gamma_1 c_{j-1} + \gamma_2 c_j + \gamma_1 c_{j+1},\\ V_j &= V(x_j) = \alpha_1 \delta_{j-1} + \alpha_2 \delta_j + \alpha_1 \delta_{j+1},\\ V'_j &= V'(x_j) = \beta_1 \delta_{j-1} + \beta_2 \delta_{j+1},\\ V''_j &= V''(x_j) = \gamma_1 \delta_{j-1} + \gamma_2 \delta_j + \gamma_1 \delta_{j+1},\end{aligned} \tag{10}$$

On substituting the approximate solution for U,V and its derivatives from Eq. (10) at the knots in Eqs. (6) and (7) yields the following difference equation with the variables $c_j(t)$ and $\delta_j(t)$.

$$A_1\delta_{j-1}^{n+1}+A_2\delta_j^{n+1}+A_3\delta_{j+1}^{n+1}+A_4c_{j-1}^{n+1}+A_5c_j^{n+1}+A_6c_{j+1}^{n+1}=A_7\delta_{j-1}^{n}+A_8\delta_j^{n}+A_7\delta_{j+1}^{n}-A_9c_{j-1}^{n}-A_{10}c_{j+1}^{n}, \quad (11)$$

$$B_1c_{j-1}^{n+1}+B_2c_j^{n+1}+B_3c_{j+1}^{n+1}+B_4\delta_{j-1}^{n+1}+B_5\delta_{j+1}^{n+1}=B_6c_{j-1}^{n}+B_7c_j^{n}+B_6c_{j+1}^{n}-B_4\delta_{j-1}^{n}-B_5\delta_{j+1}^{n}, \quad (12)$$

where

$$A_1=\alpha_1-\frac{\gamma_1}{6}+\frac{\alpha_1\Delta t}{2}z_2+\frac{\beta_1\Delta t}{2}z_1,\quad A_2=\alpha_2-\frac{\gamma_2}{6}+\frac{\alpha_2\Delta t}{2}z_2,$$
$$A_3=\alpha_1-\frac{\gamma_1}{6}+\frac{\alpha_1\Delta t}{2}z_2+\frac{\beta_2\Delta t}{2}z_1,\quad A_4=\frac{\beta_1\Delta t}{2}+\frac{\alpha_1\Delta t}{2}z_4+\frac{\beta_1\Delta t}{2}z_3,$$
$$A_5=\frac{\alpha_2\Delta t}{2}z_4,\quad A_6=\frac{\beta_2\Delta t}{2}+\frac{\alpha_1\Delta t}{2}z_4+\frac{\beta_2\Delta t}{2}z_3,$$
$$A_7=\alpha_1-\frac{\gamma_1}{6},\quad A_8=\alpha_2-\frac{\gamma_2}{6},\quad A_9=\frac{\beta_1\Delta t}{2},\quad A_{10}=\frac{\beta_2\Delta t}{2},$$
$$B_1=\alpha_1-\frac{\gamma_1}{6}+\frac{\alpha_1\Delta t}{2}z_2+\frac{\beta_1\Delta t}{2}z_1,\quad B_2=\alpha_2-\frac{\gamma_2}{6}+\frac{\alpha_2\Delta t}{2}z_2,$$
$$B_3=\alpha_1-\frac{\gamma_1}{6}+\frac{\alpha_1\Delta t}{2}z_2+\frac{\beta_2\Delta t}{2}z_1,\quad B_4=\frac{\beta_1\Delta t}{2},\quad B_5=\frac{\beta_2\Delta t}{2},$$
$$B_6=\alpha_1-\frac{\gamma_1}{6},\quad B_7=\alpha_2-\frac{\gamma_2}{6},$$
$$z_1=\alpha_1c_{j-1}+\alpha_2c_j+\alpha_1c_{j+1},\quad z_2=\beta_1c_{j-1}+\beta_2c_{j+1},$$
$$z_3=\alpha_1\delta_{j-1}+\alpha_2\delta_j+\alpha_1\delta_{j+1},\quad z_4=\beta_1\delta_{j-1}+\beta_2\delta_{j+1},$$

The system thus obtained on simplifying Eqs. (11) and (12) consists of $(2N+2)$ linear equations in the $(2N+6)$ unknowns $(c_{-1},c_{0,},\ldots,c_N,c_{N+1})^T,(\delta_{-1},\delta_{0,},\ldots,\delta_N,\delta_{N+1})^T$. To obtain a unique solution to the resulting system two additional constraints are required. These are obtained by imposing boundary conditions. Eliminating c_{-1},c_{N+1} and δ_{-1},δ_{N+1} the system get reduced to a matrix system of dimension $(2N+2)\times(2N+2)$ which is the tridiagonal system that can be solved by any algorithm.

4 Initial values

To find the initial parameters c_j^0 and δ_j^0, the initial conditions and the derivatives at the boundaries are used in the following way

$$(U')(x_0,0)=\beta_1c_{-1}+\beta_2c_1=f'(x_0),$$
$$(U'')(x_0,0)=\gamma_1c_{-1}+\gamma_2c_0+\gamma_1c_1=f''(x_0),$$
$$(U)(x_j,0)=\alpha_1c_{j-1}+\alpha_2c_j+\alpha_1c_{j+1}=f(x_j),$$
$$(U')(x_N,0)=\beta_1c_{N-1}+\beta_2c_{N+1}=f'(x_N),$$
$$(U'')(x_N,0)=\gamma_1c_{N-1}+\gamma_2c_N+\gamma_1c_{N+1}=f''(x_N),$$
$$(V')(x_0,0)=\beta_1\delta_{-1}+\beta_2\delta_1=g'(x_0),$$
$$(V'')(x_0,0)=\gamma_1\delta_{-1}+\gamma_2\delta_0+\gamma_1\delta_1=g''(x_0),$$
$$(V)(x_j,0)=\alpha_1\delta_{j-1}+\alpha_2\delta_j+\alpha_1\delta_{j+1}=g(x_j),$$
$$(V')(x_N,0)=\beta_1\delta_{N-1}+\beta_2\delta_{N+1}=g'(x_N),$$
$$(V'')(x_N,0)=\gamma_1\delta_{N-1}+\gamma_2\delta_N+\gamma_1\delta_{N+1}=g''(x_N),$$

which forms a linear block tridiagonal system for unknown initial conditions c_j^0 and δ_j^0, of order $(2N+2)$after eliminating the functions values of c and δ. This system can be solved by any algorithm. Once the initial vectors of parameters have been calculated, the numerical solution of coupled BBM system U and V can be determined from the time evaluation of the vectors c_j^n and δ_j^n, by using the recurrence relations

$$U(x_j,t_n) = \alpha_1\, c_{j-1} + \alpha_2\, c_j + \alpha_1\, c_{j+1},$$
$$V(x_j,t_n) = \alpha_1\, \delta_{j-1} + \alpha_2\, \delta_j + \alpha_1\, \delta_{j+1}.$$

5 Stability analysis of the method

The stability analysis of nonlinear partial differential equations is not easy task to undertake. Most researchers copy with the problem by linearizing the partial differential equation. Our stability analysis will be based on the Von-Neumann concept in which the growth factor of a typical Fourier mode defined as

$$c_j^n = A\zeta^n \exp(ij\phi), \quad \delta_j^n = B\zeta^n \exp(ij\phi), \tag{13}$$
$$g = \frac{\zeta^{n+1}}{\zeta^n},$$

where A and B are the harmonics amplitude, $\phi = kh$, k is the mode number, $i = \sqrt{-1}$ and g is the amplification factor of the schemes. We will be applied the stability of the cubic trigonometric schemes by assuming the nonlinear term as a constants λ_1, λ_2. This is equivalent to assuming that all the c_j^n and δ_j^n as a local constants λ_1, λ_2respectively. At $x = x_j$ systems (11) and (12) can be written as

$$a_1\delta_{j-1}^{n+1} + a_2\delta_j^{n+1} + a_3\delta_{j+1}^{n+1} + a_4 c_{j-1}^{n+1} + a_5 c_{j+1}^{n+1} = a_6\delta_{j-1}^n + a_2\delta_j^n + a_7\delta_{j+1}^n - a_4 c_{j-1}^n - a_5 c_{j+1}^n, \tag{14}$$

where

$$a_1 = \alpha_1 - \frac{\gamma_1}{6} + \frac{\beta_1\Delta t}{2}\lambda_1, \quad a_2 = \alpha_2 - \frac{\gamma_2}{6}, \quad a_3 = \alpha_1 - \frac{\gamma_1}{6} + \frac{\beta_2 k_1}{2}\lambda_1,$$
$$a_4 = \frac{\beta_1\Delta t}{2}(1+\lambda_2), \quad a_5 = \frac{\beta_2\Delta t}{2}(1+\lambda_2),$$
$$a_6 = \alpha_1 - \frac{\gamma_1}{6} - \frac{\beta_1\Delta t}{2}\lambda_1, \quad a_7 = \alpha_1 - \frac{\gamma_1}{6} - \frac{\beta_2 k_1}{2}\lambda_1,$$

$$a_1 c_{j-1}^{n+1} + a_2 c_j^{n+1} + a_3 c_{j+1}^{n+1} + b_1\delta_{j-1}^{n+1} + b_2\delta_{j+1}^{n+1} = a_6 c_{j-1}^n + a_2 c_j^n + a_7 c_{j+1}^n - b_1\delta_{j-1}^n - b_2\delta_{j+1}^n, \tag{15}$$

where

$$b_1 = \frac{\beta_1\Delta t}{2}, \quad b_2 = \frac{\beta_2\Delta t}{2},$$

Substituting (13) into the difference (14), we get

$$\zeta^{n+1}\left[B\left[2\left(\alpha_1 - \frac{\gamma_1}{6}\right)\cos\phi + \left(\alpha_2 - \frac{\gamma_2}{6}\right)\right] + i\left[\sin\varphi\left(A\left(\beta_2\Delta t\lambda_1\right) + B\left(\beta_2\Delta t\left(1+\lambda_2\right)\right)\right)\right]\right] =$$
$$\zeta^n\left[B\left[2\left(\alpha_1 - \frac{\gamma_1}{6}\right)\cos\phi + \left(\alpha_2 - \frac{\gamma_2}{6}\right)\right] - i\left[\sin\varphi\left(A\left(\beta_2\Delta t\lambda_1\right) + B\left(\beta_2\Delta t\left(1+\lambda_2\right)\right)\right)\right]\right],$$

we get

$$g = \frac{X - iY}{X + iY}, \tag{16}$$

where

$$X = B\left[2\left(\alpha_1 - \frac{\gamma_1}{6}\right)\cos\phi + \left(\alpha_2 - \frac{\gamma_2}{6}\right)\right],$$

and

$$Y = [\sin\varphi\,(A(\beta_2\Delta t\lambda_1) + B(\beta_2\Delta t\,(1+\lambda_2)))].$$

Similar substituting (13) into the difference (15), we get

$$\zeta^{n+1}\left[A\left[2\left(\alpha_1-\frac{\gamma_1}{6}\right)\cos\phi+\left(\alpha_2-\frac{\gamma_2}{6}\right)\right]+i\left[\sin\varphi\,(B(\beta_2\Delta t\lambda_1)+A(\beta_2\Delta t\,(\lambda_2)))\right]\right]=$$
$$\zeta^{n}\left[A\left[2\left(\alpha_1-\frac{\gamma_1}{6}\right)\cos\phi+\left(\alpha_2-\frac{\gamma_2}{6}\right)\right]-i\left[\sin\varphi\,(B(\beta_2\Delta t\lambda_1)+A(\beta_2\Delta t\,(\lambda_2)))\right]\right],$$

we get

$$g=\frac{X_1-iY_1}{X_1+iY_1}, \tag{17}$$

where

$$X_1 = A\left[2\left(\alpha_1-\frac{\gamma_1}{6}\right)\cos\phi+\left(\alpha_2-\frac{\gamma_2}{6}\right)\right],$$

and

$$Y_1 = [\sin\varphi\,(B(\beta_2\Delta t\lambda_1)+A(\beta_2\Delta t\,(\lambda_2)))].$$

From (16) and (17) we get $|g|\leq 1$, hence the schemes are unconditionally stable. It means that there is no restriction on the grid size, i.e. on h and Δt, but we should choose them in such a way that the accuracy of the scheme is not degraded.

6 Numerical tests and results of coupled-BBM system

In this section, we present some numerical examples to test validity of our scheme for solving coupled-BBM system.

The norms L_2-norm and L_∞-norm are used to compare the numerical solution with the analytical solution [14].

$$L_2 = \left\|u^E-u^N\right\| = \sqrt{h\sum_{i=0}^{N}(u_j^E-u_j^N)^2},$$
$$L_\infty = \max_j\left|u_j^E-u_j^N\right|, j=0,1,\cdots,N. \tag{18}$$

Where u^E is the exact solution u and u^N is the approximation solution U_N. Now we can studying our scheme from this problem.

6.1 Single soliton

Consider the coupled-BBM system (1) and (2) with the following initial and boundary conditions:

$$u(x,0)=f(x),\quad v(x,0)=g(x), a\leq x\leq b.$$

And

$$u(a,t)=0,\quad u(b,t)=0,\quad v(a,t)=0,\quad v(b,t)=0,\quad u_x(a,t)=0,\quad u_x(b,t)=0,\quad v_x(a,t)=0,\quad v_x(b,t)=0,\quad 0\leq t\leq T.$$

The exact solution is

$$u(x,t)=\left(1-\frac{g}{6}\right)+\frac{cg}{2}sech^2\left(\frac{\sqrt{g}}{2}(s+x_0-ct)\right),\quad v(x,t)=-1.$$

Now, for comparison, we consider a test problem where, $g = 6, c = \frac{1}{3}, x_0 = 0, k = 0.001$ and $-20 \le x \le 40$. the errors, at time 5 are satisfactorily small L_2-error $= 3.51191 \times 10^{-3}$ and L_∞-error $= 3.51191 \times 10^{-3}$ for approximation solution of $u(x,t)$ and L_2-error and L_∞-error approach to zero for approximation solution of $v(x,t)$ at $h = 0.1$. The Errors, at time 5 are satisfactorily small L_2-error $=1.18952 \times 10^{-3}$ and L_∞-error $=1.26181 \times 10^{-3}$ for approximation solution of $u(x,t)$ and L_2-error and L_∞-error approach to zero for approximation solution of $v(x,t)$ at $h = 0.06$. Our results are recorded in Table 2. The motion of solitary wave using our scheme is plotted at times $t = 0, 10, 20$ in Fig.1. These results illustrate that the scheme has a highest accuracy.

Table 2: L_2- norm and L_∞-norm for $t = 5.0$, $g = 6, c = \frac{1}{3}, x_0 = 0, k = 0.001$ and $-20 \le x \le 40$.

h	T	$u(x,t)$		$v(x,t)$	
		L_2- norm	L_∞- norm	L_2- norm	L_∞- norm
$h = 0.1$	0.0	0.00000000	0.00000000	0.00000000	0.00000000
	1.0	9.90537E-4	9.94496E-4	0.00000000	0.00000000
	2.0	1.96454E-3	2.30882E-3	0.00000000	0.00000000
	3.0	2.54651E-3	3.03307E-3	0.00000000	0.00000000
	4.0	3.01948E-3	3.39768E-3	0.00000000	0.00000000
	5.0	3.31466E-3	3.51191E-3	0.00000000	0.00000000
$h = 0.06$	0.0	0.00000000	0.00000000	0.00000000	0.00000000
	1.0	3.55263E-4	3.89692E-4	0.00000000	0.00000000
	2.0	6.68911E-4	8.28194E-4	0.00000000	0.00000000
	3.0	9.13812E-4	1.08811E-3	0.00000000	0.00000000
	4.0	1.08368E-3	1.22108E-3	0.00000000	0.00000000
	5.0	1.18952E-3	1.26181E-3	0.00000000	0.00000000

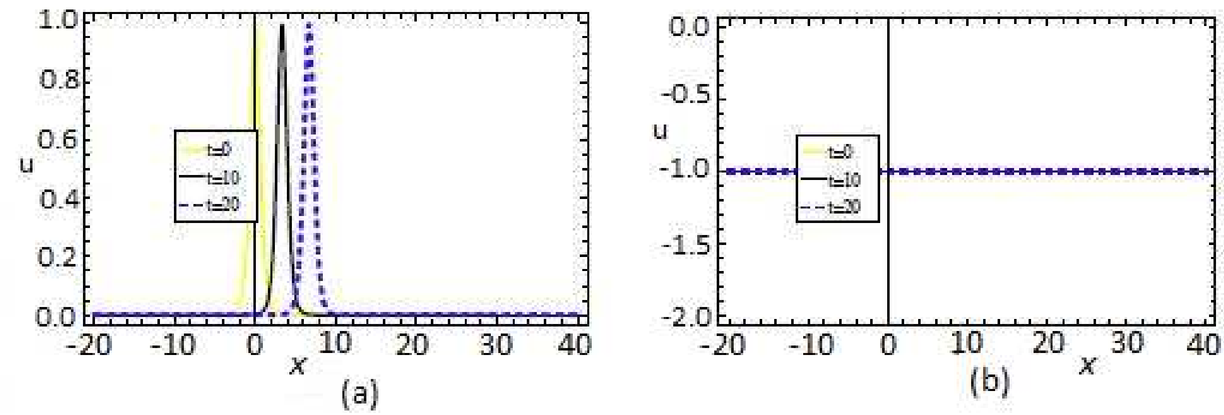

Fig. 1: Single solitory wave with $g = 6, c = \frac{1}{3}, x_0 = 0, k = 0.001$ and $-20 \le x \le 40$ at times $t = 0, 10, 20$ respectively.

Now, we consider a test problem at different constants where, $g = 6, c = \frac{1}{3}, x_0 = 0$, $k = 0.005$ and $-20 \le x \le 30$. The Errors, at time 5 are satisfactorily small L_2-error $= 8.26971 \times 10^{-4}$ and L_∞-error $= 8.77934 \times 10^{-4}$ for approximation solution of $u(x,t)$ and L_2-error and L_∞-error approach to zero for approximation solution of $v(x,t)$ at $h = 0.05$. Our results are recorded in Table 3.

Table 3: L_2- norm and L_∞- norm for $t = 5.0$, $g = 6, c = \frac{1}{3}, x_0 = 0, k = 0.005$ and $-20 \leq x \leq 30$.

h	T	$u(x,t)$		$v(x,t)$	
		L_2- norm	L_∞- norm	L_2- norm	L_∞- norm
	0.0	0.0000000	0.0000000	0.0000000	0.0000000
	1.0	2.46734E-4	3.02889E-4	0.0000000	0.0000000
	2.0	4.64639E-4	5.76178E-4	0.0000000	0.0000000
$h = 0.05$	3.0	6.34904E-4	7.59264E-4	0.0000000	0.0000000
	4.0	7.53146E-4	8.49291E-4	0.0000000	0.0000000
	5.0	8.26971E-4	8.77934E-4	0.0000000	0.0000000

Now we make comparison between our results and results in [7], [15] and [16].

Table 4: Comparison of numerical results of the problem with the results obtained from [7] for the variable u with, $g = 6, c = \frac{1}{3}, x_0 = 0, k = 0.001, -20 \leq x \leq 40$ at $t = 5$.

Schemes at$t = 5$	$u(x,t)$at $h = 0.1$	
	L_2- norm	L_2- norm
our scheme	3.31466E-3	3.51191E-3
(Shadan [7])	7.99452E-4	7.99452E-4

Table 5: Comparison of numerical results of the problem with the results obtained from [15] and [16] for the variable u with, $g = 6, c = \frac{1}{3}, x_0 = 0, k = 0.005, h = 0.05$ $-20 \leq x \leq 30$ at $t = 5$.

Schemes at$t = 5$	$u(x,t)$at $h = 0.05$	
	L_2- norm	L_2- norm
our scheme	8.26971E-4	8.77934E-4
[15] at $\lambda = 0$	-	1.23155E-3
[15] at $\lambda = -2.97 \times 10^{-3}$	-	3.69655E-4
[16] at $\lambda = 1$	-	1.35101E-3
[15] at $\lambda = 5.8339 \times 10^{-6}$	-	1.89722E-4

In tables 4 and 5 we show that our results are related with the results in [7], [15] and [16].

6.2 Interaction of two solitary waves

The interaction of two solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider Coupled-BBM system with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$u(x,0) = \sum_{j=1}^{2} \left(1 - \frac{g_j}{6}\right) c_j + \frac{c_j g_j}{2} \operatorname{sec} h^2 \left(\frac{\sqrt{g_j}}{2}(x + x_j)\right), v(x,0) = -1, \tag{19}$$

where $j = 1,2,$ g_j, x_j and $c_j$are arbitrary constants. In our computational work. Now, we choose $g_1 = 6, g_2 = 6, c_1 = 1, c_2 = \frac{1}{3}$, $x_1 = 0, x_2 = -10, h = 0.1, k = 0.01$ with interval [-20, 40]. In Fig. 2, the interactions of these solitary waves are plotted at different time levels.

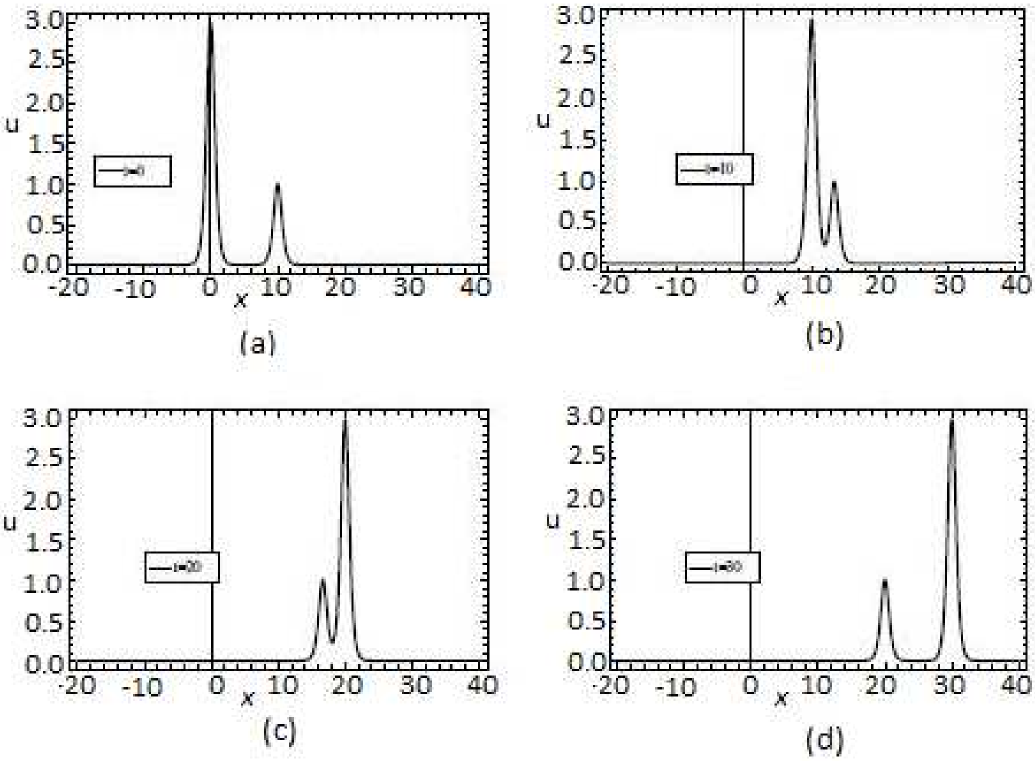

Fig. 2: Interaction two solitary waves with $g_1 = 6, g_2 = 6, c_1 = 1, c_2 = \frac{1}{3}, h = 0.1, k = 0.01, -20 \le x \le 40$ for value u at times $t = 0, 10, 20, 30$ respectively.

6.3 Interaction of three solitary waves

The interaction of three solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider Coupled-BBM system with initial conditions given by the linear sum of three well separated solitary waves of various amplitudes

$$u(x,0) = \sum_{j=1}^{3} \left(1 - \frac{g_j}{6}\right) c_j + \frac{c_j g_j}{2} \sec h^2 \left(\frac{\sqrt{g_j}}{2}(x + x_j)\right), v(x,0) = -1, \tag{20}$$

where $j = 1, 2, 3$, g_j, x_j and c_j are arbitrary constants. In our computational work. Now, we choose $g_1 = 6, g_2 = 6, g_3 = 6, c_1 = 1, c_2 = \frac{2}{3}, c_3 = \frac{1}{3}, x_1 = 0, x_2 = -5, x_3 = -10, h = 0.1$, $k = 0.01$ with interval [-20, 40]. In Fig. 3, the interactions of these solitary waves are plotted at different time levels.

7 Conclusions

In this paper a numerical treatment for the nonlinear Coupled-BBM system is proposed using a collection method with the cubic trigonometric B-spline. The stability analysis of the method is shown to be unconditionally stable. We make linearization for the nonlinear term. We tested our schemes through a single solitary wave in which the analytic solution is known, then extend it to study the interaction of solitons where no analytic solution is known during the interaction. The accuracy of our scheme was shown by calculating error norms L_2andL_∞.

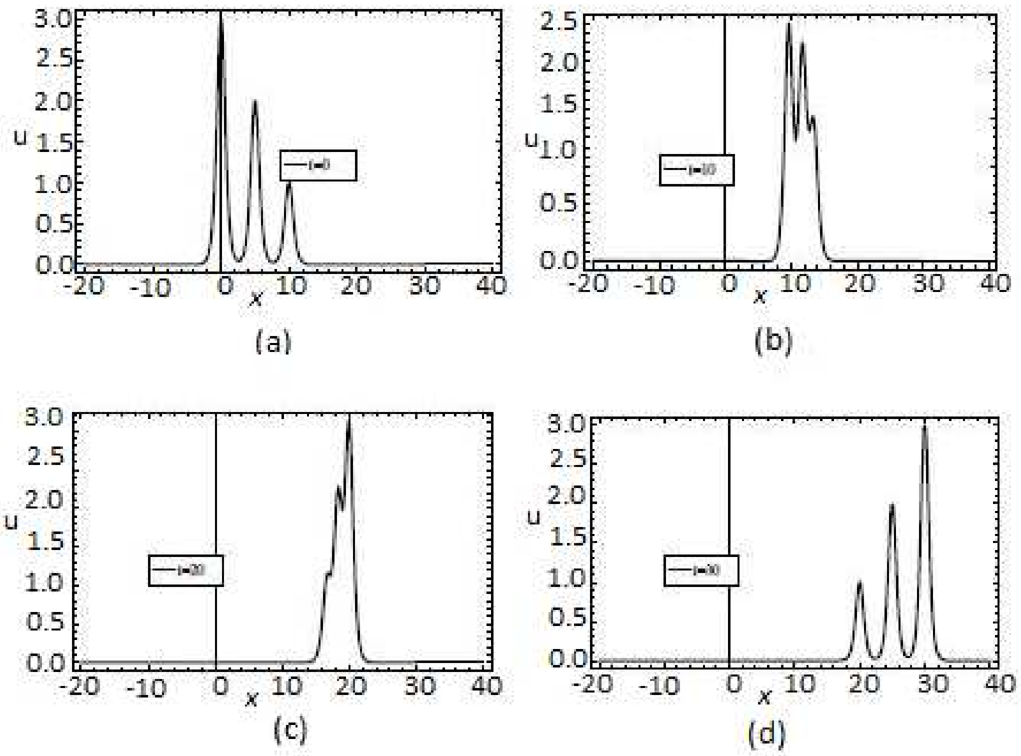

Fig. 3: interaction three solitary waves with $g_1 = 6$, $g_2 = 6$, $g_3 = 6$, $c_1 = 1$, $c_2 = \frac{2}{3}$, $c_3 = \frac{1}{3}$, $x_1 = 0$, $x_2 = -5$, $x_3 = -10$, $h = 0.1$, $k = 0.01$, $-20 \leq x \leq 40$ for values u at times $t = 0, 10, 20, 30$ respectively

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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