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Some characterizations of space curves according to Bishop frame in Euclidean 3-space E^3

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Abstract: In this study, we give some characterizations of space curves according to Bishop frame in Euclidean 3-space E^3 by using Laplacian operator and Levi-Civita connection.

Keywords: Bishop frame, slant helix, Laplacian operator.

1 Introduction

In the differential geometry, a curve of constant slope or general helix in Euclidean 3-space E^3 is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A necessary and sufficient condition that a curve to be a general helix is that the ratio of curvature to torsion be constant[18]. The study of these curves in E^3 has been given by many mathematicians. In [10], Ilarslan studied the characterizations of helices in Minkowski 3-space E_1^3 and found the differential equations according to Frenet vectors characterizing the helices in E_1^3. Then, Kocayiğit obtained the general differential equations which characterize the Frenet curves in Euclidean 3-space E^3 and Minkowski 3-space E_1^3[14].

Recently, Izumiya and Takeuchi have defined a new special curve called the slant helix in Euclidean 3-space E^3 by the property that the principal normal of a space curve γ makes a constant angle with a fixed direction [11]. The spherical images of tangent indicatrix and binormal indicatrix of a slant helix have been studied by Kula and Yayli [15]. They obtained that the spherical images of a slant helix are spherical helices. Recently, Kula et al. studied the relations between a general helix and a slant helix[18]. They have found some differential equations which characterize the slant helix[16].

Bukcu and Karacan have defined slant helix according to Bishop frame in Euclidean 3-space [6]. Furthermore, they have given some necessary and sufficient conditions for the slant helix. Ali and Turgut have studied the position vectors of slant helices in Euclidean space E^3[1]. Also, they have given the generalization of the concept of a slant helix in the Euclidean n-space E^n[2].

Furthermore Chen and Ishikawa classified biharmonic curves, the curves for which $\Delta H = 0$ holds in semi-Euclidean space E_v^n where Δ is Laplacian operator and H is mean curvature vector field of a Frenet curve [8]. After them, Kocayiğit studied biharmonic curves and 1-type curves i.e., the curves for which $\Delta H = \lambda H$ holds, where λ is constant, in Euclidean 3-space E^3 and Minkowski 3-space E_1^3. He showed the relations between 1-type curves and circular helix and the relations between biharmonic curves and geodesics[14] Boros and Gray studied curves in Euclidean space with

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harmonic mean curvature vector[4]. Further in [13], Kılıç consider the curves with 1-type mean curvature vector in Euclidean space.

In this paper, we give some characterizations of space curves according to Bishop Frame by using Laplacian operator. We find the differential equations which characterize the space curves with this frame.

2 Preliminaries

Let M be an n-dimensional smooth manifold equipped with a metric $\langle,\rangle$. A tangent space $T_p(M)$ at a point $p \in M$ is furnished with the canonical inner product. If $\langle,\rangle$ is positive definite, then M is a Riemannian manifold. A curve on a Riemannian manifold M is a smooth mapping $\alpha : I \to M$, where I is an open interval in the real line R^1. As an open submanifold of R^1, I has a coordinate system consisting of the identity map u of I. The velocity vector of α at $s \in I$ is

$$\alpha'(s) = \left.\frac{d\alpha(u)}{du}\right|_s \in T_{\alpha(s)}M.$$

A curve $\alpha(s)$ is said to be regular if $\alpha'(s)$ is not equal to zero for any s. Let $\alpha(s)$ be a curve on M. Denote by $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ the moving Frenet frame along the curve $\alpha(s)$. Then $T,\ N$ and B are the tangent, the principal normal and binormal vectors of the curve α, respectively. If α is a space curve, then this set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

$$\begin{aligned}
&\alpha'(s) = T\\
&D_T T = \kappa N\\
&D_T N = -\kappa T + \tau B\\
&D_T B = -\tau N
\end{aligned}$$

where D denotes the covariant differentiation in M. In a Riemannian manifold M, a curve is described by the Frenet formula. For example, if all curvatures of a curve are identically zero, then the curve is called a geodesic. If only the curvature κ is a non-zero constant and the torsion τ is identically zero, then the curve is called a circle.

The parallel transport frame or bishop frame is an alternative approach to defining a moving frame that is well-defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $\mathbf{T}(t)$ for a given curve model is unique, we may choose convenient arbitrary basis $(\mathbf{N}_1(t),\mathbf{N}_2(t))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $\mathbf{T}(t)$ at each point. If the derivatives of $(\mathbf{N}_1(t),\mathbf{N}_2(t))$ depend only on $\mathbf{T}(t)$ and not each other, we can make $\mathbf{N}_1(t)$ and $\mathbf{N}_2(t)$ vary smoothly throughout the path regardless of the curvature. We therefore have the alternative frame equations (1) [14].

Denote by $\{\mathbf{T},\mathbf{N}_1,\mathbf{N}_2\}$ the moving Bishop frame along the curve $\gamma(s) : I \subset R \to E^3$ in the Euclidean 3-space E^3. For an arbitrary curve $\gamma(s)$ in the space E^3, the following Bishop formulae are given,

$$\begin{bmatrix} \nabla_{\gamma'}\overrightarrow{T} \\ \nabla_{\gamma'}\overrightarrow{N}_1 \\ \nabla_{\gamma'}\overrightarrow{N}_2 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T} \\ \overrightarrow{N}_1 \\ \overrightarrow{N}_2 \end{bmatrix} \tag{1}$$

In [5], the relations between κ,τ,θ and k_1,k_2 are given as follows

$$\kappa(s) = \sqrt{k_1^2 + k_2^2} \tag{2}$$

$$\theta(s) = arc\tan\left(\frac{k_2}{k_1}\right), \; k_1 \neq 0$$

$$\tau = \theta' = \frac{k_1 k_2' - k_1' k_2}{k_1^2 + k_2^2}. \tag{3}$$

So that k_1 and k_2 effectively correspond to Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ due to the differentiation [7].

A regular curve $\alpha : I \to E^3$ is called a slant helix provided the unit vector $N_1(s)$ of α has constant angle θ with some fixed unit vector u; that is, $\langle N_1(s), u \rangle = cos\theta$, for all $s \in I$.

Let, $\gamma : I \to E^3$ be a unit speed curve with nonzero nature curvatures k_1, k_2. Then γ is a slant helix if and only if $\frac{k_1}{k_2}(s)$ is constant[6].

Let, ∇ denotes the Levi-Civita connection given by $\nabla_{\gamma'} = \frac{d}{ds}$ where s is the arclength parameter of the curve γ. The Laplacian operator of γ is defined by

$$\Delta = -\nabla_{\gamma'}{}^2 = -\nabla_{\gamma'}\nabla_{\gamma'} \tag{4}$$

(see [8,9,17]).

3 Characterizations of space curves with respect to Bishop frame

In this section we will give the characterizations of the space curves according to Bishop frame in Euclidean 3-space E^3. Furthermore, we will obtain the general differential equations which characterize the space curves according to the vectors the $\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2$ in Euclidean 3-space.

Theorem 1. *Let γ be a unit speed curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N}_1, \overrightarrow{N}_2\right\}$, curvature k_1, torsion k_2. The differential equation characterizing according to the tangent vector T is given by*

$$\lambda_4 \nabla_{\gamma'}^3 \overrightarrow{T} + \lambda_3 \nabla_{\gamma'}^2 \overrightarrow{T} + \lambda_2 \nabla_{\gamma'} \overrightarrow{T} + \lambda_1 \overrightarrow{T} = 0$$

where

$$\lambda_4 = f,$$
$$\lambda_3 = g,$$
$$\lambda_2 = (k_1 k_2 z + t - k_2' k_1^3 + k_1' k_2^3),$$
$$\lambda_1 = 3fh + g(k_1^2 + k_2^2),$$

and

$$f = \left(\frac{k_1}{k_2}\right)' k_2^2, \; g = k_2'' k_1 - k_1'' k_2, \; h = k_1 k_1' + k_2 k_2', \; z = k_1 k_1' - k_2 k_2', \; t = k_1'' k_2' - k_2'' k_1'$$

Proof. Let γ be a unit speed curve with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N}_1, \overrightarrow{N}_2\right\}$ and k_1, k_2 be the curvature and torsion of the curve, respectively. By differentiating $\overrightarrow{T}$ three times with respect to s we find the followings,

$$\nabla_{\gamma'} \overrightarrow{T} = k_1 \overrightarrow{N}_1 + k_2 \overrightarrow{N}_2 \tag{5}$$

$$\nabla^2_{\gamma'}\overrightarrow{T} = -(k_1^2+k_2^2)\overrightarrow{T} + k_1'\overrightarrow{N}_1 + k_2'\overrightarrow{N}_2 \tag{6}$$

$$\nabla^3_{\gamma'}\overrightarrow{T} = -3(k_1k_1'+k_2k_2')\overrightarrow{T} - (k_1^3+k_1k_2^2-k_1'')\overrightarrow{N}_1 - (k_1^2k_2+k_2^3-k_2'')\overrightarrow{N}_2. \tag{7}$$

From (5) and (6) we have

$$\overrightarrow{N}_1 = \frac{-k_2(k_1^2+k_2^2)}{k_2'k_1-k_2k_1'}\overrightarrow{T} + \frac{k_2'}{k_2'k_1-k_2k_1'}\nabla_{\gamma'}\overrightarrow{T} - \frac{k_2}{k_2'k_1-k_2k_1'}\nabla^2_{\gamma'}\overrightarrow{T} \tag{8}$$

and

$$\overrightarrow{N}_2 = \frac{-k_1(k_1^2+k_2^2)}{k_2k_1'-k_2'k_1}\overrightarrow{T} + \frac{k_1'}{k_2k_1'-k_2'k_1}\nabla_{\gamma'}\overrightarrow{T} - \frac{k_1}{k_2k_1'-k_2'k_1}\nabla^2_{\gamma'}\overrightarrow{T}. \tag{9}$$

By substituting (8) and (9) in (7) we get

$$f\nabla^3_{\gamma'}\overrightarrow{T} + g\nabla^2_{\gamma'}\overrightarrow{T} + (k_1k_2z+t+k_1'k_2^3-k_2'k_1^3)\nabla_{\gamma'}\overrightarrow{T} + (3hf+g(k_1^2+k_2^2))\overrightarrow{T} = 0 \tag{10}$$

where $f=\left(\frac{k_1}{k_2}\right)'k_2^2$, $g=k_2''k_1-k_1''k_2$, $h=k_1k_1'+k_2k_2'$, $z=k_1k_1'-k_2k_2'$, $t=k_1''k_2'-k_2''k_1'$. By writing

$$\begin{aligned}
\lambda_4 &= f,\\
\lambda_3 &= g,\\
\lambda_2 &= (k_1k_2z+t-k_2'k_1^3+k_1'k_2^3),\\
\lambda_1 &= (3fh+g\kappa^2),
\end{aligned}$$

from (10) we get

$$\lambda_4\nabla^3_{\gamma'}\overrightarrow{T} + \lambda_3\nabla^2_{\gamma'}\overrightarrow{T} + \lambda_2\nabla_{\gamma'}\overrightarrow{T} + \lambda_1\overrightarrow{T} = 0$$

which is desired equation.

Theorem 2. *Let γ be a unit speed curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T},\overrightarrow{N}_1,\overrightarrow{N}_2\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing γ according to the vector $\overrightarrow{N}_1$ is given by*

$$\nabla^3_{\gamma'}\overrightarrow{N}_1 + \beta_3\nabla^2_{\gamma'}\overrightarrow{N}_1 + \beta_2\nabla_{\gamma'}\overrightarrow{N}_1 + \beta_1\overrightarrow{N}_1 = 0$$

where

$$\begin{aligned}
\beta_3 &= (-2\tfrac{k_1'}{k_1}-\tfrac{k_2'}{k_2}),\\
\beta_2 &= (2\tfrac{(k_1')^2}{k_1^2}+\tfrac{k_1'k_2'}{k_1k_2}-\tfrac{k_1''}{k_1}-k_2^2-k_1^2),\\
\beta_1 &= (k_1k_1'-\tfrac{k_1^2k_2'}{k_2}).
\end{aligned}$$

Proof. Let γ be a unit speed curve with Bishop frame $\left\{\overrightarrow{T},\overrightarrow{N}_1,\overrightarrow{N}_2\right\}$ and k_1, k_2 be curvature and torsion of the curve, respectively. By differentiating $\overrightarrow{N}_1$ three times with respect to s we find the followings,

$$\nabla_{\gamma'}\overrightarrow{N}_1 = -k_1\overrightarrow{T} \tag{11}$$

$$\nabla^2_{\gamma'}\overrightarrow{N}_1 = -k_1'\overrightarrow{T} - k_1^2\overrightarrow{N}_1 - k_1k_2\overrightarrow{N}_2 \tag{12}$$

$$\nabla^3_{\gamma'}\overrightarrow{N}_1 = (-k_1'' + k_1^3 + k_1k_2^2)\overrightarrow{T} + (-3k_1k_1')\overrightarrow{N}_1 + (-2k_1'k_2 - k_1k_2')\overrightarrow{N}_2. \tag{13}$$

From (11) and (12) we have

$$\overrightarrow{T} = -\frac{1}{k_1}\nabla_{\gamma'}\overrightarrow{N}_1 \tag{14}$$

and

$$\overrightarrow{N}_2 = -\frac{1}{k_1k_2}\nabla^2_{\gamma'}\overrightarrow{N}_1 + \frac{k_1'}{k_1^2k_2}\nabla_{\gamma'}\overrightarrow{N}_1 - \frac{k_1}{k_2}\overrightarrow{N}_1. \tag{15}$$

By substituting (14) and (15) in (13) we get

$$\nabla^3_{\gamma'}\overrightarrow{N}_1 + \left(-2\frac{k_1'}{k_1} - \frac{k_2'}{k_2}\right)\nabla^2_{\gamma'}\overrightarrow{N}_1 + \left(2\frac{(k_1')^2}{k_1^2} + \frac{k_1'k_2'}{k_1k_2} - \frac{k_1''}{k_1} + k_1^2 + k_2^2\right)\nabla_{\gamma'}\overrightarrow{N}_1 + \left(k_1k_1' - \frac{k_1^2k_2'}{k_2}\right)\overrightarrow{N}_1 = 0 \tag{16}$$

By writing

$$\begin{aligned}\beta_3 &= (-2\tfrac{k_1'}{k_1} - \tfrac{k_2'}{k_2}),\\ \beta_2 &= (2\tfrac{(k_1')^2}{k_1^2} + \tfrac{k_1'k_2'}{k_1k_2} - \tfrac{k_1''}{k_1} - k_2^2 - k_1^2),\\ \beta_1 &= (k_1k_1' - \tfrac{k_1^2k_2'}{k_2}).\end{aligned}$$

From (16) we get

$$\nabla^3_{\gamma'}\overrightarrow{N}_1 + \beta_3\nabla^2_{\gamma'}\overrightarrow{N}_1 + \beta_2\nabla_{\gamma'}\overrightarrow{N}_1 + \beta_1\overrightarrow{N}_1 = 0$$

which is desired equation.

If γ is a slant helix in E^3, then $\frac{k_1}{k_2} = const.$, that is $\frac{k_1'}{k_1} = \frac{k_2'}{k_2}$. In this case, we have $\beta_1 = 0$. Therefore, we give the following corollary.

Corollary 1. *Let γ be a slant helix in E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N}_1, \overrightarrow{N}_2\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing γ according to the vector $\overrightarrow{N}_1$ is given by*

$$\nabla^3_{\gamma'}\overrightarrow{N}_1 + \left(-3\frac{k_1'}{k_1}\right)\nabla^2_{\gamma'}\overrightarrow{N}_1 + \left(3\frac{(k_1')^2}{k_1^2} - \frac{k_1''}{k_1} + k_1^2 + k_2^2\right)\nabla_{\gamma'}\overrightarrow{N}_1 = 0$$

Theorem 3. *Let γ be a unit speed curve in Euclidean 3-space E^3 with Bishop frame $\left\{\overrightarrow{T}, \overrightarrow{N}_1, \overrightarrow{N}_2\right\}$, curvature k_1 and torsion k_2. The differential equation characterizing γ according to the vector $\overrightarrow{N}_2$ is given by*

$$\nabla^3_{\gamma'}\overrightarrow{N}_2 + \eta_3\nabla^2_{\gamma'}\overrightarrow{N}_2 + \eta_2\nabla_{\gamma'}\overrightarrow{N}_2 + \eta_1\overrightarrow{N}_2 = 0$$

where

$$\begin{aligned}\eta_3 &= (-2\tfrac{k_2'}{k_2} - \tfrac{k_1'}{k_1}),\\ \eta_2 &= (2\tfrac{(k_2')^2}{k_2^2} + \tfrac{k_1'k_2'}{k_1k_2} - \tfrac{k_2''}{k_2} - k_2^2 - k_1^2),\\ \eta_1 &= (k_2k_2' - \tfrac{k_2^2k_1'}{k_1}).\end{aligned}$$

*Proof.*Let γ be a unit speed curve with Bishop frame $\left\{\overrightarrow{T},\overrightarrow{N}_1,\overrightarrow{N}_2\right\}$ and k_1, k_2 be curvature and torsion of the curve, respectively. By differentiating $\overrightarrow{N}_2$ three times with respect to s we find the followings,

$$\nabla_{\gamma'}\overrightarrow{N}_2=-k_2\overrightarrow{T} \tag{17}$$

$$\nabla^2_{\gamma'}\overrightarrow{N}_2=-k_2'\overrightarrow{T}-k_2k_1\overrightarrow{N}_1-k_2^2\overrightarrow{N}_2 \tag{18}$$

$$\nabla^3_{\gamma'}\overrightarrow{N}_2=(-k_2''+k_2^3+k_2k_1^2)\overrightarrow{T}+(-2k_2'k_1-k_2k_1')\overrightarrow{N}_1+(-3k_2k_2')\overrightarrow{N}_2. \tag{19}$$

From (17) and (18) we have

$$\overrightarrow{T}=-\frac{\nabla_{\gamma'}\overrightarrow{N}_2}{k_2} \tag{20}$$

and

$$\overrightarrow{N}_1=-\frac{1}{k_1k_2}\nabla^2_{\gamma'}\overrightarrow{N}_2+\frac{k_2'}{k_2^2k_1}\nabla_{\gamma'}\overrightarrow{N}_2-\frac{k_2}{k_1}\overrightarrow{N}_2. \tag{21}$$

By substituting (20) and (21) in (19) we get

$$\nabla^3_{\gamma'}\overrightarrow{N}_2+\left(-2\frac{k_2'}{k_2}-\frac{k_1'}{k_1}\right)\nabla^2_{\gamma'}\overrightarrow{N}_2+\left(2\frac{(k_2')^2}{k_2^2}+\frac{k_1'k_2'}{k_1k_2}-\frac{k_2''}{k_2}+k_1^2+k_2^2\right)\nabla_{\gamma'}\overrightarrow{N}_2+\left(k_2k_2'-\frac{k_2^2k_1'}{k_1}\right)\overrightarrow{N}_2=0 \tag{22}$$

By writing

$$\begin{aligned}\eta_3&=(-2\tfrac{k_2'}{k_2}-\tfrac{k_1'}{k_1}),\\ \eta_2&=(2\tfrac{(k_2')^2}{k_2^2}+\tfrac{k_1'k_2'}{k_1k_2}-\tfrac{k_2''}{k_2}-k_2^2-k_1^2),\\ \eta_1&=(k_2k_2'-\tfrac{k_2^2k_1'}{k_1}).\end{aligned}$$

From (20) we get

$$\nabla^3_{\gamma'}\overrightarrow{N}_2+\eta_3\nabla^2_{\gamma'}\overrightarrow{N}_2+\eta_2\nabla_{\gamma'}\overrightarrow{N}_2+\eta_1\overrightarrow{N}_2=0$$

which is desired equation.

If γ is a slant helix in E^3 i.e., if $\frac{k_1'}{k_1}=\frac{k_2'}{k_2}$ then $\eta_1=0$. Therefore, we obtain the following corollary.

Corollary 2. *Let γ be a slant helix in E^3 with Bishop frame $\left\{\overrightarrow{T},\overrightarrow{N}_1,\overrightarrow{N}_2\right\}$, curvature k_1, torsion k_2. The differential equation characterizing γ according to the vector $\overrightarrow{N}_2$ is given by*

$$\nabla^3_{\gamma'}\overrightarrow{N}_2+\left(-3\frac{k_2'}{k_2}\right)\nabla^2_{\gamma'}\overrightarrow{N}_2+\left(3\frac{(k_2')^2}{k_2^2}-\frac{k_2''}{k_2}+k_1^2+k_2^2\right)\nabla_{\gamma'}\overrightarrow{N}_2=0.$$

4 Space curves with harmonic 1-type T,N_1,N_2 vectors in Euclidean 3-space E^3

In this section we will give the characterizations of the space curves with Harmonic 1-type $\overrightarrow{T},\overrightarrow{N}_1,\overrightarrow{N}_2$ vectors in Euclidean 3-space E^3.

Definition 1. *A regular curve γ in E^3 said to have harmonic tangent vector if*

$$\Delta \overrightarrow{T} = 0 \tag{23}$$

holds. Further, a regular curve γ in E^3 said to have harmonic 1-type tangent vector if

$$\Delta \overrightarrow{T} = \lambda \overrightarrow{T},\ \lambda \in R, \tag{24}$$

holds. First we prove the following theorem.

Theorem 4. *Let γ be a unit speed curve in E^3 with Bishop frame $\{T, N_1, N_2\}$. Then, γ is of harmonic 1-type tangent vector if and only if the curvature k_1 and the torsion k_2 of the curve γ satisfy the followings,*

$$\begin{cases} \lambda = k_1^2 + k_2^2, \\ k_1 = c_1, \\ k_2 = c_2. \end{cases} \tag{25}$$

where λ, c_1, c_2 are constants.

Proof. Let γ be a unit speed curve in E^3 with tangent vector $\overrightarrow{T}$ and let Δ be the Laplacian associated with ∇. One can use (5) and (6) to compute

$$\Delta \overrightarrow{T} = (k_1^2 + k_2^2)\overrightarrow{T} - k_1' \overrightarrow{N}_1 - k_2' \overrightarrow{N}_2 \tag{26}$$

We assume that the curve γ is of harmonic 1-type tangent vector. Substituting (26) in (24) we have (25).

Conversely, if the equations (25) satisfy for the constant λ, then it is easy to show that γ is of harmonic 1-type tangent vector.

Corollary 3. *Let γ be a unit speed curve in E^3 with Bishop frame $\{T, N_1, N_2\}$. Then, γ is of harmonic 1-type tangent vector if and only if γ is a slant helix, with constant curvature and constant torsion.*

Corollary 4. *Let γ be a unit speed curve in E^3 with Bishop frame $\{T, N_1, N_2\}$. Then,it has a harmonic tangent vector, if and only if*

$$k_1(s) = k_2(s) = 0.$$

Let, now consider the characterization of the curve γ according to the vector $\overrightarrow{N}_1$. Similar to Definition 1., we can give the following definition.

Definition 2. *A regular curve γ in E^3 said to have harmonic vector $\overrightarrow{N}_1$ if*

$$\Delta \overrightarrow{N}_1 = 0 \tag{27}$$

holds. Further, a regular curve γ in E^3 said to have harmonic 1-type vector $\overrightarrow{N}_1$ if

$$\Delta \overrightarrow{N}_1 = \lambda \overrightarrow{N}_1, \lambda \in R \tag{28}$$

holds.

Theorem 5. *Let γ be a unit speed curve in E^3 with Bishop frame $\{T, N_1, N_2\}$. Then, γ is of harmonic 1-type vector $\overrightarrow{N}_1$ if and only if the curvature k_1 and the torsion k_2 of the curve γ satisfy the followings,*

$$\begin{cases} \lambda = k_1^2, \\ k_1 = const., \\ k_2 = 0. \end{cases} \tag{29}$$

Proof. Let γ be a unit speed curve in E^3 with vector $\overrightarrow{N}_1$ and let Δ be the Laplacian associated with ∇. One can use (5) and (6) to compute

$$\Delta \overrightarrow{N}_1 = k_1' \overrightarrow{T} + k_1^2 \overrightarrow{N}_1 + k_1 k_2 \overrightarrow{N}_2 \tag{30}$$

We assume that the curve γ is of harmonic 1-type vector $\overrightarrow{N}_1$. Substituting (30) in (28) we have (29).

Conversely, if the equations (29) satisfy for the constant λ, then it is easy to show that γ is of harmonic 1-type vector $\overrightarrow{N}_1$.

Corollary 5. *Let γ be a unit speed curve in E^3 with Bishop frame $\{T, N_1, N_2\}$. Then, γ has harmonic vector $\overrightarrow{N}_1$, if and only if*

$$k_1(s) = 0.$$

Finally, let give the characterization of the curve γ according to the vector $\overrightarrow{N}_2$.

Definition 3. *A regular curve γ in E^3 said to have harmonic vector $\overrightarrow{N}_2$ if*

$$\Delta \overrightarrow{N}_2 = 0 \tag{31}$$

holds. Further, a regular curve γ in E^3 said to have harmonic 1-type vector $\overrightarrow{N}_2$ if

$$\Delta \overrightarrow{N}_2 = \lambda \overrightarrow{N}_2, \lambda \in R \tag{32}$$

holds.

Theorem 6. *Let γ be a unit speed curve in E^3 with Bishop frame T, N_1, N_2 Then, γ is of harmonic 1-type vector N_2 if and only if the curvature and the torsion of the curve satisfy the followings,*

$$\begin{cases} \lambda = k_2^2, \\ k_2 = const., \\ k_1 = 0. \end{cases} \tag{33}$$

Proof. Let γ be a unit speed curve in with vector E^3 and let Δ be the Laplacian associated with ∇. One can use (5) and (6) to compute

$$\Delta \overrightarrow{N}_2 = k_2' \overrightarrow{T} + k_1 k_2 \overrightarrow{N}_1 + k_2^2 \overrightarrow{N}_2 \tag{34}$$

We assume that the curve γ is of harmonic 1-type vector $\mathbf{N}_2$. Substituting (34) in (32) we have (33).

Conversely, if the equations (32) satisfy for the constant λ, then it is easy to show that γ is of harmonic 1-type vector $\overrightarrow{N}_2$.

Corollary 6. *Let γ be a unit speed curve in E^3 with Bishop frame $\{T, N_1, N_2\}$. Then, γ has harmonic vector $\overrightarrow{N}_2$, if and only if*

$$k_2(s) = 0.$$

Let now consider the general characterizations of a Bishop curve γ according to the Laplacian operator Δ. Then, by considering the vectors $\overrightarrow{T}$, $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ we obtain the followings.

Theorem 7. *Let γ be a unit speed curve in E^3 with Bishop frame $\{T, N_1, N_2\}$. Then,*

$$\Delta \overrightarrow{T} + \lambda \nabla_{\gamma} \mathbf{T} + \mu \overrightarrow{T} = 0, \tag{35}$$

holds along the curve γ for the constants λ and μ, if and only if γ is a slant helix, with curvature and the torsion

$$k_1 = ck_2,$$

$$k_2 = \mp i \sqrt{\frac{\mu}{c^2+1}},$$

where c is constants.

*Proof.*Assume that (35) holds along the curve γ. Then from the equalities (5), (6) and (35) we have

$$\begin{array}{l} k_1^2 + k_2^2 + \mu = 0 \\ -k_1' + \lambda k_1 = 0 \\ -k_2' + \lambda k_2 = 0 \end{array} \tag{36}$$

The second and third equation of the system (36) gives that $\frac{k_1}{k_2}$ is constant, i.e., γ is a slant helix. Furthermore, from the equations of the system (36) we get

$$k_1 = ck_2 \tag{37}$$

and

$$k_2 = \mp i \sqrt{\frac{\mu}{c^2+1}} \tag{38}$$

where c is constant.

Conversely, if γ is a slant helix with curvature k_1 and torsion k_2 given by (37) and (38), respectively, it is easily seen that (36) holds.

Theorem 8. *Let γ be a unit speed curve in E^3 and μ be a nonzero constant. Then,*

$$\Delta \overrightarrow{N}_1 + \mu \mathbf{N}_1 = 0, \tag{39}$$

holds along the curve γ if and only if,

$$k_1 = \mp i\sqrt{\mu}, k_2 = 0 \tag{40}$$

Proof. Assume that (39) holds along the curve γ. Then from the equality (30) we have

$$\begin{array}{l} k_1^2 + \mu = 0 \\ k_1 \cdot k_2 = 0 \end{array} \tag{41}$$

From the equations of the system (41) we get $k_1 = \mp i\sqrt{\mu}$ and $k_2 = 0$.

Conversely, if (41) holds then (39) is satisfied.

Theorem 9. *Let γ be a unit speed curve in E^3 and ρ be a nonzero constant. Then,*

$$\Delta \overrightarrow{N}_2 + \rho \overrightarrow{N}_2 = 0, \tag{42}$$

holds along the curve γ if and only if

$$k_2 = \mp i\sqrt{\rho}, k_1 = 0. \tag{43}$$

Proof. Assume that (39) holds along the curve γ. Then from the equality (34) we have

$$\begin{aligned} k_2^2 + \rho &= 0 \\ k_1 \cdot k_2 &= 0 \end{aligned} \tag{44}$$

From the equations of the system (44) we get

$$k_2 = \mp i\sqrt{\rho}, k_1 = 0 \tag{45}$$

Conversely if (45) holds, then it is easily seen that (39) is satisfied.

Example 1. Let consider the curve $\gamma(s) : I \to E^3$ given by the parameterization $\gamma(s) = \left(\frac{4}{5}\cos s,\ 1 - \sin s,\ -\frac{3}{5}\cos s\right)$. The curvature and the torsion of $\gamma(s)$ are $\kappa = 1,\ \ \tau = 0$, respectively. Then from (2) and (3) we have $\theta = 0$, $\frac{k_2}{k_1} = 0$ i.e., $k_2 = 0$. Thus $\gamma(s)$ is a slant helix according to Bishop Frame. Furthermore, from Corollary 4.4, $\gamma(s)$ is of harmonic type vector $\overrightarrow{N}_2$.

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