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On Anti Fuzzy relations in modules

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Abstract: In this paper we initiate anti fuzzy relations on modules by using the idea of anti fuzzy ideals and explore a few useful outcome.

Keywords: Fuzzy set; Modules; Anti Fuzzy relations.

1 Introduction

The primary theory of a fuzzy set, introduce by L. A. Zadeh in his manuscript [3] of 1965, it provides a usual frame for generalize numerous vital ideas of algebra. Fuzzy subgroups and its significant properties were defined and recognized by Rosenfeld [4]. After this time it was necessary to define ideal of a ring. The notion of a fuzzy ideal of a ring was introduced by Liu [17,18]. The concept of a fuzzy relation on a set was introduced by Zadeh [3]. Bhattacharya and Mukherjee [15] have studied fuzzy relation on groups. Malik and Mordeson [6] studied fuzzy relation on rings. Fuzzy submodules of M over R were first introduced by Negoita and Ralescu [4]. Pan [7,8] studied fuzzy finitely generated modules and fuzzy quotient modules. Fuzzy relations on modules were introduced by Bayram and Dan in [15].

Biswas introduced the concept of anti fuzzy subgroups of groups [2] of 1990. In 2007, Akram and Dar defined anti fuzzy left h-ideals in hemirings [9]. In 2009, M. Shabir and Y. Nawaz had characterized semigroups by the properties of their anti fuzzy ideals [10]. In [11], M. Shabir and N. Rehman initiated anti fuzzy ideals in ternary semigroups and characterized ternary semigroup by anti fuzzy ideals. More on anti fuzzy ideals see [20,21].

In this paper we initiate anti fuzzy relations on modules by using the idea of anti fuzzy ideals and explore a few useful outcome. In this paper M, N are R-modules and R is a commutative ring. An anti fuzzy relation on R is the fuzzy subset of $R\times R$. Then we prove some theorems on anti fuzzy relations by using our definition. That is if α and β are anti fuzzy submodules of M, then the Cartesian product of α and β, $\alpha\times\beta$ is an anti fuzzy submodules of $M\times M$. Also if $\alpha\times\beta$ is an anti fuzzy submodules of $M\times M$, then α and β are anti fuzzy submodules of M. Thus from these we concluded that if α and β are anti fuzzy submodules of M then the cartesian product of α and β, $\alpha\times\beta$ is an anti fuzzy submodules of $M\times M$ if and only if $(\alpha\times\beta)_t$ is a submodule of $M\times M$ for all $t\in[0,1]$ is anti fuzzy submodule of $M\times M$. Moreover If α is anti fuzzy submodule of M and $\alpha=\alpha_\beta$ then $\beta=\beta_\alpha$. finally we have shown the one-one correspondence between all anti fuzzy submodules $M\times M$ and $N\times N$. That is if $f: M\times M\to N\times N$ and $\alpha\times\beta$ be f invariant of $M\times M$ and $N\times N$. If If $\alpha\times\beta$ is an anti fuzzy submodules of $M\times M$, then $f(\alpha\times\beta)$ is an anti fuzzy submodules of $N\times N$, and if $f: M\times M\to N\times N$ and $\alpha\times\beta$ be f invariant of $M\times M$ and $N\times N$. If If $\alpha\times\beta$ is an anti fuzzy submodules of $M\times M$,

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then $f^{-1}(\alpha\times\beta)$ is an anti fuzzy submodules of $M\times M$. Therefore if $f: M\times M\rightarrow N\times N$ is an homomorphism and α is an f invariant of $M\times M$ and $N\times N$. The mapping $\alpha\rightarrow f(\alpha)$ defines a one-one correspondence between the set of all anti fuzzy submodules of $M\times M$ and the set of all anti fuzzy submodules of $N\times N$.

2 Preliminaries

Let R denotes commutative ring and M denotes R-module throughout the paper.

Definition 2.1. [3] A fuzzy subset α of R is a function $\alpha: R\rightarrow[0,1]$.

Definition 2.2. [1]A fuzzy subset α of R is called a fuzzy left (right) ideal of R if

i) $\alpha(x-y)\geq\min(\alpha(x),\alpha(y))$
ii) $\alpha(xy)\geq\alpha(x)$,$(\alpha(xy)\geq\alpha(y))$ for all $x,y\in R$.

A fuzzy subset α of R is called a fuzzy ideal of R if it is a fuzzy left and a fuzzy right ideal of R.

Definition 2.3. [1]If α is a fuzzy subset of R, then for any $t\in\Im\alpha$, the set $\alpha_t=\{x\in R;\alpha(x)\geq t\}$ is called the level subset of R with respect to α.

Definition 2.4. [7] A fuzzy relation α on M is a fuzzy subset of $M\times M$.

Definition 2.5. [7] Let α is a fuzzy relation on M and let β be a fuzzy subset of M. Then α is called a fuzzy relation on β if for all $x,y\in M$,

$$\alpha(x-y)\leq\min(\alpha(x),\alpha(y)).$$

Definition 2.6. [7] Let α be a fuzzy subset of M, then the strongest fuzzy subset on M that is a fuzzy relation on β is α_β defined by, for all $x,y\in M$ $\alpha_\beta=\beta\times\beta(x,y)=\min\{\beta(x),\beta(y)\}$.

Definition 2.7. [7] Let β be a fuzzy relation on M, then the weakest fuzzy subset of M on which α is a fuzzy relation is β_α defined by, for all $x\in M$

$$\beta_\alpha(x)=\sup\{\max\alpha(x,y),\alpha(y,x)\}=\beta\times\beta(x,y)=\min\{\beta(x),\beta(y)\}.$$

Definition 2.8. [7] Let α and β are fuzzy subset of M, then for all $x\in M$

$$\alpha\circ\beta(x,y)=\sup_{x=yz}\{\alpha(y),\beta(z)\}.$$

Definition 2.9 [1] Let $f: M\rightarrow N$ and α be a fuzzy subset of M. The fuzzy subset $f(u)$ of N defined as follows; for all $y\in N$,

$$f(u)(y)=\left\{\begin{array}{c}\vee\{(u)(x):\ x\in M,\ f(x)=y\}\ \text{if}\ f^{-}(y)\neq\emptyset\\ 0,\qquad\text{otherwise}\end{array}\right\}$$ is called the fuzzy image of α under f.

Definition 2.10 [1] Let $f: M\rightarrow N$ and α be a fuzzy subset of N. The fuzzy subset $f^{-}(u)$ of N defined as follows; for all $x\in M$, $f^{-}(u)(x)=(u)f(x)$ is called the fuzzy preimage of α under f.

Definition 2.11. [1] A fuzzy submodule of M is a fuzzy subset of M such that

i) $\alpha(1) = 0$
ii) $\alpha(rx) \geq \alpha(x)$ for all $r \in R$ and for all $x \in M$.
iii) $\alpha(x+y) \geq \min\{\alpha(x), \alpha(y)\}$ for all $x, y \in M$.

3 Anti Fuzzy relations in modules

Definition 3.1. A fuzzy subset α of R is called anti fuzzy left (right) ideal of R if
i) $\alpha(x-y) \leq \max(\alpha(x), \alpha(y))$
ii) $\alpha(xy) \leq \alpha(x), \alpha(xy) \leq \alpha(y)$ for all $x, y \in R$.

A fuzzy subset α of R is called anti fuzzy ideal of$R$ if it is a anti fuzzy left and a anti fuzzy right ideal of R.

Definition 3.2. If α is a fuzzy subset of R, then for any $t \in Im\alpha$, the set $\alpha_t = \{x \in R; \alpha(x) \leq t\}$ is called the lower level subset of R with respect to α.

Theorem 3.3. If α is a fuzzy subset of R, then α is an anti fuzzy ideal of R if and only if α_t is an ideal of R for all $t \in Im\alpha$.

Proof. Easy to prove. □

Definition 3.4. An anti fuzzy relation α on M is a fuzzy subset of $M \times M$.

Definition 3.5. Let α be an anti fuzzy relation on M and let β is a fuzzy subset of M. Then α is called anti fuzzy relation on β if for all $x, y \in M$,

$$\alpha(x,y) \geq \max(\alpha(x), \alpha(y)).$$

Definition 3.6. Let α be a fuzzy subset of M, then the strongest fuzzy subset on M that is anti fuzzy relation on β is α_β defined by, for all $x, y \in M$ $\alpha_\beta = \beta \times \beta(x,y) = \max\{\beta(x), \beta(y)\}$.

Definition 3.7. Let β be anti fuzzy relation on M, then the weakest fuzzy subset of M on which α is anti fuzzy relation is β_α defined by, for all $x \in M$ Let α and β arefuzzy subset of M, then for all $x \in M$

$$\alpha \circ \beta(x,y) = \inf_{x-yz}\{\alpha(y), \beta(z)\}.$$

Definition 3.8. Let $f : M \to N$ and α be a fuzzy subset of M. The fuzzy subset $f(u)$ of N defined as follows; for all $y \in N$,

$$f(u)(y) = \left\{ \begin{array}{c} \wedge\{(u)(x) : \ x \in M, \ f(x) = y\} \text{ if } f^-(y) \neq \emptyset \\ 0 \text{ otherwise} \end{array} \right\}$$ is called the anti fuzzy image of α under f.

Definition 3.9. Let $f : M \to N$ and α be a fuzzy subset of N. The fuzzy subset $f^-(u)$ of N defined as follows; for all $x \in M$,

$$f^-(u)(x) = (u)f(x)$$

is called the anti fuzzy preimage of α under f.

Definition 3.10. A anti fuzzy submodule of M is a fuzzy subset of M such that

i) $\alpha(0) = 1$
ii) $\alpha(rx) \leq \alpha(x)$ for all $r \in R$ and for all $x \in M$.
ii) $\alpha(x+y) \leq \max\{\alpha(x), \alpha(y)\}$ for all $x, y \in M$.

Theorem 3.11. If α and β are anti fuzzy submodules of M, then $\alpha \times \beta$ is an anti fuzzy submodules of $M \times M$.

Proof. i) Since $\alpha(1) = 0 = \beta(1)$. Consider

$$\alpha \times \beta(1,1) = \max\{\alpha(1), \beta(1)\} = 1.$$

ii)

$$\begin{aligned} \alpha \times \beta(rx, ry) &= \max\{\alpha(rx), \beta(ry)\} \\ &\leq \max\{\alpha(x), \beta(y)\} \\ &= \alpha \times \beta(x, y). \end{aligned}$$

iii)

$$\begin{aligned} \alpha \times \beta\{(x,y) + (z,t)\} &= \alpha \times \beta(x+z, y+t) \\ &= \max\{\alpha(x+z), \beta(y+t)\} \\ &\leq \max\{\alpha(x), \alpha(z)\} \max\{\beta(y), \beta(t)\} \\ &= \max\{\alpha(x), \beta(y)\} \max\{\alpha(z), \beta(t)\} \\ &= \max\{\alpha \times \beta(x,y), \alpha \times \beta(z,t)\}. \end{aligned}$$

Thus $\alpha \times \beta$ is an anti fuzzy submodules of $M \times M$. $\square$

Theorem 3.12. If $\alpha \times \beta$ is an anti fuzzy submodules of $M \times M$, then α and β are anti fuzzy submodules of M.

Proof. i) $\alpha \times \beta(1,1) = \max\{\alpha(1), \beta(1)\} = \alpha(1) = 1$ and $\beta(1) = 1$.

ii)

$$\begin{aligned} \alpha(rx) &= \alpha \times \beta(rx, 1) \\ &= \alpha \times \beta\{(r,1)(x,1)\} \\ &\leq \alpha \times \beta(x, 1) \\ &= \max\{\alpha(x), \beta(y)\} \\ &= \alpha(x) \end{aligned}$$

iii)

$$\begin{aligned} \alpha(x+y) &= \alpha \times \beta(x+y, 1) \\ &= \alpha \times \beta\{(x,1) + (y,1)\} \\ &\leq \max\{\alpha \times \beta(x,1), \alpha \times \beta(y,1)\} \\ &= \max\max\{\alpha(x), \beta(y)\}. \end{aligned}$$

Thus α is anti fuzzy submodule of M. Similarly by the same way β is anti fuzzy submodule of M. □

Theorem 3.13. If α and β are anti fuzzy submodules of M then $\alpha\times\beta$ is an anti fuzzy submodules of $M\times M$ if and only if $(\alpha\times\beta)_t$ is a submodule of $M\times M$ for all $t\in[0,1]$.

Proof. Trival. □

Lemma 3.14. If α is anti fuzzy submodule of M and $\alpha=\alpha_\beta$ then $\beta=\beta_\alpha$.

Proof. Consider

$$\begin{aligned}\beta_\alpha(x) &= \alpha\times\alpha(x,1)\\ &= \max\{\alpha(x),\alpha(1)\}\\ &= \max\{\beta\times\beta(x,1),\beta\times\beta(1,1)\}\\ &\quad \max\{\beta(x),\beta(1)\}\\ &= \beta(x).\ \square\end{aligned}$$

Corollary 3.15. Let β be a subset of M. Then α_β is an anti fuzzy submodule of $M\times M$ if and only if β is anti fuzzy submodule of M.

Proof. Easy. □

Now we show the one-one correspondence between all anti fuzzy submodules $M\times M$ and $N\times N$.

Theorem 3.16. Let $f:M\times M\to N\times N$ and $\alpha\times\beta$ be f invariant of $M\times M$ and $N\times N$. If If $\alpha\times\beta$ is an anti fuzzy submodules of $M\times M$, then $f(\alpha\times\beta)$ is an anti fuzzy submodules of $N\times N$.

Proof. i)

$$\begin{aligned}f(\alpha\times\beta)(1,1) &= \inf\{(\alpha\times\beta)(1,1) \qquad f(x,y)=(1,1)\ \exists(x,y)\in M\times M\}\\ &= \inf\{\max\{\alpha(1),\beta(1)\} \qquad f(x,y)=(1,1)\}\\ &= \inf\{1,1\}\\ &= 1.\end{aligned}$$

ii) For $r_1,r_2r_3\in R$, $x_1,x_2\in N$ and $x_1x_2\in M$. Consider

$$\begin{aligned}f(\alpha\times\beta)(r_1x_1,r_2x_2) &= \inf\{(\alpha\times\beta)(r_1x_1,r_2x_2) \qquad f(r_1x_1,r_2x_2)=(r_1x_1,r_2x_2)\}\\ &= \inf\{\max\{\alpha(r_1x_1),\beta(r_2x_2)\} \quad f(r_1x_1,r_2x_2)=(r_1x_1,r_2x_2)\}\\ &\le \inf\{\max\{\alpha(x_1),\beta(x_2)\}\}\\ &= \inf\{(\alpha\times\beta)(x_1,x_2) \qquad f(x_1,x_2)=(rx_1,x_2)\}\\ &= f(\alpha\times\beta)(x_1,x_2).\end{aligned}$$

iii) For $x_1,y_1,x_2,y_2\in N$ and $x_1,y_1,x_2,y_2\in M$, consider

$$f(\alpha\times\beta)\{(x_1,x_2)+(y_1,y_2)\}=f(\alpha\times\beta)(x_1+y_1)+(x_2+y_2)$$

$$
\begin{aligned}
&= \inf\{(\alpha\times\beta)(x_1+y_1)+(x_2+y_2)\ \ f(x_1+y_1)+(x_2+y_2)=(x_1+y_1)+(x_2+y_2)\}\\
&= \inf\{\max\{\alpha(x_1+y_1),\beta(x_2+y_2)\}\}\\
&\leq \inf\{\max\{\alpha(x_1),\alpha(y_1)\}\max\{\beta(x_2),\beta(y_2)\}\}\\
&= \inf\{\max\{\alpha(x_1),\beta(x_2)\}\max\{\alpha(y_1),\beta(y_2)\}\}\\
&= \inf\{(\alpha\times\beta)(x_1,x_2),(\alpha\times\beta)(y_1,y_2)\qquad f(x_1,x_2)=(x_1,x_2),f(y_1,y_2)=(y_1,y_2)\}\\
&= \max\{f(\alpha\times\beta)(x_1,x_2),f(\alpha\times\beta)(y_1,y_2)\}.
\end{aligned}
$$

Therefore $(\alpha\times\beta)$ is an anti fuzzy submodules of $N\times N$. □

Theorem 3.17. Let $f:M\times M\to N\times N$ and $\alpha\times\beta$ be f invariant of $M\times M$ and $N\times N$. If If $\alpha\times\beta$ is an anti fuzzy submodules of $N\times N$, then $f^{-1}(\alpha\times\beta)$ is an anti fuzzy submodules of $M\times M$.

Proof. i) Consider

$$
\begin{aligned}
f^{-1}(\alpha\times\beta)(1,1) &= (\alpha\times\beta)(f(1,1))\\
&= \alpha\times\beta(1,1)\\
&= \max\{\alpha(1),\beta(1)\}\\
&= 1.
\end{aligned}
$$

ii) For $r_1,r_2r_3\in R$, $x_1x_2\in N$ and $x_1,x_2\in M$. Consider

$$
\begin{aligned}
f^{-1}(\alpha\times\beta)(r_1x_1,r_2x_2) &= (\alpha\times\beta)f(r_1x_1,r_2x_2)\\
&= (\alpha\times\beta)(r_1x_1,r_2x_2)\\
&= \max\{\alpha(r_1x_1),\beta(r_2x_2)\}\\
&\leq \max\{\alpha(x_1),\beta(x_2)\}\\
&= (\alpha\times\beta)f(x_1,x_2)\\
&= f^{-1}(\alpha\times\beta)(x_1,x_2).
\end{aligned}
$$

iii) For $x_1,y_1,x_2,y_2\in N$ and $x_1,y_1,x_2,y_2\in M$, consider

$$
\begin{aligned}
f^{-1}(\alpha\times\beta)\{(x_1,x_2)+(y_1,y_2)\} &= f^{-1}(\alpha\times\beta)((x_1+y_1)+(x_2+y_2))\\
&= (\alpha\times\beta)(f((x_1+y_1)+(x_2+y_2)))\\
&= \alpha\times\beta((x_1+y_1)+(x_2+y_2))\\
&= \max(\alpha(x_1+y_1)+\beta(x_2+y_2))\\
&\leq \max(\alpha(x_1),\alpha(y_1)),\max(\beta(x_2),\beta(y_2))\\
&= \alpha\times\beta((x_1,x_2)),\alpha\times\beta(y_1,y_2)\\
&= \alpha\times\beta(f(x_1,x_2)),\alpha\times\beta(f(y_1,y_2))\\
&= f^{-1}(\alpha\times\beta)(x_1,x_2)+f^{-1}(\alpha\times\beta)(y_1,y_2).
\end{aligned}
$$

Hence $f^{-1}(\alpha\times\beta)$ is an anti fuzzy submodules of $N\times N$. □

Corollary 3.18. Let $f:M\times M\to N\times N$ is an homomorphism and α is an f invariant of $M\times M$ and $N\times N$. The mapping $\alpha\to f(\alpha)$ defines a one-one correspondence between the set of all anti fuzzy submodules of $M\times M$ and the set of all anti fuzzy submodules of $N\times N$.

Proof. Follows from theorem 3. □

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