New Trends in Mathematical Sciences
http://www.ntmsci.com

Bernoulli collocation method for high-order generalized pantograph equations

Aysegul Akyuz-Dascioglu[1] *and Mehmet Sezer*[2]

[1]Department of Mathematics, Faculty of Science, Pamukkale University, Denizli, Turkey
[2]Department of Mathematics, Faculty of Science, Celal Bayar University, Manisa, Turkey

Received: 30 January 2015, Revised: 31 January 2015, Accepted: 10 February 2015
Published online: 14 February 2015

Abstract: In this paper, an approximate method based on Bernoulli polynomials has been presented to obtain the solution of generalized pantograph equations with linear functional arguments. Both initial and boundary value problems have been solved by this collocation technique. Approximate solution can also be corrected with the residual function. Some numerical examples have been given to illustrate the reliability and efficiency of the method.

Keywords: Pantograph equations, Functional differential equations, Bernoulli polynomials, Collocation method.

1 Introduction

Pantograph equations are a kind of functional differential equations, the name of which is originated from the study on the collection of current by the pantograph head of an electric locomotive [1]. These equations play an important role in many areas such as economy, biology, astrophysics, control and electrodynamics [1-4]. Properties of the analytic solution of pantograph equations have been given by several authors [5-11]. A numerical solution of such equations with high-order has not been studied in detail. In recent years, there has been a growing interest in the numerical solution of pantograph equations. Taylor methods have been used to find the approximate solutions of difference, differential-difference, integro-difference and pantograph equations [12-16]. High-order pantograph equations with initial conditions have been also studied by Taylor method [17], Adomian decomposition method [18], differential transform method [19] and homotopy perturbation method [20]. Besides, a subdivision approach for second-order functional differential equations with boundary conditions has been given in [21].

In this study, our purpose is to develop a method based on the Bernoulli polynomials to obtain the solution of the generalized pantograph equation

$$y^{(m)}(t)+\sum_{j=0}^{J}\sum_{k=0}^{m-1}p_{jk}(t)y^{(k)}(\alpha_j t+\beta_j)=f(t), \quad a\leq t\leq b \tag{1}$$

with initial conditions

$$\sum_{k=0}^{m-1}c_{ik}y^{(k)}(c)=\lambda_i, \quad c\in[a,b] \tag{2}$$

* Corresponding author e-mail: aysegulakyuz@yahoo.com

or boundary conditions

$$\sum_{k=0}^{m-1} a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b)=\mu_i, \quad i=1,2,\ldots,m. \tag{3}$$

Here $p_{jk}(t)$ and $f(t)$ are known functions defined on the finite interval $[a,b]$, c_{ik}, a_{ik}, b_{ik}, λ_i, μ_i, $\alpha_j$and β_j are real or complex constants.

We assume that the solution is expressed by the Bernoulli series

$$y(t)=\sum_{n=0}^{\infty} y_n B_n(t)$$

where $B_n(t)$ are the Bernoulli polynomials defined as

$$B_n(t)=\sum_{r=0}^{n}\binom{n}{r}B_r t^{n-r}, \quad B_r=B_r(0) \text{ (Bernoulli numbers)},$$

and y_n are unknown coefficients.

2 Fundamental matrix relations

Firstly, we consider the desired solution $y(t)$ of Eq. (1) approximated by a truncated Bernoulli series expansion of the form

$$y(t)\cong y_N(t)=\sum_{n=0}^{N} y_n B_n(t). \tag{4}$$

Here N is any positive integer such that $N\geq m$. Then we can put the finite series (4) in the matrix form as

$$y(t)\cong B(t)\mathrm{Y}$$

where $\mathrm{B}(t)=\left[B_0(t)\ B_1(t)\ \cdots\ B_N(t)\right]$,

$$\mathrm{Y}=\left[y_0\ y_1\ \cdots\ y_N\right]^T.$$

Differentiating and using the following relation [22]

$$B_n{}'(x)=nB_{n-1}(x),$$

we obtain

$$y'(t)\cong B(t)\mathrm{AY}$$

so that $B'(t)=B(t)\mathrm{A}$ and

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\mathrm{x}(N+1)}.$$

Similarly, the matrix form of the second derivative becomes

$$y''(t) \cong \mathrm{B}(t)\mathrm{A}^2\mathrm{Y}.$$

By continuing, we get the derivatives of the unknown function in the forms

$$y^{(m)}(t) \cong B(t)\mathrm{A}^m Y \tag{5}$$

and

$$y^{(k)}(\alpha_j t+\beta_j) \cong B(\alpha_j t+\beta_j)\mathrm{A}^k Y,\ k=0,1,2,\ldots,m-1. \tag{6}$$

3 Method of solution

For constructing the main matrix equation, we first substitute the matrix relations (5) and (6) into Eq. (1). Then, by using the collocation points t_s $(s=0,1,...,S)$, we obtain the system of the matrix equations as

$$B(t_s)\mathrm{A}^m Y+\sum_{j=0}^{J}\sum_{k=0}^{m-1} p_{jk}(t_s)B(\alpha_j t_s+\beta_j)\mathrm{A}^k Y = f(t_s).$$

Thus, the main matrix equation becomes

$$\left\{\mathrm{B}\mathrm{A}^m+\sum_{j=0}^{J}\sum_{k=0}^{m-1}\mathrm{P}_{jk}\mathrm{B}_j\mathrm{A}^k\right\}Y=F \tag{7}$$

where

$$\mathrm{P}_{jk}=\begin{bmatrix} p_{jk}(t_0) & 0 & \cdots & 0 \\ 0 & p_{jk}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{j_k}(t_S) \end{bmatrix},\ F=\begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_S) \end{bmatrix},\ \mathrm{B}=\begin{bmatrix} B_0(t_0) & B_1(t_0) & \cdots & B_N(t_0) \\ B_0(t_1) & B_1(t_1) & \cdots & B_N(t_1) \\ \vdots & \vdots & & \vdots \\ B_0(t_S) & B_1(t_S) & \cdots & B_N(t_S) \end{bmatrix}$$

$$\mathrm{B}_j=\begin{bmatrix} B(\alpha_j t_0+\beta_j) \\ B(\alpha_j t_1+\beta_j) \\ \vdots \\ B(\alpha_j t_S+\beta_j) \end{bmatrix}=\begin{bmatrix} B_0(\alpha_j t_0+\beta_j) & B_1(\alpha_j t_0+\beta_j) & \cdots & B_N(\alpha_j t_0+\beta_j) \\ B_0(\alpha_j t_1+\beta_j) & B_1(\alpha_j t_1+\beta_j) & \cdots & B_N(\alpha_j t_1+\beta_j) \\ \vdots & \vdots & & \vdots \\ B_0(\alpha_j t_S+\beta_j) & B_1(\alpha_j t_S+\beta_j) & \cdots & B_N(\alpha_j t_S+\beta_j) \end{bmatrix}.$$

Hence, the fundamental matrix equation (7) for Eq. (1) can be written in the compact form

$$\mathrm{WY} = \mathrm{F}or[\mathrm{W};\mathrm{F}] \tag{8}$$

This corresponds to a system of $S+1$ linear algebraic equations with the unknown Bernoulli coefficients $y_0, y_1, \ldots, y_N$. To determine the coefficients uniquely, m equations are needed. These equations are obtained from the conditions as follows:

By means of the relation (6), matrix forms of the conditions (2) and (3) can be written, respectively

$$\sum_{k=0}^{m-1} c_{ik} B(c) \mathrm{A}^k Y = \lambda_i,\ i = 1,2,...,m,$$

$$\sum_{k=0}^{m-1} a_{ik} B(a) A^k Y + b_{ik} B(b) A^k Y = \mu_i,$$

or compactly,

$$U_i Y = \eta_i, \quad [U_i;\eta_i] \equiv \left[u_{i,0}\ u_{i,1}\ u_{i,2}\ \ldots\ u_{i,N}\ ;\ \eta_i \right] \tag{9}$$

where η_i is the right side any of the conditions (2) or (3) to be used.

To obtain the solution of Eq. (1) under the given conditions, we combine the matrices (8) and (9). Adding the elements of the row matrices (9) to the end of the matrix (8), we have the new augmented matrix $\left[\tilde{\mathrm{W}};\tilde{\mathrm{F}}\right]$. Alternatively, we can replace m rows of the augmented matrix (8) with the row matrices (9), and denote the new augmented matrix by $[W^*, F^*]$. These linear systems can be solved by the standard methods such as matrix inversion, Gauss elimination and LU decomposition. Thus, the matrix $\mathbf{Y}$, which gives the coefficients of the Bernoulli polynomial solution (4), is determined. If the solution of the problem is a polynomial of degree at most N, then the method determines its coefficients. Otherwise, an approximate polynomial solution is obtained.

If the number of the collocation points equals to N-m+1, say S=N-m, we obtain a squared system with dimension N+1 by adding the entries belong to the conditions. However, if S=N, this system becomes rectangular. In this case, in order to form a linear system of $N+1$ equation with $N+1$ unknown, we replace some rows instead of adding. Note that the location of the rows to be deleted affects the numerical results.

4 Accuracy of solution and error estimation with residual correction

We can easily check the accuracy of the method as follows. As the truncated Bernoulli series (4) is an approximate solution of Eq. (1), the function $y_N(t)$ must satisfy the pantograph equation approximately; that is, the remainder term is

$$R_N(t) = y_N^{(m)}(t) + \sum_{j=0}^{J} \sum_{k=0}^{m-1} p_{jk}(t) y_N^{(k)}(\alpha_j t + \beta_j) - f(t) \tag{10}$$

Numerical results at the collocation points are affected only by round-off errors that we consider as negligible. Thus, the remainder term must be zero at the collocation points t_s, that is $R_N(t_s) = 0$, since $y^{(k)}(t_s) = y_N^{(k)}(t_s), \quad k = 0,1,\ldots,m.$

When N is sufficiently large enough, the error decreases. If $d-1$ exact decimal digits are required for the solution, then the truncation limit N is increased until

$$\max_n |R_N(t_n)| \le 10^{-d},\, t_n \in [a,b],$$

where t_n are any points different from the collocation points.

If the exact solution is known, then the error function is the difference between the approximate and the exact solutions defined by

$$e_N(t) = y(t) - y_N(t),$$

otherwise it can be estimated by following.Here we analyze the error function of $y_N(t)$and show that it can be obtained using the residual function of the operator equation,

$$L[y(t)] = f(t), \quad L[y(t)] = y^{(m)}(t) + \sum_{j=0}^{J}\sum_{k=0}^{m-1} p_{jk}(t)y^{(k)}(\alpha_j t + \beta_j).$$

We know the residual function $R_N(t)$

$$L[y_N(t)] - f(t) = R_N(t), \quad a \le t \le b.$$

Since L is a linear operator, the error function $e_N(t)$ satisfies the equation

$$L[e_N(t)] = -R_N(t), \quad a \le t \le b$$

with homogenous initial conditions

$$\sum_{k=0}^{m-1} c_{ik}{e_N}^{(k)}(c) = 0, \quad c \in [a,b],$$

or homogenous boundary conditions

$$\sum_{k=0}^{m-1} a_{ik}{e_N}^{(k)}(a) + b_{ik}{e_N}^{(k)}(b) = 0, \quad i = 1,2,\ldots,m.$$

Solving the new equation with homogenous conditions, we approximate the error function. Denoting this by $e_{N,M}(t)$, $y_N(t)$ can be improved. So, we have a new approximation $y_{N,M}(t)$ given by

$$y_{N,M}(t) = y_N(t) + e_{N,M}(t).$$

Consequently, we construct a new error function

$$E_{N,M}(t) = e_N(t) - e_{N,M}(t) = y(t) - y_{N,M}(t),$$

where $e_{N,M}(t)$ is the estimated error function and $E_{N,M}(t)$ is the corrected error function.
The residual correction method is similar to the deferred correction method [23]. We could repeat the procedure:
i. form the residual function,
ii. solve the error equation,
iii. form a new corrected approximation.
Note that, if the error equation are solved again by Bernoulli collocation method, the new truncation limit M must be chosen greater than N. If we choose $M = N$, the collocation points are taken different from the previous one. Otherwise, we

obtain trivial solution because of $R_N(t_s)=0$ and homogeneous conditions, and therefore we can not have any estimation. For example, at the first approximation, taking the collocation pointst_s defined by

$$t_s = a + s(b-a)/(N-m) \quad s = 0,1,\ldots,N-m,$$

and then choosing the new collocation points by

$$t_s = a + s(b-a)/M, \quad s = 0,1,\ldots,M,$$

we obtain the improved results. However, dimensions of the linear algebraic system are different although the truncation limits are the same. Dimension of the augmented matrix W̃ is (N+1)x(N+1) at the first approach whereas (N+m+1)x(N+1) at the second approach.

5 Numerical examples

The presented method is useful in finding the numerical solution of generalized pantograph equations in terms of Bernoulli polynomials. We illustrate the numerical solution with the following six examples. All the numerical computations have been done using a program written in Mathcad 15. The results obtained by the presented method are compared with the known results. The accuracy of solutions and the error analysis are performed as well.

Example 1. Let us first consider the pantograph equation of the second order

$$y''(t) = \tfrac{3}{4}y(t) + y\left(\tfrac{1}{2}t\right) - t^2 + 2, \quad y(0) = 0,\, y'(0) = 0, \quad 0 \le t \le 1$$

with the exact solution $y(t) = t^2$. Here $m = 2$, $p_{00}(t) = \frac{-3}{4}$, $p_{10}(t) = -1$, $\alpha_0 = 1$, $\alpha_1 = \frac{1}{2}$, $\beta_0 = \beta_1 = 0$, $f(t) = -t^2 + 2$. Following the procedure in Section 3, the fundamental matrix equation of the problem is

$$WY = \left\{\mathrm{BA}^2 - \tfrac{3}{4}\mathrm{B}_0 - \mathrm{B}_1\right\} Y = F.$$

Taking $N = 2$ and using the collocation points $t_0 = 0, t_1 = 1/2, t_2 = 1$, we have the augmented matrix

$$[W,F] = \begin{bmatrix} -7/4 & 7/8 & 41/24\,; & 2 \\ -7/4 & 1/4 & 25/12\,; & 7/4 \\ -7/4 & -3/8 & 47/24\,; & 1 \end{bmatrix},$$

and the matrix forms of the initial conditions are

$$[U_1;\lambda_1] = \left[1 \;\; -\tfrac{1}{2} \;\; \tfrac{1}{6} \;;\; 0\right], \quad [U_2;\lambda_2] = \left[0 \;\; 1 \;\; -1 \;;\; 0\right].$$

We can combine the matrix form of the pantograph equation with matrix form of the conditions in different ways. One is to replace the last two rows of $[W,F]$ with $[U_i;\lambda_i]$. Solving this system $[W^*,F^*]$ or the other system $\left[\tilde{W},\tilde{F}\right]$, Bernoulli coefficients matrix is obtained as

$$Y = \left[1/3 \;\; 1 \;\; 1\right]^T.$$

Thereby, we find the exact solution.

Average relative errors by the wavelet method [24] for N=15 and N= 255 at the points $t_s = s/16$, $s = 1,2,\ldots,16$ are

4.92E-4 and 1.96E-6, respectively. However, our average relative errors for all N (solved until 100) at the same points are zero with the zero tolerance 50. Although the exact solution is obtained for N=4 by Taylor method [17], it is found for N=2 by the presented method.

Example 2. Consider the linear delay differential equation of the first order

$$y'(t)+y(t)+y(0.8t)=0, y(0)=1. \tag{11}$$

By means of the Eq. (7), the fundamental matrix equation of the problem is

$$\{BA+B+B_1\}Y=0$$

where $\mathbf{B}_1$ is a matrix for $\alpha_1=0.8,\ \beta_1=0$. Following the procedure in Section 3, we find the solution of the problem for different N. The obtained results, using the collocation points $t_s=s/N-1\ (s=0,1,...,N-1)$, are given in Table 1. The previous results of (11) by Walsh series approach [25], by Laguerre series approach [26], by delayed unit step function series approach [27], and by Taylor series approach [17] are also given in Table 2 for comparison. The Bernoulli method seems more rapidly convergent than the other methods. Although accuracy is obtained at six decimal places for N=7 by the presented method, the same accuracy is obtained for N=11 by Taylor method and for N=29 by Laguerre method. Besides, Walsh and DUSF series method can not reach the same accuracy even for N=99. Accuracy of the solution for N=19 shown in Table 1 is better than the Taylor method shown in Table 2, as well.

Table 1. Bernoulli series solutions of Eq. (11) and accuracy of these solutions.

	Bernoulli Series Solution $y_N(t)$				Accuracy of Solutions $\|R_N(t)\|$			
t	N=7	N=11	N=19	N=20	N=7	N=11	N=19	N=20
0	1.000000	1.000000 0000	1.000000 0000 00000	1.0000000000 00000	0.00	1.11E-16	2.22E-16	0.00
0.2	0.664691	0.664691 0008	0.664691 0008 28908	0.6646910008 28912	1.75E-7	1.11E-16	1.11E-16	0.00
0.4	0.433561	0.433560 7788	0.433560 7787 76339	0.4335607787 76341	1.20E-7	3.33E-16	0.00	1.11E-16
0.6	0.276482	0.276482 3302	0.276482 3302 22267	0.2764823302 22268	1.20E-7	2.22E-16	2.22E-16	1.11E-16
0.8	0.171484	0.171484 1120	0.171484 1119 76061	0.1714841119 76062	1.75E-7	4.44E-16	5.55E-16	0.00
1	0.102670	0.102670 1266	0.102670 1265 74419	0.1026701265 74413	2.22E-16	1.11E-16	1.11E-16	2.22E-16

Table 2. Comparison of the solutions of Eq. (11).

	Walsh Series	DUSF Series	Laguerre Series		Taylor Series Solutions			
t	N = 99	N = 99	N = 19	N = 29	N = 8	N = 11	N = 19	$R_{19}(t)$
0	1.000000	1.000000	0.999971	1.000000	1.000000	1.000000	1.000000 0000 00000	8.44 E-15
0.2	0.665621	0.664677	0.664703	0.664691	0.664691	0.664691	0.664691 0008 28909	1.38 E-14
0.4	0.432426	0.433540	0.433555	0.433561	0.433561	0.433561	0.433560 7787 76339	3.22 E-14
0.6	0.275140	0.276460	0.276471	0.276482	0.276483	0.276482	0.276482 3302 22267	1.25 E-14
0.8	0.170320	0.171464	0.171482	0.171484	0.171494	0.171484	0.171484 1119 76062	7.38 E-15
1	0.100856	0.102652	0.102679	0.102670	0.102744	0.102670	0.102670 1265 74418	1.55 E-14

Example 3. Consider the following multi pantograph equation with variable coefficients

$$y'(t)=\tfrac{1}{2}e^{t/2}y\left(\tfrac{1}{2}t\right)+\tfrac{1}{2}y(t) \tag{12}$$

which has the exact solution $y(t)=e^t$. The fundamental matrix equation of the problem is

$$\left\{\mathrm{BA}-\tfrac{1}{2}\left(\mathrm{B}+P_1\mathrm{B}_1\right)\right\}Y=0$$

where $\mathbf{B}_1$ is a matrix for $\alpha_1=0.5,\ \beta_1=0$ and P_1 is a diagonal matrix for $p_1(t)=e^{t/2}$ defined in Eq. (7).

Table 3. Average relative errors of Eq. (12).

$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=7$	$N=8$	$N=9$	$N=10$	$N=11$	$N=12$
4.37E-2	1.46E-3	9.86E-5	3.94E-6	1.99E-7	6.97E-9	2.79E-10	8.38E-12	2.77E-13	7.12E-15	1.76E-16

Table 4. Comparison of absolute errors$|e_N(t)|$ for Example 3.

t	N=3	N=4	N=5	N=6	N=7	N=8	N=9	N=10	N=11	N=12
0.2	1.84E-03	1.38E-04	7.48E-06	3.22E-07	1.17E-08	3.85E-10	1.18E-11	3.46E-13	9.33E-15	0
0.4	4.52E-03	2.04E-04	7.27E-06	2.90E-07	1.23E-08	4.58E-10	1.48E-11	4.36E-13	1.18E-14	0
0.6	5.36E-03	1.59E-04	9.08E-06	4.34E-07	1.51E-08	5.33E-10	1.79E-11	5.34E-13	1.47E-14	0
0.8	4.06E-03	2.45E-04	1.31E-05	4.21E-07	2.03E-08	6.41E-10	2.17E-11	6.61E-13	1.87E-14	0
1	3.26E-03	4.51E-04	7.75E-06	8.45E-07	1.32E-08	1.15E-09	1.55E-11	1.12E-12	1.24E-14	0

Table 5. Comparison of corrected errors $E_{7,M}(t)$ with residual corrections.

t	$R_7(t)$	$e_7(t)$	M=7	M=8	M=9	M=10	M=11	M=12
0.2	2.04E-08	1.17E-08	8.00E-09	-2.50E-10	7.73E-12	-2.33E-13	6.88E-15	0
0.4	-1.41E-08	1.23E-08	8.59E-09	-3.36E-10	1.04E-11	-2.98E-13	8.22E-15	0
0.6	1.43E-08	1.51E-08	1.05E-08	-3.48E-10	1.26E-11	-3.76E-13	1.04E-14	0
0.8	-2.12E-08	2.03E-08	1.38E-08	-4.60E-10	1.45E-11	-4.68E-13	1.29E-14	0
1	0	1.32E-08	9.22E-09	-7.95E-10	1.08E-11	-7.77E-13	9.33E-15	0

The average relative errors of solution obtained by the presented method at the collocation points are given in Table 3 for different N. The errors of the wavelet method [24] for N=15 and N=255 are respectively 4.37E-4 and 2.05E-5. The average relative error of the variational iteration method [28] is also found 4.92E-3 at the second iteration. Besides, Taylor methods [16] and [17] have respectively the errors 1.73E-8, 2.11E-8 for N=9. It is obvious that the presented method has better results than wavelet, variational iteration and Taylor methods. However, average relative errors of Adomian method [18] and homotopy perturbation method [20] are also 5.33E-16 and 2.78E-16 for N=12, respectively. These results are similar to our error, yet these are higher than the presented method. For N=11, collocation points in our method are $t_s=s/10,\ s=0,1,...,10$, that is, step size h=0.1. The average relative errors of spline methods [29], [30] and [31] for h=0.01 are 4.45E-6, 2E-10, 6.87E-11 respectively. In addition, although the absolute errors in the exponential approximation method [32] for N=3, N=5 and N=10 are around 10^{-2}, 10^{-3} and 10^{-5}, respectively, they are around 10^{-3}, 10^{-6} and 10^{-13} in our method. Table 3 and Table 4 show that presented method has better results than the other methods.

Residual functions must be equal to zero at the collocation points, i.e. $R_N(t_s)=0,\ t_s=s/(N-1),\ \ s=0,1,\ldots,N-1$, but some of these are computed accuracy of 16 digits in Mathcad. To get rid of this error caused by computer, we took the zero tolerance 15 in Table 4 and Table 5. We observe from Table 4 that 15 digits accuracy are obtained at N=12. The errors are corrected with the residual function as in Section 4. In Table 5, the new collocation points are chosen by $t_k=k/M,\ \ k=0,1,\ldots,M$and N is taken seven at the first approach. Table 5 shows that corrected errors are better than the previous results, yet 15 digit accuracy are obtained M=15. For this reason, we do not need to have any correction, we already have good approximation.

Example 4. Consider the boundary value problem,

$$y''(t) = y(|t|), \quad y(0) = 1 \text{and} y(1) = e,$$

has a unique solution $y(t) = e^t$. In Table 6, the absolute errors obtained by the present method are compared with the errors for $h = 1/10^n$, $n = 1,2,3$ given by collocation method in [21]. Collocation points for N=12 are $t_s = s/10$, $s = 0,1,...,10$. in presented method, that is mesh size is h=0.1. Compared the first and the last columns of the Table 6, it seems that Bernoulli collocation method has much better result than the other collocation method. Here zero tolerance is fifty. From numerical computations, presented method converges quite fast.

Table 6. Comparison of absolute errors of Example 4.

	Subdivision Approach			Present method					
t	h=0.1	h=0.01	h=0.001	N=5	N=6	N=9	N=10	N=11	N=12
0.1	6.079E-7	8.785E-11	4.752E-13	9.921E-6	3.418E-7	1.467E-11	3.428E-13	1.044E-14	0.000
0.2	1.086E-6	1.609E-10	4.511E-12	1.296E-5	2.887E-7	9.938E-12	1.628E-13	8.438E-15	2.220E-16
0.3	1.562E-6	2.177E-10	9.976E-12	9.854E-6	6.311E-8	1.067E-11	1.315E-13	8.660E-15	0.000
0.4	1.914E-6	2.571E-10	1.455E-11	5.088E-6	5.701E-8	1.115E-11	4.818E-14	8.216E-15	0.000
0.5	2.070E-6	2.772E-10	1.777E-11	3.081E-6	2.356E-8	1.022E-11	1.554E-14	8.882E-15	0.000
0.6	1.999E-6	2.760E-10	2.071E-11	5.670E-6	9.104E-9	1.142E-11	7.949E-14	8.438E-15	2.220E-16
0.7	1.690E-6	2.508E-10	2.008E-11	1.111E-5	1.179E-7	1.117E-11	1.648E-13	9.770E-15	4.441E-16
0.8	1.205E-6	1.988E-10	1.782E-11	1.473E-5	3.571E-7	1.068E-11	1.945E-13	9.326E-15	4.441E-16
0.9	7.159E-7	1.162E-10	9.839E-12	1.139E-5	4.041E-7	1.602E-11	3.828E-13	1.155E-14	0.000

Example 5. Consider the pantograph equation of the third order

$$y'''(t) - ty''(2t) + y'(t) + y\left(\tfrac{t}{2}\right) = t\cos(2t) + \cos\left(\tfrac{t}{2}\right) \tag{13}$$

under the following two cases:
i. initial conditions; $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$,
ii. boundary conditions; $y(0) = 1$, $y(\pi) = -1$, $y'(\pi) = 0$.
The exact solution of the problem with conditions is $y(t) = \cos t$, and the main matrix equation of (13) is

$$\left\{BA^3 + P_{02}B_0A^2 + BA + B_2\right\}Y = F$$

where $\mathbf{B}_0$ and$\mathbf{B}_2$ are the matrices corresponding to $\alpha_0 = 2$, $\beta_0 = 0$ and $\alpha_2 = 0.5$, $\beta_2 = 0$, respectively, P_{02} is a matrix defined in Eq.(7) for $p_{02}(t) = -t$.

The approximate solutions obtained by using the collocation points $t_s = s\pi/(N-m)$ in $[0,\pi]$ for N=4, N=5, N= 6, N=7 are compared with the exact solution in Figure 1 and Figure 2. Absolute errors in $[0,\pi]$ for N=8, N=9, N= 10, N=11, N=12 are shown in Figure 3 and Figure 4, as well. From the Figures, it is obvious that the results get better as increase N. Besides, absolute errors of the problem with the initial conditions at the selected points in $[0,1]$ are tabulated in Table 7. Obtained results are good.

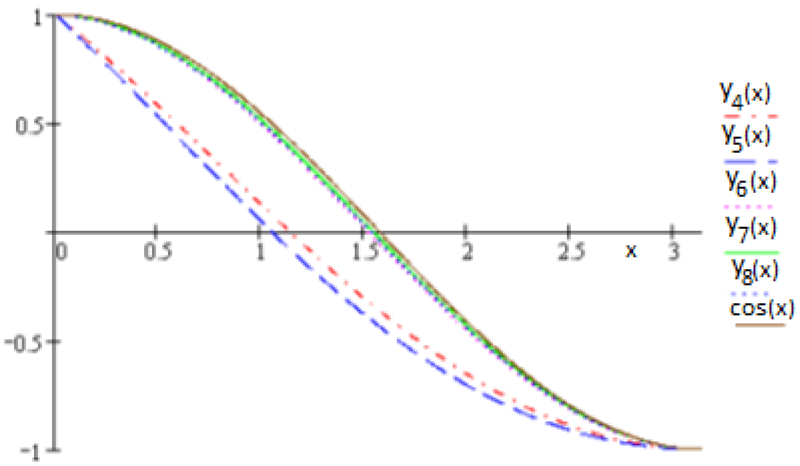

Figure 1. Comparison the solutions of Eq. (13) with initial conditions.

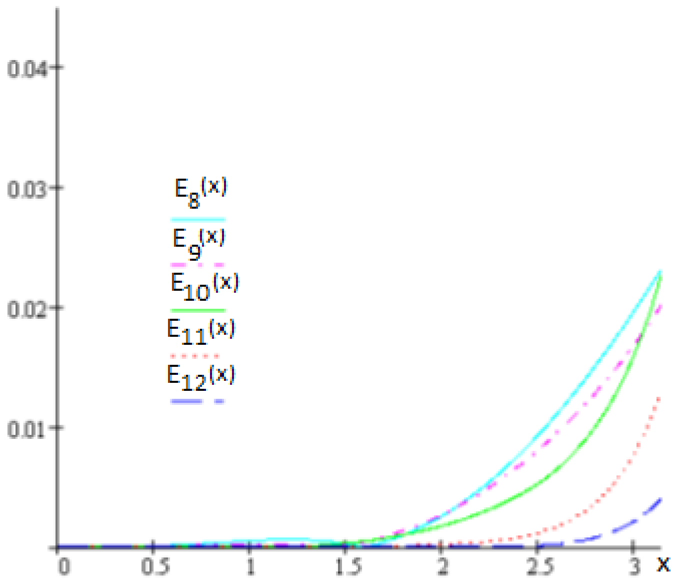

Figure 2. Comparison the solutions of Eq. (13) with boundary conditions.

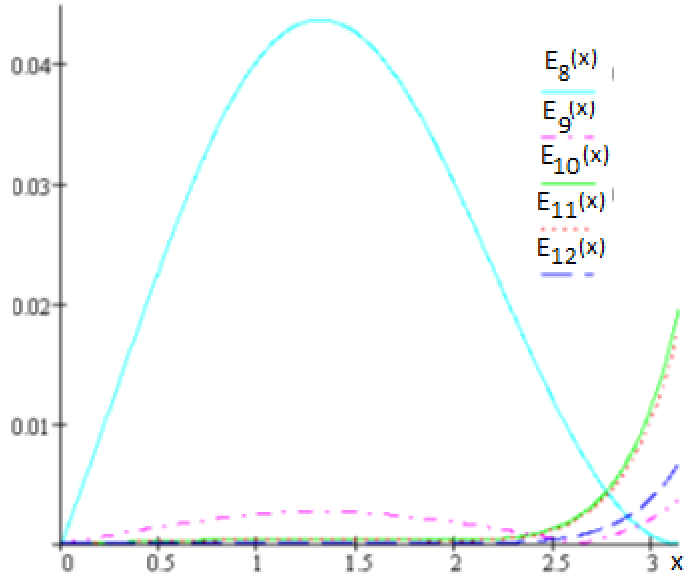

Figure 3. Error analysis of Eq.13 with initial conditions.

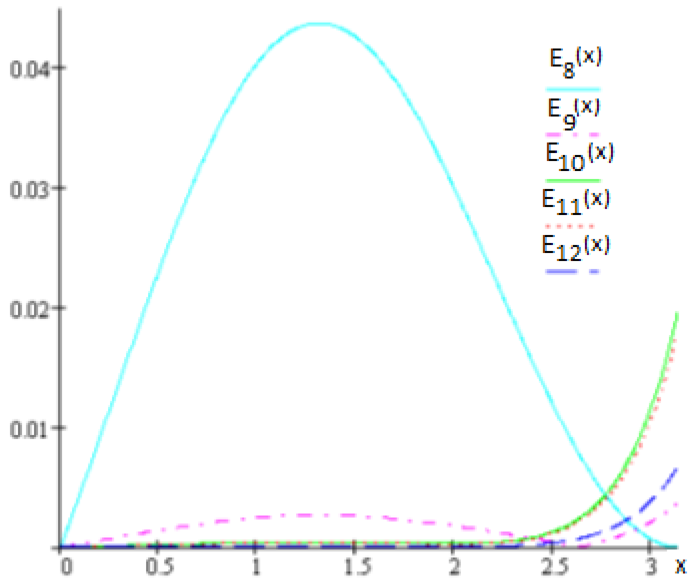

Figure 4. Error analysis of Eq.13 with boundary conditions.

Table 7. Absolute errors $|e_N(t)|$ for Example 5.

t	N=12	N=13	N=14	N=15	N=16
0.2	2.220E-14	1.332E-15	3.331E-16	2.220E-16	1.110E-16
0.4	1.845E-13	2.653E-14	5.773E-15	3.331E-16	0.000
0.6	9.717E-12	7.372E-13	1.624E-13	1.099E-14	7.772E-16
0.8	8.851E-11	6.486E-12	1.432E-12	9.592E-14	7.438E-15
1	4.041E-10	2.943E-11	6.495E-12	4.354E-13	3.375E-14

Example 6. Finally, let us consider the pantograph equation of the third order

$$y'''(t)+y(t)+y(t-0.3)=e^{-t-0.3}, \quad y(0)=1, \; y'(0)=-1, \; y''(0)=1.$$

The exact solution is $y=e^{-t}$. Maximum absolute errors and mean errors are tabulated in Table 8, by applying the presented method for different collocation points and truncation limit. Also, the last three rows replaced with the entries of the conditions in the system $[\mathbf{W}^*,\mathbf{F}^*]$. The numerical results of the trigonometric points are better than the others. In addition to this, the results are improved by the residual correction in Section 4. For example, let the collocation points be $t_k=k/(N-3)$ at the first approximation, its maximum errors can be seen in the first column of the Table 8. After calculated the residual function for N=5, the new system $\left[\tilde{W},\tilde{F}\right]$for the error function with homogenous conditions can be formed using the new collocation points $t_j=j/M$ $(j=0,1,...,M)$. Solving this system, estimated error function $e_{5,M}$ are obtained. In Table 9, maximum absolute errors of the corrected function, that is max $\left|E_{5,M}\right|$, are given. It can be seen these results are better than the results in Table 8. For N=8, maximum error is nearly 10^{-8} by the presented method, whereas 10^{-6} and 10^{-7} by Taylor [17] and Hermite [33] methods, respectively. Furthermore, maximum error for N=26 is about 10^{-8} in Jacobi method [34].

Table 8. Comparison of the maximum and mean errors for Example 6.

	$t_k=\frac{k}{N-3},k=0,1,...,N-3$		$t_i=\frac{i}{N},i=0,1,...,N$		$t_i=\frac{1}{2}\left[1+\cos\frac{(N-i)\pi}{N}\right]$	
N	max	mean	max	mean	max	mean
3	3.45E-2	1.05E-2	3.45E-2	1.05E-2	3.45E-2	1.05E-2
3	3.45E-2	1.05E-2	3.45E-2	1.05E-2	3.45E-2	1.05E-2
4	7.77E-3	2.29E-3	2.32E-3	5.26E-4	4.21E-3	1.02E-3
5	4.63E-4	1.51E-4	2.04E-4	4.12E-5	2.98E-4	5.71E-5
6	2.16E-5	7.31E-6	1.06E-5	1.72E-6	1.60E-5	2.55E-6
7	9.88E-7	3.35E-7	6.52E-7	9.90E-8	6.90E-7	9.29E-8
8	4.09E-8	1.39E-8	2.69E-8	3.40E-9	2.54E-8	2.97E-9
9	1.53E-9	5.22E-10	1.25E-9	1.51E-10	7.97E-10	8.19E-11
10	5.14E-11	1.75E-11	4.32E-11	4.44E-12	2.30E-11	2.22E-12
11	1.50E-12	5.11E-13	1.56E-12	1.47E-13	5.44E-13	5.19E-14
12	3.59E-14	1.23E-14	4.94E-14	4.39E-15	1.40E-14	1.32E-15
13	0	0	1.33E-15	0	0	0

Table 9. Comparison of the maximum absolute errors with the residual correction.

M=6	M=7	M=8	M=9	M=10	M=11	M=12	M=13
1.01E-5	4.72E-7	1.78E-8	6.15E-10	1.87E-11	4.59E-13	6.22E-15	0

6 Conclusions

A collocation method based on the truncated Bernoulli series is developed to numerically solve higher order pantograph equations with initial or boundary conditions. This method is applied to any finite interval. It is obvious that N^{th} order Bernoulli series approximation gives the exact solution when the solution is polynomial of degree less than or equal to N. However, more terms of the Bernoulli series are required for accurate calculation for large t, if the solution is not polynomial. Bernoulli approximation converges to the exact solution as N increases, but the truncation limit N must be chosen large enough. Besides, the residual error estimation is given for the method. If the exact solution of the problem is unknown, residual function can be used to estimate the error. The corrected approximate solution can also be computed by summing the estimated error function and approximate solution obtained by the Bernoulli collocation method.

Bernoulli collocation method provides two main advantages: it is very simple to construct the main matrix equation and it is very easy for computer programming. Another considerable advantage is that computational time of the method is too short. Besides, our method produces much better results than the other methods in the examples.

References

[1] J.R. Ockendon, A.B. Tayler, The dynamics of a current collection system for an electric locomotive, Proc. Roy. Soc. London, Ser. A 322 (1971) 447-468.
[2] L. Fox, D.F. Mayers, J.A. Ockendon, A.B. Tayler, On a functional differential equation, J. Inst. Math. Appl. 8 (1971) 271-307.
[3] W.G. Ajello, H.I. Freedman, J.Wu, A model of stage structured population growth with density depended time delay, SIAM J. Appl. Math. 52 (1992) 855-869.
[4] M.D. Buhmann, A. Iserles, Stability of the discretized pantograph differential equation, Math. Comput. 60 (1993) 575-589.
[5] G.R. Morris, A. Feldstein, E.W. Bowen, The Phragmen-Lindel' of principle and a class of functional-differential equations, in: Proceedings of NRL-MRC Conference on Ordinary Differential Equations, 1972, pp. 513-540.
[6] G. Derfel, On compactly supported solutions of a class of functional-differential equations, in: Modern Problems of Functions Theory and Functional Analysis, Karaganda University Press. 1980 (in Russian).
[7] G. Derfel, N. Dyn, D. Levin, Generalized refinement equation and subdivision process, J. Approx. Theory 80 (1995) 272-297.
[8] G.A. Derfel, F. Vogl, On the asymptotic of solutions of a class of linear functional-differential equations, European J. Appl. Maths. 7 (1996) 511-518.
[9] G. Derfel, A.Iserles, The pantograph equation in the complex plane, J. Math. Anal. Appl. 213 (1997) 117-132.
[10] A. Feldstein, Y. Liu, On neutral functional-differential equations with variable time delays, Math. Proc. Cambridge Philos. Soc. 124 (1998) 371-384.
[11] M.Z. Liu, D. Li, Properties of analytic solution and numerical solution and multi-pantograph equation, Appl. Math. Comput. 155 (2004) 853-871.
[12] M. Gülsu, M. Sezer, The approximate solution of high-order linear difference equation with variable coefficients in terms of Taylor polynomials, Appl. Math. Comput. 168 (1) (2005) 76-88.
[13] M. Sezer, M. Gülsu, A new polynomial approach for solving difference and Fredholm integro-difference equation with mixed argument, Appl. Math. Comput. 171 (1) (2005) 332-344.
[14] M. Gülsu, M. Sezer, A Taylor polynomial approach for solving differential-difference equations, J. Comput. Appl. Math. 186 (2) (2006) 349-364.
[15] M. Sezer, A. Akyüz-Daşcıoğlu, Taylor polynomial solutions of general linear differential–difference equations with variable coefficients, Appl. Math. Comput. 174 (2006) 1526–1538.
[16] M. Sezer, S. Yalçınbaş, N. Şahin, Approximate solution of multi-pantograph equation with variable coefficients, J. Comput. Appl. Math. 214 (2008) 406-416.

[17] M. Sezer, A. Akyüz-Daşcıoğlu, A Taylor method for numerical solution of generalized pantograph equations with linear functional argument, J. Comput. Appl. Math. 200 (2007) 217-225.
[18] D.J. Evans, K.R. Raslan, The Adomian decomposition method for solving delay differential equation, Int. J. Comput. Math. 82 (1) (2005) 49-54.
[19] Y. Keskin, A. Kurnaz, M.E. Kiris, G. Oturanc, Approximate solutions of generalized pantograph equations by the differential transform method, Int. J. Nonlinear Sci. Num. Simulation 8 (2) (2007) 159-164.
[20] E. Yusufoğlu, An efficient algorithm for solving generalized pantograph equations with linear functional argument, Appl. Math. Comput. 217 (2010) 3591–3595.
[21] R. Qu, R.P. Agarwal, A subdivision approach to the construction of approximate solutions of boundary-value problems with deviating arguments, Computers Math. Applic. 35 (1998) 121-135.
[22] N. M. Temme, Special Functions, an introduction to the classical functions of mathematical physics, John Wiley and Sons Inc., New York, USA, 1996.
[23] P. E. Zadunaisky, On the estimation of errors propagated in the numerical integration of ordinary differential equations, Numer. Math. 27 (1976) 21-39.
[24] C. H. Hsiao, Wavelet approach to time-varying functional differential equations, Int. J. Comput. Math. 8732 (2010) 528-540.
[25] G.P. Rao, K.R. Palanisamy, Walsh stretch matrices and functional differential equations, IEEE Trans. Autom. Control 27 (1982) 272-276.
[26] C. Hwang, Y.-P. Shih, Laguerre series solution of a functional differential equation, Int. J. Systems Sci. 13 (7) (1982) 783-788.
[27] C. Hwang, Solution of a functional differential equation via delayed unit step functions, Int. J. Systems Sci. 14 (9) (1983) 1065-1073.
[28] Zhan-Hua Yu, Variational iteration method for solving the multi-pantograph delay equation, Physics Letters A 372 (2008) 6475-6479.
[29] A. El-Safty, M. Shadia, On the application of spline function to initial value problems with retarded argument, Int. J. Comput. Math. 32 (1990) 173-179.
[30] M. Shadia, Numerical solution of delay differential and neutral differential equations using spline methods, Ph.D. Thesis, Assuit University, 1992.
[31] A. El-Safty, M. S. Salim, M. A. El-Khatib, Convergence of the spline function for delay dynamic system, Intern. J. Comput. Math. 80 (4) (2003) 509-518.
[32] Ş.Yüzbaşı, M. Sezer, An exponential approximation for solutions of generalized pantograph-delay differential equations, Applied Mathematical Modelling 37 (2013) 9160–9173.
[33] S. Yalçinbaş, M. Aynigul, M. Sezer, A collocation method using Hermite polynomials for approximate solution of pantograph equations, Journal of the Franklin Institute 348 (2011) 1128–1139.
[34] E.H. Doha, A.H. Bhrawy, D. Baleanu, R.M. Hafez, A new Jacobi rational–Gauss collocation method solution of generalized pantograph equations, Applied Numerical Mathematics, 77 (2014) 43–54.