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On the Fibonacci and Lucas curves

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Abstract: In this study, we have constituted the Frenet vector fields of Fibonacci and Lucas curves which are created with the help of Fibonacci and Lucas sequences hold a very important place in nature defined by Horadam and Shannon, [1]. Based on these vector fields, we have investigated notions of evolute, pedal and parallel curves and obtained the graphics of these curves.

Keywords: Fibonacci and Lucas curves, evolute curve, parallel curve, pedal curve.

1 Introduction

One of the most fundamental structure of differential geometry is the theory of curves. An increasing interest of this theory makes a development of special curves to be investigated. A way for classification and characterization of curves is the relationship between the Frenet vectors of the curves. By means of this notions, characterizations of special curves have been revealed, ([3], [5], [8], [9]). Some of these are involute-evolute, parallel and pedal curves etc. C. Boyer discovered involutes while trying to build a more accurate clock, [6]. The involute of a plane curve is constructed from the unit tangent at each point of $F(s)$, multiplied by the difference between the arc length to a constant and the arc length to $F(s)$. The evolute and involute of a plane curve are inverse operations similar to Differentiation and Integration. Also, in the 17th-century, the firstly pedal curve was found by G. Roberval, [7]. Only in 19th century, more than 500 studies have been conducted on the pedal curves, [10, 11, 13]. Given two curves, one is a parallel curve (also known as an offset curve) of the other if the points on the first curve are equidistant to the corresponding points in the direction of the second curve's normal. Alternatively, a parallel of a curve can be defined as the envelope of congruent circles whose centers lie on the curve. A parallel of a curve is the envelope of a family of congruent circles centered on the curve. It generalizes the concept of parallel lines. It can be also defined as a curve whose points are at a fixed normal distance of a given curve. More information on the pedal and parallel curves can be found in [14].

The Fibonacci numbers was introduced by Leonardo of Pisa in his book Liber Abaci, [12]. These numbers was used as a model for investigate the growth of rabbit populations. The Fibonacci numbers are closely related to Lucas numbers in that they are a complementary pair of Lucas sequences. They are intimately connected with the golden ratio.

The Fibonacci and Lucas sequence are defined as

$$F_0 = 1,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$$

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and

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2}$$

for $n \geq 2$, respectively. Here, F_n is n^{th} Fibonacci number and L_n is n^{th} Lucas number, ([1], [2], [4], [15], [16]).

2 Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space $\mathbb{R}^2$ are briefly presented. We restrict our study to local theory of plane curves.

Let $r : I \to \mathbb{R}^2$ be a planar curve, smooth and regular. Firstly, we give without details the two representations of a plane curve.

a) The parametric representation of the plane curve is $x = x(s)$, $y = y(s)$, $x'(s)^2 + y'(s)^2 > 0,\ \forall s \in I$;

b) The implicit representation is $F(x, y) = 0,\ F : D \to \mathbb{R},\ D \subseteq \mathbb{R}^2,\ F$ has $F_x'^2 + F_y'^2 > 0,\ \forall (x, y) \in D$, ([13], [16]).

In this study, we have dealt with the plane curves which defined as parametrical. In parametric form, the equation of the curvature is

$$k(s) = \frac{x'(s)\, y''(s) - x''(s)\, y'(s)}{\left((x')^2(s) + (y')^2(s)\right)^{\frac{3}{2}}}.$$

If the curve is parametrized by arc-length, the curvature is $k(s) = x'(s)\, y''(s) - x''(s)\, y'(s)$.

Let $r : I \to \mathbb{R}^2$ be a plane curve. The condition $r'(s) \neq 0$ is equivalent to the existence of the distinguished Frenet frame. We shall always choose this frame as the moving 2-frame on the curve r. So, let $\{\boldsymbol{T}, \boldsymbol{N}\}$ be Frenet vector field of the planar curve r and k be curvature of this curve. Also, Frenet formulas of the planar curve r are as follows

$$\begin{aligned} \boldsymbol{T}' &= vk\boldsymbol{N} \\ \boldsymbol{N}' &= -vk\boldsymbol{T} \end{aligned} \tag{1}$$

where $\|r'\| = v$, ([15], [16]).

Let consider a plane curve r and the family of normal to the curve. The envelope of this family is called the evolute of the curve r. If the curve is given in parametric form: $x = x(s)$, $y = y(s)$, then the equation of the normal at an arbitrary point $r(s)$ that

$$y - y(s) = -\frac{x'(s)}{y'(s)}\left(x - x(s)\right).$$

Using the method described in previous paragraph, we can determine the parametric equations of the evolute that

$$\begin{aligned} X &= x(s) - \frac{y'(s)\left(x'^2(s) + y'^2(s)\right)}{x'(s)\, y''(s) - x''(s)\, y'(s)}, \\ Y &= y(s) + \frac{x'(s)\left(x'^2(s) + y'^2(s)\right)}{x'(s)\, y''(s) - x''(s)\, y'(s)}. \end{aligned}$$

Let $r : I \to \mathbb{R}^2$ be a planar curve. The curve which generated by perpendicular foot reflected tangents of the curve r from the constant point P is positive pedal curve according to point P of curve. So, the Pedal curve of the parameter curve r is that

$$Pd_r(s) = r(s) + \frac{r'(s).(A - r(s))}{\|r'(s)\|^2} r'(s) \tag{2}$$

where A is a constant point. If the curve Pd_r is the Pedal curve of the curve r, then the curve r is negative Pedal curve of the curve Pd_r, [9].

The parametric equations of the Fibonacci and Lucas curves are given by Horadam and Shannon, [1]. In this study, after we have obtained the Frenet vector fields of the Fibonacci and Lucas taking advantage of [1] we have researched notions of Evolute, Pedal and Parallel curves for this curves.

3 The Fibonacci Curves

Let $I \subseteq \mathbb{R}$ be a open interval of $\mathbb{R}$. So, the ***Fibonacci curve*** is that

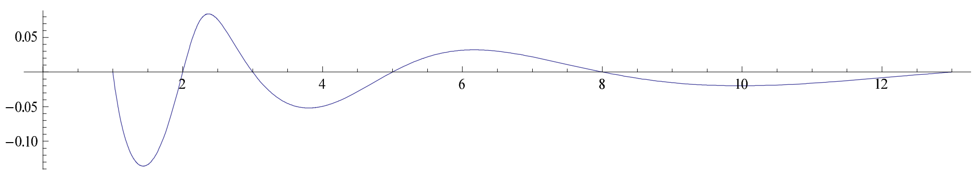
$$\begin{aligned} r : I &\to \mathbb{R}^2 \\ \theta &\to r(\theta) = (x(\theta), y(\theta)). \end{aligned}$$

If we consider $\alpha = \frac{1+\sqrt{5}}{2}$, it is obvious that

$$x(\theta) = \frac{\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi)}{\sqrt{5}}, \tag{3}$$

$$y(\theta) = \frac{-\alpha^{-\theta}\sin(\theta\pi)}{\sqrt{5}} \tag{4}$$

and the graph of the Fibonacci curve is that



Firstly, if we take the derivative of the equations (3.1) and (3.2), with respect to θ, we obtain that

$$\frac{dx}{d\theta} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{-\theta}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2\theta}\log\left(\frac{1+\sqrt{5}}{2}\right) + \log\left(\frac{1+\sqrt{5}}{2}\right)\cos(\theta\pi) + \pi\sin(\theta\pi)\right]}{\sqrt{5}} \tag{5}$$

and

$$\frac{dy}{d\theta} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{-\theta}\left[-\pi\cos(\theta\pi) + \log\left(\frac{1+\sqrt{5}}{2}\right)\sin(\theta\pi)\right]}{\sqrt{5}} \tag{6}$$

Considering $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\log(\frac{1+\sqrt{5}}{2}) =$ s in the equations (3.3) and (3.4), we can find

$$\frac{dx}{d\theta} = x'(\theta) = \frac{\alpha^{-\theta}\left[\alpha^{2\theta}s + s\cos(\theta\pi) + \pi\sin(\theta\pi)\right]}{\sqrt{5}}, \tag{7}$$

$$\frac{dy}{d\theta} = y'(\theta) = \frac{\alpha^{-\theta}\left[-\pi\cos(\theta\pi) + s\sin(\theta\pi)\right]}{\sqrt{5}}. \tag{8}$$

Again taking the derivative of the equations (3.5) and (3.6) with respect to θ, we have

$$\frac{d^2x}{d\theta^2} = x''(\theta) = \frac{\alpha^{-\theta}\left[\left(\pi^2 - s^2\right)\cos(\theta\pi) + \alpha^{2\theta}s^2 - 2\pi s\sin(\theta\pi)\right]}{\sqrt{5}}, \tag{9}$$

$$\frac{d^2y}{d\theta^2} = y''(\theta) = \frac{\alpha^{-\theta}\left[2\pi s\cos(\theta\pi) + \left(\pi^2 - s^2\right)\sin(\theta\pi)\right]}{\sqrt{5}}. \tag{10}$$

Now, we will construct the orthonormal frame of the Fibonacci curve at its any points. Considering $v_1 = \{x'(\theta), y'(\theta)\}$ and $v_2 = \{x''(\theta), y''(\theta)\}$, we find the orthonormal set benefit from the linear set $\{v_1, v_2\}$. So, we can write $v_1 = u_1 = \{x'(\theta), y'(\theta)\}$ and $u_2 = v_2 - \frac{\langle u_1, v_2\rangle}{\langle u_1, u_2\rangle}u_1$. If we make necessary arrangement in the latter equation, we get

$$u_2 = \{G_1, G_2\}.$$

where,

$$G_1 = \frac{\alpha^{-\theta}\left(\pi cos(\pi\theta) - ssin(\pi\theta)\right)\left(\pi(\pi^2+s^2) + \alpha^{2\theta}s\left[3\pi scos(\pi\theta) + (\pi^2 - 2s^2)sin(\pi\theta)\right]\right)}{\sqrt{5}\left\{\pi^2 + (1+\alpha^{4\theta})s^2 + 2\alpha^{2\theta}s\left[cos(\pi\theta)s + \pi sin(\pi\theta)\right]\right\}}.$$

$$G_2 = \frac{\alpha^{-\theta}(Zk_1 + Zk_2)}{2\sqrt{5}\left\{\pi^2 + s^2\left(1+\alpha^{4\theta}\right) + 2s\alpha^{2\theta}\left[scos(\pi\theta) + \pi sin(\pi\theta)\right]\right\}},$$

including,

$$Zk_1 = 2\pi\left(\pi^2+s^2\right)\left[scos(\pi\theta) + \pi sin(\pi\theta)\right] + 2s^2\alpha^{4\theta}\left[3\pi scos(\pi\theta) + \left(\pi^2 - 2s^2\right)sin(\pi\theta)\right],$$

and

$$Zk_2 = -s\alpha^{2\theta}\left[-3\pi\left(\pi^2+s^2\right) + \pi\left(\pi^2 - 5s^2\right)cos(2\pi\theta) + 2s\left(-2\pi^2 + s^2\right)sin(2\pi\theta)\right].$$

Considering $\boldsymbol{T} = \frac{u_1}{\|u_1\|}$ and $\boldsymbol{N} = \frac{u_2}{\|u_2\|}$, we can find the orthonormal basis $\{\boldsymbol{T}, \boldsymbol{N}\}$ as follows

$$\boldsymbol{T} = \left\{\frac{\alpha^{-\theta}\left(s\alpha^{2\theta} + scos(\pi\theta) + \pi sin(\pi\theta)\right)}{\sqrt{2s^2cos(\pi\theta) + \alpha^{-2\theta}\left(\pi^2 + s^2\left(1+\alpha^{4\theta}\right) + l_{s\alpha}\right)}}, \frac{\alpha^{-\theta}\left(-\pi cos(\pi\theta) + ssin(\pi\theta)\right)}{\sqrt{2s^2cos(\pi\theta) + \alpha^{-2\theta}\left(\pi^2 + s^2\left(1+\alpha^{4\theta}\right) + l_{s\alpha}\right)}}\right\} \tag{11}$$

where

$$l_{s\alpha} = 2\pi s\alpha^{2\theta}sin(\pi\theta),$$

$$\boldsymbol{N} = \left\{\frac{\pi cos(\pi\theta) - ssin(\pi\theta)}{\sqrt{\pi^2 + s^2\left(1+\alpha^{4\theta}\right) + 2s\alpha^{2\theta}\left(scos(\pi\theta) + \pi sin(\pi\theta)\right)}}, \frac{s\alpha^{2\theta} + scos(\pi\theta) + \pi sin(\pi\theta)}{\sqrt{\pi^2 + s^2\left(1+\alpha^{4\theta}\right) + 2s\alpha^{2\theta}\left(scos(\pi\theta) + \pi sin(\pi\theta)\right)}}\right\}. \tag{12}$$

Now, let $r : I \to \mathbb{R}^2$ be the Fibonacci curve and $\{\boldsymbol{T},\boldsymbol{N}\}$, k be Frenet vector fields and curvature of this curve, respectively. So, the relationship between the curvature of the Frenet vector fields is that

$$\begin{aligned} \boldsymbol{T}' &= vk\boldsymbol{N} \\ \boldsymbol{N}' &= -vk\boldsymbol{T} \end{aligned} \tag{13}$$

where $\|r'\| = v$. The formula $k = \frac{x'y''-x'y''}{\left[(x')^2+(y')^2\right]^{\frac{3}{2}}}$ is valid for planar curves. Thus, the curvature function of the Fibonacci curve is that

$$k(\theta) = \frac{\sqrt{5}\left[\pi\left(\pi^2+s^2\right)\alpha^{-2\theta}+3\pi s^2 cos(\pi\theta)+s\left(\pi^2-2s^2\right)sin(\pi\theta)\right]}{\left[\alpha^{-2\theta}\left\{\pi^2+s^2\left(1+\alpha^{4\theta}\right)+2s\alpha^{2\theta}\left(scos(\pi\theta)+\pi sin(\pi\theta)\right)\right\}\right]^{3/2}}. \tag{14}$$

Also, the parametric equation of evolute of the Fibonacci curve $r(\theta) = (x(\theta), y(\theta))$ is that

$$p(\theta) = (X(\theta), Y(\theta))$$

where

$$X(\theta) = \frac{\alpha^{-\theta}\left\{-2s\left(\pi^2+s^2\right)sin(\pi\theta)+2s\alpha^{4\theta}l_{s\pi}+\pi\alpha^{2\theta}\left[2\pi^2-s^2+s^2cos(2\pi\theta)+\pi ssin(2\pi\theta)\right]\right\}}{2\sqrt{5}\left\{\pi\left(\pi^2+s^2\right)+s\alpha^{2\theta}\left[3\pi scos(\pi\theta)+\left(\pi^2-2s^2\right)sin(\pi\theta)\right]\right\}},$$

$$l_{s\pi} = \left[4\pi scos(\pi\theta)+\left(\pi^2-3s^2\right)sin(\pi\theta)\right],$$

$$Y(\theta) = \frac{-\left\{s\alpha^{-\theta}\left\{-2S_{\pi\alpha}cos(\pi\theta)+\alpha^{2\theta}\left[-3\pi^2-2s^2\left(3+\alpha^{4\theta}\right)+\pi^2cos(2\pi\theta)-2\pi s\left(3\alpha^{2\theta}+cos(\pi\theta)\right)sin(\pi\theta)\right]\right\}\right\}}{2\sqrt{5}\left\{\pi\left(\pi^2+s^2\right)+s\alpha^{2\theta}\left[3\pi scos(\pi\theta)+\left(\pi^2-2s^2\right)sin(\pi\theta)\right]\right\}},$$

$$S_{\pi\alpha} = \left[\pi^2+s^2\left(1+3\alpha^{4\theta}\right)\right].$$

And the relationship between the Fibonacci curve and its evolute is given in Figure 1.

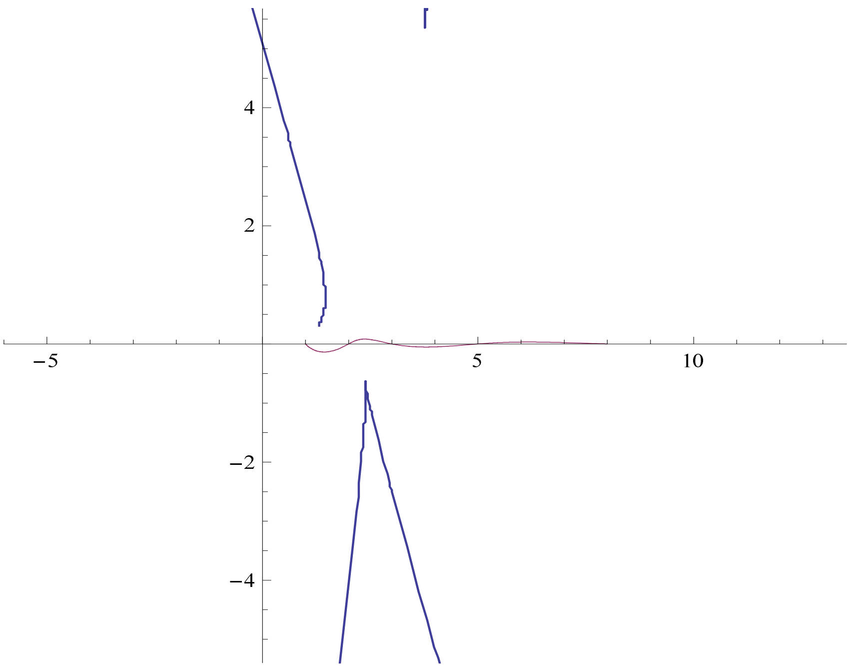


Figure 1. The Fibonacci Curve and its Evolute.

Also, the pedal curve of the Fibonacci curve $r(\theta) = (x(\theta), y(\theta))$ given in the equation (2.2) is as follows. In the Figure 2, the Pedal curves $Pd_r(\theta)$ of the Fibonacci curve $r(\theta)$ at the points (3,2), (4,2) and (8,2) are plotted.

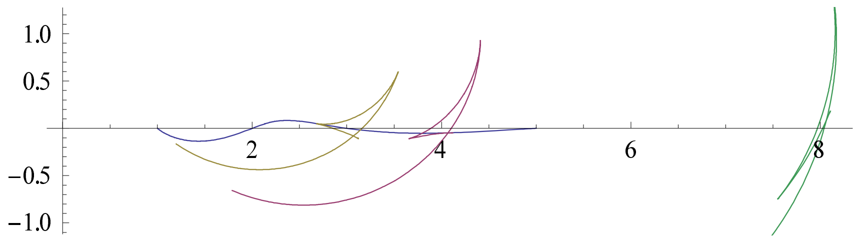


Figure 2. The Pedal Curves of the Fibonacci Curve.

Likewise, the Parallel curves $Pl_r(\theta)$ of the Fibonacci curve $r(\theta) = (x(\theta), y(\theta))$ given in the equation (2.3) is as follows. In the Figure 3, the distances between the Parallel curves and the Fibonacci curve are -0.5, 1 and 1,5, respectively.

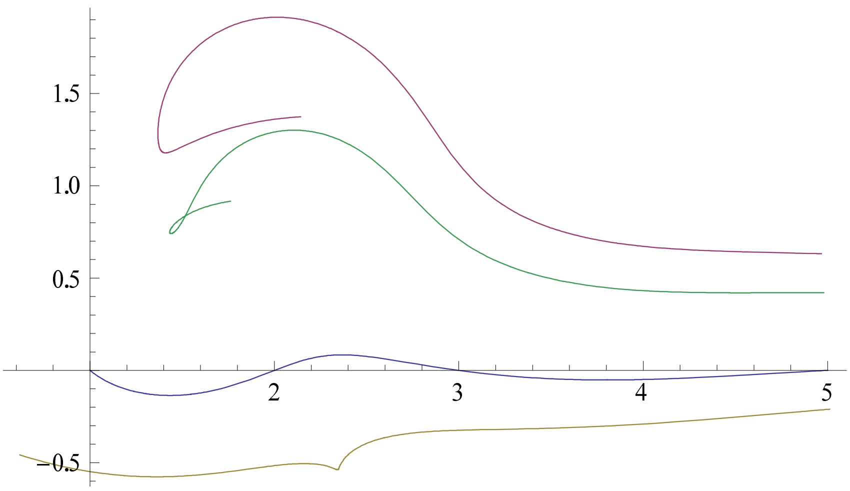


Figure 3. The Parallel Curves of the Fibonacci Curve.

4 The Lucas Curves

Let $I \subseteq \mathbb{R}$ be open interval of $\mathbb{R}$. So, the ***Lucas curve*** is that

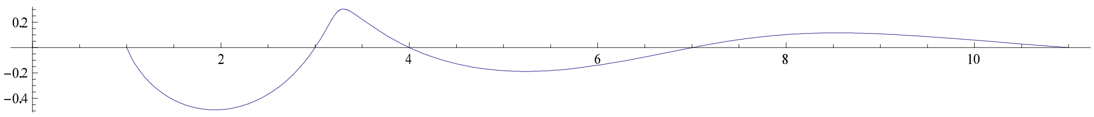
$$\begin{aligned} r : I &\rightarrow \mathbb{R}^2 \\ \theta &\rightarrow r(\theta) = (x(\theta), y(\theta)). \end{aligned}$$

If we consider $\alpha = \frac{1+\sqrt{5}}{2}$, it is obvious that

$$x(\theta) = \alpha^{\theta} + \alpha^{-\theta} cos(\theta\pi), \tag{15}$$

$$y(\theta) = \alpha^{-\theta} sin(\theta\pi). \tag{16}$$

And the graph of the Lucas curve is that



Firstly, if we take the derivative of the equations (4.1) and (4.2) with respect to θ, we can find that

$$\frac{dx}{d\theta} = \left(\frac{1+\sqrt{5}}{2}\right)^{-\theta}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2\theta}\log\left(\frac{1+\sqrt{5}}{2}\right) - \log\left(\frac{1+\sqrt{5}}{2}\right)cos\left(\pi\theta\right) - \pi sin\left(\pi\theta\right)\right] \tag{17}$$

$$\frac{dy}{d\theta} = \left(\frac{1+\sqrt{5}}{2}\right)^{-\theta}\left[\pi cos\left(\pi\theta\right) - \log\left(\frac{1+\sqrt{5}}{2}\right)sin\left(\pi\theta\right)\right]. \tag{18}$$

Considering $\alpha = \frac{1+\sqrt{5}}{2}$ and $\log(\frac{1+\sqrt{5}}{2}) = \text{s}$ in the equations (4.3) and (4.4) we have

$$\frac{dx}{d\theta} = x'\left(\theta\right) = \alpha^{-\theta}\left[s\alpha^{2\theta} - scos\left(\pi\theta\right) - \pi sin\left(\pi\theta\right)\right], \tag{19}$$

$$\frac{dy}{d\theta} = y'\left(\theta\right) = \alpha^{-\theta}\left[\pi cos\left(\pi\theta\right) - ssin\left(\pi\theta\right)\right]. \tag{20}$$

Again calculating the derivative of the equations (4.5) and (4.6) with respect to θ, we have

$$\frac{d^2x}{d\theta^2} = x''\left(\theta\right) = \alpha^{-\theta}\left[\left(-\pi^2+s^2\right)cos\left(\pi\theta\right) + s\left(s\alpha^{2\theta} + 2\pi sin\left(\pi\theta\right)\right)\right], \tag{21}$$

$$\frac{d^2y}{d\theta^2} = y''\left(\theta\right) = \alpha^{-\theta}\left[-2\pi scos\left(\pi\theta\right) + (-\pi^2+s^2)sin\left(\pi\theta\right)\right]. \tag{22}$$

Now, we will construct the Frenet vector fields of the Lucas curve.

Let $v_1 = \{x'\left(\theta\right), y'\left(\theta\right)\}$ and $v_2 = \{x''\left(\theta\right), y''\left(\theta\right)\}$ be linear independent vectors. We obtain the orthogonal set $\{u_1, u_2\}$ from the linear independent set $\{v_1, v_2\}$ using the Gram-Schmidt method. So, we can write $v_1 = u_1 = \{x'\left(\theta\right), y'\left(\theta\right)\}$ and $u_2 = v_2 - \frac{\langle u_1, v_2\rangle}{\langle u_1, u_2\rangle}u_1$. If we make necessary arrangement in the latter equation, we get

$$u_2 = \left[u_{21}, \frac{\alpha^{-\theta}(u_{22} + -s\alpha^{2\theta}\left[-3\pi\left(\pi^2+s^2\right) + \pi\left(\pi^2-5s^2\right)cos\left(2\pi\theta\right) + 2s\left(-2\pi^2+s^2\right)sin\left(2\pi\theta\right)\right])}{2\left[\pi^2+s^2\left(1+\alpha^{4\theta}\right) - 2s\alpha^{2\theta}\left(scos\left(\pi\theta\right) + \pi sin\left(\pi\theta\right)\right)\right]}\right\},$$

where

$$u_{21} = -\frac{\alpha^{-\theta}\left[\pi cos\left(\pi\theta\right) - ssin\left(\pi\theta\right)\right]\left[\pi\left(\pi^2+s^2\right) + s\alpha^{2\theta}\left[-3\pi scos\left(\pi\theta\right) - \left(\pi^2-2s^2\right)sin\left(\pi\theta\right)\right]\right]}{\pi^2+s^2\left(1+\alpha^{4\theta}\right) - 2s\alpha^{2\theta}\left(scos\left(\pi\theta\right) + \pi sin\left(\pi\theta\right)\right)},$$

and

$$u_{22}=-2\pi\left(\pi^2+s^2\right)\left(scos\left(\pi\theta\right)+\pi sin\left(\pi\theta\right)\right)-2s^2\alpha^{4\theta}\left(3\pi scos\left(\pi\theta\right)+\left(\pi^2-2s^2\right)sin\left(\pi\theta\right)\right).$$

Considering $\boldsymbol{T}=\frac{u_1}{\|u_1\|}$ and $\boldsymbol{N}=\frac{u_2}{\|u_2\|}$, we can find the orthonormal basis $\{\boldsymbol{T},\boldsymbol{N}\}$ as follows

$$\boldsymbol{T}=\left\{\frac{\alpha^{-\theta}\left(s\alpha^{2\theta}-scos\left(\pi\theta\right)-\pi sin\left(\pi\theta\right)\right)}{\sqrt{\alpha^{-2\theta}\left(\pi^2+s^2\left(1+\alpha^{4\theta}\right)\right)-2s\left(scos\left(\pi\theta\right)+\pi sin\left(\pi\theta\right)\right)}},\frac{\alpha^{-\theta}\left(\pi cos\left(\pi\theta\right)-ssin\left(\pi\theta\right)\right)}{\sqrt{\alpha^{-2\theta}\left(\pi^2+s^2\left(1+\alpha^{4\theta}\right)\right)-2s\left(scos\left(\pi\theta\right)+\pi sin\left(\pi\theta\right)\right)}}\right\} \tag{23}$$

and

$$\boldsymbol{N}=\left\{\frac{-\pi cos\left(\pi\theta\right)+ssin\left(\pi\theta\right)}{\sqrt{\pi^2+s^2+s^2\alpha^{4\theta}-2s^2\alpha^{2\theta}cos\left(\pi\theta\right)-2\pi s\alpha^{2\theta}sin\left(\pi\theta\right)}},\frac{st^{2\theta}-scos\left(\pi\theta\right)-\pi sin\left(\pi\theta\right)}{\sqrt{\pi^2+s^2\left(1+\alpha^{4\theta}\right)-2s\alpha^{2\theta}\left(scos\left(\pi\theta\right)+\pi sin\left(\pi\theta\right)\right)}}\right\}. \tag{24}$$

If we make necessary arrangements, we can easily find the curvature function of the Lucas curve that

$$k\left(\theta\right)=\frac{\pi\left(\pi^2+s^2\right)\alpha^{-2\theta}-3\pi s^2cos\left(\pi\theta\right)+s\left(-\pi^2+2s^2\right)sin\left(\pi\theta\right)}{\left[\alpha^{-2\theta}\left(\pi^2+s^2\left(1+\alpha^{4\theta}\right)\right)-2s\left(scos\left(\pi\theta\right)+\pi sin\left(\pi\theta\right)\right)\right]^{3/2}} \tag{25}$$

So, the relationship between the curvature of the Frenet vector fields is that

$$\begin{aligned}\boldsymbol{T}'&=vk\boldsymbol{N}\\ \boldsymbol{N}'&=-vk\boldsymbol{T}\end{aligned}. \tag{26}$$

where $\|l'\|=v$. Also, the parametric equation of the evolute of the Lucas curve $l\left(\theta\right)=\left(x\left(\theta\right),y\left(\theta\right)\right)$ is that $c\left(\theta\right)=\left(X\left(\theta\right),Y\left(\theta\right)\right)$ where

$$X\left(\theta\right)=\frac{-2s\left(\pi^2+s^2\right)sin\left(\pi\theta\right)+2s\alpha^{4\theta}\left[4\pi scos\left(\pi\theta\right)+\left(\pi^2-3s^2\right)sin\left(\pi\theta\right)\right]-\pi\alpha^{2\theta}\left[2\pi^2-s^2+s^2cos\left(2\pi\theta\right)+\pi ssin\left(2\pi\theta\right)\right]}{-2\pi\left(\pi^2+s^2\right)\alpha^{\theta}+2s\alpha^{3\theta}\left[3\pi scos\left(\pi\theta\right)+\left(\pi^2-2s^2\right)sin\left(\pi\theta\right)\right]},$$

$$Y\left(\theta\right)=\frac{s\left[-2\left[\pi^2+s^2\left(1+3\alpha^{4\theta}\right)\right]cos\left(\pi\theta\right)+\alpha^{2\theta}\left\{3\pi^2+2s^2\left(3+\alpha^{4\theta}\right)-\pi^2cos\left(2\pi\theta\right)+2\pi s\left[-3\alpha^{2\theta}+cos\left(\pi\theta\right)\right]sin\left(\pi\theta\right)\right\}\right]}{2\pi\left(\pi^2+s^2\right)\alpha^{\theta}-2s\alpha^{3\theta}\left[3\pi scos\left(\pi\theta\right)+\left(\pi^2-2s^2\right)sin\left(\pi\theta\right)\right]}.$$

Besides, the relation between the Lucas curve and its evolute is given in Figure 4 that

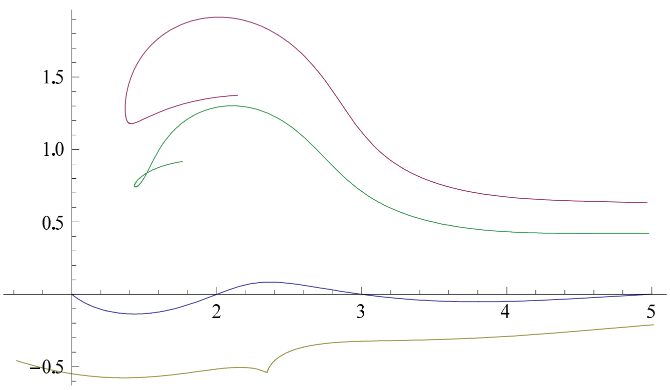


Figure 4. The Lucas Curves and its Evolute.

Also, the Pedal curve $Pd_l(\theta)$ of the Lucas curve $l(\theta) = (x(\theta), y(\theta))$ given in the equation (2.2) is as follows. In the Figure 5, the Pedal curves $Pd_l(\theta)$ of the Lucas curve at the points $(3,2)$ and $(10,2)$ are plotted.

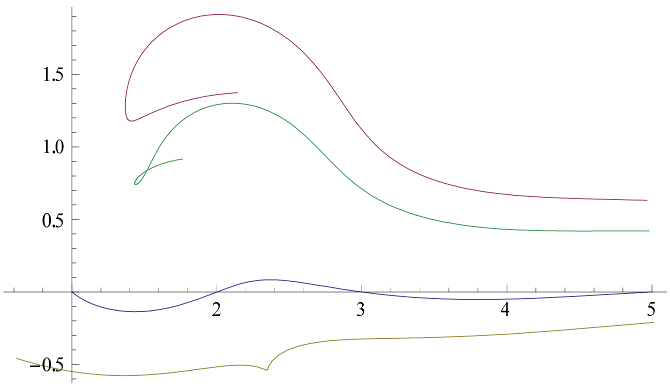


Figure 5. The Pedal Curves of the Lucas Curve.

Similarly, the Parallel curves $Pl_l(\theta)$ of the Lucas curve $l(\theta) = (x(\theta), y(\theta))$ given in the equation (2.3) is as follows. In the Figure 6, the distances between the Parallel curves and the Lucas curve are -0.5, 1 and 1,5, respectively.

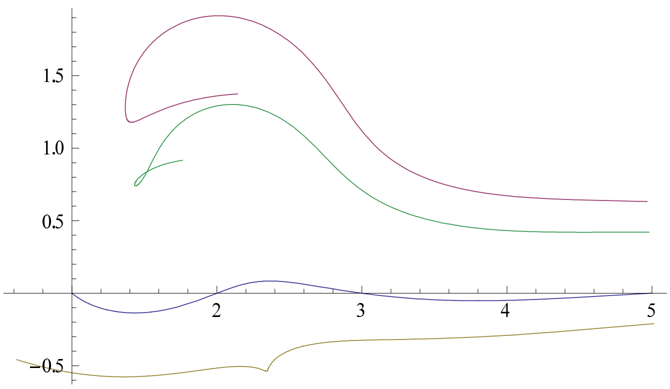


Figure 6. The Parallel Curves of the Lucas Curve.

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