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Characterizations of dual spacelike curves of constant breadth in dual Lorentzian space $\mathbb{D}_1^3$

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Abstract: In this paper, we study dual curves of constant breadth in dual Lorentzian Space $\mathbb{D}_1^3$. We obtain the differential equations characterizing dual curves of constant breadth in $\mathbb{D}_1^3$ and we introduce some special cases for these dual curves. Furthermore, we obtain that the total torsion of a closed dual spacelike curve of constant breadth is zero while the total torsion of a simple closed dual timelike curve is equal to $2n\pi$, $(n \in \mathbb{Z})$.

Keywords: Dual Lorentzian space, dual Frenet frame, dual curve of constant breadth.

1 Introduction

The properties of plane convex curves have been studied by many geometers so far. Two brief reviews of the most important publications on this subject have been published by Struik [19]. Also, a number of properties of plane curves of constant breadth are included in the works of Euler [8], Ball [2], Barbier [3], Blaschke [4] and Mellish [15].

A space curve of constant breadth was defined by Fujivara [9]. He considered a closed curve whose normal plane at a point P has only one more point Q in common with the curve, and for which $d(P,Q)$ is constant. For such curves PQ is also normal at Q. Furthermore, spherical curve of constant breadth was defined by Blaschke [5]. Köse presented some concept for space curves of constant breadth in Euclidean 3-space [12, 13]. Furthermore, differential equations characterizing space curves of constant breadth were obtained by Sezer [18]. Similar characterization of space curves of constant breadth in Euclidean 4-space were given by Mağden and Köse [14]. Also, the curves of constant breadth have been studied in Minkowski 3-space spacelike and timelike curves of constant breadth are normal curves, helices and spherical curves in some special cese. In [10], Kocayiğit and Önder showed that in Minkowski 3-space spacelike and timelike curves of constant breadth in are normal curves, helices and spherical curves in some special ceses. Moreover, Önder, Kocayiğit and Candan studied differential equations characterizing timelike and spacelike curves of constant breadth in Minkowski 3-space E_1^3 and gave a criterion for a timelike or spacelike curve of to be the curve of constant breadth in E_1^3 [17]. Spacelike curves of constant breadth in Minkowski 4-space were given by Kazaz, Önder and Kocayiğit [11]. Furthermore, Yılmaz have studied dual timelike curves of constant breadth in dual Lorentzian space [23], and in [24] Yılmaz et. al have given some characterizations of closed dual spacelike curves of constant breadth in dual Lorentzian space $\mathbb{D}_1^3$.

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In this paper we study dual curves of constant breadth in dual Lorentzian space $\mathbb{D}_1^3$. We obtain a third order differential equation which characterizes dual curves of constant breadth in $\mathbb{D}_1^3$ and give some special cases.

2 Dual curves and dual Lorentzian space

The Minkowski 3-space E_1^3 is the real vector space $\mathbb{R}^3$ provided with the standart flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $\overrightarrow{v} = (v_1, v_2, v_3)$ in E_1^3 can have one of three Lorentzian casual characters; it can be spacelike if $g(\overrightarrow{v}, \overrightarrow{v}) > 0$ or $\overrightarrow{v} = 0$, timelike if $g(\overrightarrow{v}, \overrightarrow{v}) < 0$ and null (lightlike) if $g(\overrightarrow{v}, \overrightarrow{v}) = 0$ and $\overrightarrow{v} \neq 0$. Similarly, an arbitrary curve $\overrightarrow{\alpha} = \overrightarrow{\alpha}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\overrightarrow{\alpha}'(s)$ are respectively spacelike, timelike or null (lightlike) [16,22]. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\overrightarrow{x} = (x_1, x_2, x_3)$ and $\overrightarrow{y} = (y_1, y_2, y_3)$ in E_1^3, in the meaning Lorentz vector product of $\overrightarrow{x}$ and $\overrightarrow{y}$ is defined by

$$\overrightarrow{x} \times \overrightarrow{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2)$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}) \ and \ e_1 \times e_2 = -e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = -e_2.$$

The Lorentzian sphere and hyperbolic sphere of radius r and center 0 in E_1^3 are given by

$$S_1^2 = \{\overrightarrow{x} = (x_1, x_2, x_3) \in E_1^3 : g(\overrightarrow{x}, \overrightarrow{x}) = r^2\}$$

and

$$H_0^2 = \{\overrightarrow{x} = (x_1, x_2, x_3) \in E_1^3 : g(\overrightarrow{x}, \overrightarrow{x}) = -r^2\}$$

respectively [20].

Let $\mathbb{D} = \mathbb{R} \times \mathbb{R} = \{\overline{a} = (a, a^*) : a, a^* \in \mathbb{R}\}$ be the set of the pairs (a, a^*). For $\overline{a} = (a, a^*)$, $\overline{b} = (b, b^*) \in \mathbb{D}$ the following operations are defined on $\mathbb{D}$:

Equality: $\overline{a} = \overline{b} \Leftrightarrow a = b,\ a^* = b^*$, addition: $\overline{a} + \overline{b} = (a + b, a^* + b^*)$ and multiplication: $\overline{ab} = (ab, ab^* + a^*b)$.

The element $\varepsilon = (0,1) \in \mathbb{D}$ satisfies the relationships, $\varepsilon \neq 0,\ \ \varepsilon^2 = 0,\ \ \varepsilon 1 = 1\varepsilon = \varepsilon$.

Let consider the element $\overline{a} \in \mathbb{D}$ of the form $\overline{a} = (a, 0)$. Then the mapping $f : \mathbb{D} \to \mathbb{R},\ f(a,0) = a$ is a isomorphism. So, we can write $a = (a, 0)$. By the multiplication rule we have that

$$\overline{a} = (a, a^*) = (a, 0) + (0, a^*) = (a, 0) + (0, 1)(a^*, 0) = a + \varepsilon a^*.$$

Then $\overline{a}=a+\varepsilon a^*$ is called dual number and ε is called dual unit. Thus the set of dual numbers is given by

$$\mathbb{D}=\{\overline{a}=a+\varepsilon a^*: a,a^*\in\mathbb{R},\varepsilon^2=0\}.$$

The set $\mathbb{D}$ forms a commutative group under addition [6,7]. The associative laws hold for multiplication. Dual numbers are distributive and form a ring over the real number field. Dual function of dual number presents a mapping of a dual numbers space on itself. The general expression for dual analytic (differentiable) function as follows

$$f(\overline{x})=f(x+\varepsilon x^*)=f(x)+\varepsilon x^* f'(x),$$

where $f'(x)$ is derivative of $f(x)$ and $x,x^*\in\mathbb{R}$.

Let $\mathbb{D}^3=\mathbb{D}\times\mathbb{D}\times\mathbb{D}$ be the set of all triples of dual numbers, i.e.,

$$\mathbb{D}^3=\{\tilde{a}=(\overline{a}_1,\overline{a}_2,\overline{a}_3):\overline{a}_i\in\mathbb{D},i=1,2,3\}.$$

Then the set $\mathbb{D}^3$ is called dual space. The elements of $\mathbb{D}^3$ are called dual vectors. Similar to the dual numbers, a dual vector $\tilde{a}$ may be expressed in the form $\tilde{a}=\overrightarrow{a}+\varepsilon\overrightarrow{a^*}=(\overrightarrow{a},\overrightarrow{a^*})$, where $\overrightarrow{a}$ and $\overrightarrow{a^*}$ are the vectors of $\mathbb{R}^3$. Then for any vectors $\tilde{a}=\overrightarrow{a}+\varepsilon\overrightarrow{a^*}$ and $\tilde{b}=\overrightarrow{b}+\varepsilon\overrightarrow{b^*}$ of $\mathbb{D}^3$, the scalar product and the vector product are defined by

$$<\tilde{a},\tilde{b}>=<\overrightarrow{a},\overrightarrow{b}>+\varepsilon(<\overrightarrow{a},\overrightarrow{b^*}>+<\overrightarrow{a^*},\overrightarrow{b}>),$$

and

$$\tilde{a}\times\tilde{b}=\overrightarrow{a}\times\overrightarrow{b}+\varepsilon(\overrightarrow{a}\times\overrightarrow{b^*}+\overrightarrow{a^*}\times\overrightarrow{b}),$$

respectively, where $<\overrightarrow{a},\overrightarrow{b}>$ and $\overrightarrow{a}\times\overrightarrow{b}$ are the inner product and the vector product of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in $\mathbb{R}^3$, respectively. The norm of a dual vector $\tilde{a}$ is given by

$$\|\tilde{a}\|=\|\overrightarrow{a}\|+\varepsilon\frac{<\overrightarrow{a},\overrightarrow{a^*}>}{\|\overrightarrow{a}\|},(\overrightarrow{a}\neq 0).$$

A dual vector $\tilde{a}$ with norm $1+\varepsilon 0$ is called dual unit vector. The set of dual unit vectors is given by

$$\tilde{S}^2=\{\tilde{a}=(a_1,a_2,a_3)\in\mathbb{D}^3:<\tilde{a},\tilde{a}>=1+\varepsilon 0\},$$

and called dual unit sphere (For details [6,7,21]).

The Lorentzian inner product of two dual vectors $\tilde{a}=\overrightarrow{a}+\varepsilon\overrightarrow{a^*}$, $\tilde{b}=\overrightarrow{b}+\varepsilon\overrightarrow{b^*}\in\mathbb{D}^3$ is defined by

$$<\tilde{a},\tilde{b}>=<\overrightarrow{a},\overrightarrow{b}>+\varepsilon(<\overrightarrow{a},\overrightarrow{b^*}>+<\overrightarrow{a^*},\overrightarrow{b}>),$$

where $<\overrightarrow{a},\overrightarrow{b}>$ is the Lorentzian inner product of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in the Minkowski 3-space $\mathbb{R}^3_1$. Then a dual vector $\tilde{a}=\overrightarrow{a}+\varepsilon\overrightarrow{a^*}$ is said to be timelike if $\overrightarrow{a}$ is timelike, spacelike if $\overrightarrow{a}$ is spacelike or $\overrightarrow{a}=0$ and lightlike (null) if $\overrightarrow{a}\neq 0$[20].

The set of all dual Lorentzian vectors is called dual Lorentzian space and it is denoted by $\mathbb{D}_1^3$,

$$\mathbb{D}_1^3 = \{\tilde{a} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*} : \overrightarrow{a}, \overrightarrow{a^*} \in \mathbb{R}_1^3\}.$$

The Lorentzian cross product of dual vectors $\tilde{a}, \tilde{b} \in \mathbb{D}_1^3$ is defined by

$$\tilde{a} \times \tilde{b} = \overrightarrow{a} \times \overrightarrow{b} + \varepsilon(\overrightarrow{a^*} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{b^*}),$$

where $\overrightarrow{a} \times \overrightarrow{b}$ is the Lorentzian cross pruduct in $\mathbb{R}_1^3$.

Let $\tilde{a} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*} \in \mathbb{D}_1^3$. Then $\tilde{a}$ is said to be dual unit timelike (resp. spacelike) vector if the vectors $\overrightarrow{a}$ and $\overrightarrow{a^*}$ satisfy the following equations:

$$< \overrightarrow{a}, \overrightarrow{a} >= -1 \; (resp. < \overrightarrow{a}, \overrightarrow{a} >= 1), \;\; < \overrightarrow{a}, \overrightarrow{a^*} >= 0.$$

The set of all unit dual timelike vectors is called the dual hyperbolic unit sphere, and is denoted by $\tilde{H}_0^2$,

$$\tilde{H}_0^2 = \{\tilde{a} = (a_1, a_2, a_3) \in \mathbb{D}_1^3 :< \tilde{a}, \tilde{a} >= -1 + \varepsilon 0\}.$$

Similarly, the set of all unit dual spacelike vectors is called the dual Lorentzian unit sphere, and is denoted by $\tilde{S}_1^2$,

$$\tilde{S}_1^2 = \{\tilde{a} = (a_1, a_2, a_3) \in \mathbb{D}_1^3 :< \tilde{a}, \tilde{a} >= 1 + \varepsilon 0\}.$$

(For details see [20]).

Let $\tilde{\phi} : \mathbf{I} \subset \mathbb{R} \to \mathbb{D}_1^3$ be a dual curve with are lenght parameter $\bar{s}$. Then the unit tangent vector is defined $\dot{\tilde{\phi}} = \frac{d\tilde{\phi}}{ds} = \tilde{t}$ and the principal normal is $\tilde{n} = \frac{\dot{\tilde{t}}}{\kappa}$ where $\bar{\kappa}$ is never pure-dual and "." denotes the derivative with respect to $\bar{s}$. The function $\bar{\kappa}(\bar{s}) = \| \dot{\tilde{t}} \| = \kappa(s) + \varepsilon \kappa^*(s)$ is called dual curvature of dual curve $\tilde{\phi}$. Then the binormal vector of $\tilde{\phi}$ is given by the dual vector $\tilde{b} = \pm \tilde{t} \times \tilde{n}$. Hence, the triple $\{\tilde{t}, \tilde{n}, \tilde{b}\}$ is called dual Frenet frame of $\tilde{\phi}$ and Frenet formulas are given by

$$\begin{bmatrix} \dot{\tilde{t}} \\ \dot{\tilde{n}} \\ \dot{\tilde{b}} \end{bmatrix} = \begin{bmatrix} 0 & \bar{\kappa} & 0 \\ -\xi_1\xi_2\bar{\kappa} & 0 & \bar{\tau} \\ 0 & -\xi_2\xi_3\bar{\tau} & 0 \end{bmatrix} \begin{bmatrix} \tilde{t} \\ \tilde{n} \\ \tilde{b} \end{bmatrix} \tag{1}$$

where $< \tilde{t}, \tilde{t} >= \xi_1, \; < \tilde{n}, \tilde{n} >= \xi_2, \; < \tilde{b}, \tilde{b} >= \xi_3$ and $< \tilde{t}, \tilde{n} >=< \tilde{t}, \tilde{b} >=< \tilde{n}, \tilde{b} >= 0$.

Here $\bar{\tau}(\bar{s}) = \tau(s) + \varepsilon \tau^*(s)$ is dual torsion of dual curve $\tilde{\phi}$ and we suppose, as the curvature $\bar{\tau}$ is never pure-dual [1].

3 Dual curves of constant breadth in $\mathbb{D}_1^3$

In this section, we consider dual curves of constant breadth in dual Lorentzian space $\mathbb{D}_1^3$.

Definition 1. *Let C be a dual curve in $\mathbb{D}_1^3$ with dual position vector $\tilde{\phi} = \tilde{\phi}(\bar{s})$. If C has parallel tangents in opposite directions at corresponding points and if the distance between these points is always constant then C is called a dual curve of constant breadth in $\mathbb{D}_1^3$. Moreover, a pair of dual curves C and C_ζ for which the tangents at the corresponding points $\tilde{\phi}(\bar{s})$ and $\tilde{\zeta}(\bar{s_\zeta})$ are parallel and in opposite directions and the distance between these points is always constant is called a dual curve pair of constant breadth in $\mathbb{D}_1^3$ where $\tilde{\zeta}(\bar{s_\zeta})$ is dual position vector of dual curve C_ζ in $\mathbb{D}_1^3$.*

Let now C and C_ζ be a pair of unit speed dual curves of class C^3 with nonzero dual curvature and dual torsion in $\mathbb{D}_1^3$ and let those dual curves have parallel tangents in opposite directions at corresponding points. Then we may write the position vector of dual curve C_ζ as follows

$$\tilde{\zeta} = \tilde{\phi} + \overline{\gamma}\tilde{t} + \overline{\delta}\tilde{n} + \overline{\lambda}\tilde{b} \tag{2}$$

where $\overline{\gamma}$, $\overline{\delta}$ and $\overline{\lambda}$ are arbitrary functions of $\overline{s}$. By differentiating (2) with respect to $\overline{s}$ we get

$$\frac{\overline{\zeta}}{d\overline{s}_\zeta}\frac{d\overline{s}_\zeta}{d\overline{s}} = (1+\frac{d\overline{\gamma}}{d\overline{s}} - \xi_1\xi_2\overline{\kappa}\overline{\delta})\tilde{t} + (\overline{\kappa}\overline{\gamma} + \frac{d\overline{\delta}}{d\overline{s}} - \xi_2\xi_3\overline{\tau}\overline{\lambda})\tilde{n} + (\overline{\tau}\overline{\delta} + \frac{d\overline{\lambda}}{d\overline{s}})\tilde{b}. \tag{3}$$

Since we have $\tilde{t}_\zeta = -\tilde{t}$ from (3) it follows

$$\begin{cases} \frac{d\overline{\gamma}}{d\overline{s}} = -1 - \frac{d\overline{s}_\zeta}{d\overline{s}} + \xi_1\xi_2\overline{\kappa}\overline{\delta} \\ \frac{d\overline{\delta}}{d\overline{s}} = -\overline{\kappa}\overline{\gamma} + \xi_2\xi_3\overline{\tau}\overline{\lambda} \\ \frac{d\overline{\lambda}}{d\overline{s}} = -\overline{\tau}\overline{\delta} \end{cases} \tag{4}$$

If we call $\overline{\varphi}$ as the dual angle between the tangent of dual curve C at the point $\tilde{\phi}$ and a given fixed direction and consider $\frac{d\overline{\varphi}}{d\overline{s}} = \overline{\kappa} = \frac{1}{\overline{\rho}}$ and $\frac{d\overline{\varphi}}{d\overline{s}_\zeta} = \overline{\kappa}_\zeta = \frac{1}{\overline{\rho}_\zeta}$ then the system (4) can be given as follows

$$\begin{cases} \frac{d\overline{\gamma}}{d\overline{\varphi}} = -\xi_1\xi_2\overline{\delta} - f(\overline{\varphi}) \\ \frac{d\overline{\delta}}{d\overline{\varphi}} = -\overline{\gamma} + \xi_2\xi_3\overline{\tau}\overline{\rho}\overline{\lambda} \\ \frac{d\overline{\lambda}}{d\overline{\varphi}} = -\overline{\tau}\overline{\rho}\overline{\delta} \end{cases} \tag{5}$$

where $f(\overline{\varphi}) = \overline{\rho} + \overline{\rho}_\zeta$. Using the system of ordinary differential equations (5), we have the following dual third order differential equation with respect to $\overline{\gamma}$

$$\frac{1}{\xi_2\xi_3}\left\{\frac{d}{d\overline{\varphi}}\left[\frac{\overline{\kappa}}{\overline{\tau}}\frac{1}{\xi_1\xi_2}\left(\frac{df}{d\overline{\varphi}} + \frac{d^2 f}{d\overline{\varphi}^2}\right)\right]\right\} + \frac{\overline{\tau}}{\overline{\kappa}}\frac{1}{\xi_1\xi_2}\left(f + \frac{d\overline{\gamma}}{d\overline{\varphi}}\right) - \frac{d}{d\overline{\varphi}}\left(\frac{\overline{\kappa}}{\overline{\tau}}\overline{\gamma}\right) = 0. \tag{6}$$

Then we can give the following corollary.

Corollary 1. *The dual differential equation given in (6) is a characterization for dual curve C_ζ and the position vector of this dual curve can be determined by the solution of this equation.*

Let now investigate the solution of equation (6) in a special case. Assume that $\overline{\kappa}(\overline{s})$, $\overline{\tau}(\overline{s})$ be constants i.e., κ, κ^*, τ and τ^* be constant. It means that dual curve C is a dual helix. In the case equation (6) has the form

$$\frac{1}{\xi_2\xi_3}\frac{d^3\overline{\gamma}}{d\overline{\varphi}^3} + \left(\frac{\overline{\tau}^2}{\overline{\kappa}^2} - \xi_1\xi_2\right)\frac{d\overline{\gamma}}{d\overline{\varphi}} + f = 0.$$

Then we have the following corollary.

Corollary 2. *The dual differential equation characterizing dual helix curves of constant breadth in $\mathbb{D}_1^3$ is given as follows.*

$$\frac{1}{\xi_2\xi_3}\frac{d^3\overline{\gamma}}{d\overline{\varphi}^3}+\left(\frac{\overline{\tau}^2}{\overline{\kappa}^2}-\xi_1\xi_2\right)\frac{d\overline{\gamma}}{d\overline{\varphi}}+f=0. \tag{7}$$

Now, there are three cases for solution of equation (7).

Case 1. Let C be a dual timelike helix. Then we have $\xi_1=-1$, $\xi_2=\xi_3=1$. In this case equation (7) becomes

$$\frac{d^3\overline{\gamma}}{d\overline{\varphi}^3}+\left(\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1\right)\frac{d\overline{\gamma}}{d\overline{\varphi}}+f \tag{8}$$

and solution of this equation gives us followings

$$\begin{cases}\overline{\gamma}=Acos\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)+Bsin\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)-\frac{\overline{\kappa}^2\overline{\varphi}}{\overline{\kappa}^2+\overline{\tau}^2}f\\ \overline{\delta}=\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\left[Bcos\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)-Asin\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)\right]+\frac{\overline{\tau}^2}{\overline{\kappa}^2+\overline{\tau}^2}f\\ \overline{\lambda}=-\frac{\overline{\tau}}{\overline{\kappa}}\left[Acos\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)+Bsin\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)\right]-\frac{1}{\overline{\tau}}\left(\frac{\overline{\kappa}^3\overline{\varphi}}{\overline{\kappa}^2+\overline{\tau}^2}f-\frac{\overline{\tau}}{\overline{\kappa}^2+\overline{\tau}^2}\frac{df}{d\overline{\varphi}}\right)\end{cases} \tag{9}$$

where A and B are constants. Then we have the following theorem.

Theorem 1. *Dual time like helices C and C_ζ form a dual curve pair of constant breadth in $\mathbb{D}_1^3$ if and only if there exists the following relationship between the dual position vectors of the curves*

$$\begin{aligned}\tilde{\zeta}=\tilde{\phi}&+\left[Acos\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)+Bsin\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)-\frac{\overline{\kappa}^2\overline{\varphi}}{\overline{\kappa}^2+\overline{\tau}^2}f\right]\tilde{t}\\ &+\left[\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\left[Bcos\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)-Asin\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)\right]+\frac{\overline{\tau}^2}{\overline{\kappa}^2+\overline{\tau}^2}f\right]\tilde{n}\\ &+\left[-\frac{\overline{\tau}}{\overline{\kappa}}\left[Acos\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)+Bsin\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)\right]-\frac{1}{\overline{\tau}}\left(\frac{\overline{\kappa}^3\overline{\varphi}}{\overline{\kappa}^2+\overline{\tau}^2}f-\frac{\overline{\tau}}{\overline{\kappa}^2+\overline{\tau}^2}\frac{df}{d\overline{\varphi}}\right)\right]\tilde{b}\end{aligned}$$

Case 2. Let now C be a dual spacelike helix with a dual timelike principal normal $\tilde{n}$. Then we have $\xi_1=\xi_3=1$, $\xi_2=-1$. In this case (7) gives us

$$\frac{d^3\overline{\gamma}}{d\overline{\varphi}^3}+\left(\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1\right)\frac{d\overline{\gamma}}{d\overline{\varphi}}-f=0 \tag{10}$$

and it follows from the solution of (10) that

$$\begin{cases}\overline{\gamma}=Kcosh\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)+Lsinh\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)-\frac{\overline{\kappa}^2\overline{\varphi}}{\overline{\kappa}^2+\overline{\tau}^2}f\\ \overline{\delta}=\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2+1}}\left[Lcosh\left(\varphi\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)+Ksinh\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)\right]+\frac{\overline{\tau}^2}{\overline{\kappa}^2+\overline{\tau}}f\\ \overline{\lambda}=-\frac{\overline{\kappa}}{\overline{\tau}}\left[Kcosh\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)+Lsinh\left(\overline{\varphi}\sqrt{\frac{\overline{\tau}^2}{\overline{\kappa}^2}+1}\right)\right]+\frac{\overline{\kappa}}{\overline{\tau}}\left(\frac{\overline{\kappa}^2\overline{\varphi}}{\overline{\kappa}^2+\overline{\tau}^2}f-\frac{\overline{\tau}^2}{\overline{\kappa}^2+\overline{\tau}^2}\frac{df}{d\varphi}\right)\end{cases} \tag{11}$$

where K and L are constants.

From (11) we have the following theorem,

Theorem 2. *Let C has dual timelike principal normal $\tilde{n}$. Then dual spacelike helices C and C_ζ form a dual curve pair of constant breadth in $\mathbb{D}_1^3$ if and only if there exists the following relationship between the dual position vectors of the curves*

$$\begin{aligned}\tilde{\zeta} = \tilde{\phi} &+ \left[K\cosh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}+1}\right) + L\sinh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}+1}\right) - \frac{\bar{\kappa}^2\bar{\varphi}}{\bar{\kappa}^2+\bar{\tau}^2} f \right]\tilde{t} \\ &+ \left[\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}+1} \left[L\cosh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}+1}\right) + K\sinh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}+1}\right)\right] + \frac{\bar{\tau}^2}{\bar{\kappa}^2+\bar{\tau}^2} f \right]\tilde{n} \\ &+ \frac{\bar{\kappa}}{\bar{\tau}}\left[-K\cosh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}+1}\right) - L\sinh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}+1}\right) + \frac{\bar{\kappa}^2\bar{\varphi}}{\bar{\kappa}^2+\bar{\tau}^2} f - \frac{\bar{\tau}^2}{\bar{\kappa}^2+\bar{\tau}^2}\frac{df}{d\bar{\varphi}} \right]\tilde{b}.\end{aligned}$$

Case 3. Let now C be a dual spacelike helix with a dual timelike binormal $\tilde{b}$. Then $\xi_1 = \xi_2 = 1$, $\xi_3 = -1$ and from (7) it follows

$$\frac{d^3\bar{\gamma}}{d\bar{\varphi}^3} - \left(\frac{\bar{\tau}^2}{\bar{\kappa}^2} - 1\right)\frac{d\bar{\gamma}}{d\bar{\varphi}} - f = 0. \tag{12}$$

The solution of (12) depends on the sign of $\frac{\bar{\tau}^2}{\bar{\kappa}^2} - 1$. Then we have the following subcases.

(i) $\kappa = \tau$ and $\kappa^* = \tau^*$.Thus $\frac{d^3\bar{\gamma}}{d\bar{\varphi}^3} = f$. By this way we have the components

$$\bar{\gamma} = \frac{\bar{\varphi}^3}{6} f, \quad \bar{\delta} = -\left(1 + \frac{\bar{\varphi}^2}{2}\right) f, \quad \bar{\lambda} = \frac{\bar{\kappa}}{\bar{\tau}}\left[\left(\bar{\varphi} - \frac{\bar{\varphi}^3}{6}\right) f + \frac{df}{d\bar{\varphi}}\right]. \tag{13}$$

(ii) $\frac{\kappa}{\tau} = \frac{\kappa^*}{\tau^*}$ and $\tau > \kappa$.In this case from (12) we obtain

$$\begin{cases} \bar{\gamma} = K\cosh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}-1}\right) + L\sinh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}-1}\right) + \frac{\bar{\kappa}^2}{\bar{\kappa}^2-\bar{\tau}^2}\bar{\varphi} f \\ \bar{\delta} = -\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}-1}\left[L\cosh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}-1}\right) + K\sinh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}-1}\right)\right] + \frac{2\bar{\kappa}^2-\bar{\tau}^2}{\bar{\kappa}^2-\bar{\tau}^2} f \\ \bar{\lambda} = \frac{\bar{\tau}^2-\bar{\kappa}^2}{\bar{\tau}}\left[K\cosh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}-1}\right) + L\sinh\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\bar{\kappa}^2}-1}\right)\right] - \frac{\bar{\kappa}}{\bar{\tau}}\left(\frac{2\bar{\kappa}^2-\bar{\tau}^2}{\bar{\kappa}^2-\bar{\tau}^2}\frac{df}{d\varphi}\right) \end{cases} \tag{14}$$

where K and L are constants.

(iii) $\frac{\kappa}{\tau} = \frac{\kappa^*}{\tau^*}$ and $\tau < \kappa$.Then, by means of solution of (12), we have the components

$$\begin{cases} \bar{\gamma} = A\cos\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\kappa}-1}\right) + B\sin\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\kappa}-1}\right) - \frac{\bar{\kappa}^2\bar{\varphi}}{\bar{\kappa}^2-\bar{\tau}^2} f \\ \bar{\delta} = \sqrt{\frac{\bar{\tau}^2}{\kappa}-1}\left[B\cos\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\kappa}-1}\right) - A\sin\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\kappa}-1}\right)\right] + \frac{\bar{\tau}^2}{\bar{\kappa}^2-\bar{\tau}^2} f \\ \bar{\lambda} = -\left[A\cos\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\kappa}-1}\right) + B\sin\left(\bar{\varphi}\sqrt{\frac{\bar{\tau}^2}{\kappa}-1}\right)\right] + \frac{\bar{\kappa}}{\bar{\tau}}\left(\frac{\bar{\kappa}^2\varphi}{\bar{\kappa}^2-\bar{\tau}^2} f - \frac{\bar{\tau}}{\bar{\kappa}^2-\bar{\tau}^2}\frac{df}{d\varphi}\right) \end{cases} \tag{15}$$

Then we give the following theorem.

Theorem 3. *Let C has dual timelike binormal vector $\tilde{b}$.Then dual spacelike helices C and C_ζ form a dual curve pair of constant breadth in $\mathbb{D}_1^3$ if and only if the relationship between dual position vectors of the curves determined by the components obtained (13), (14) and (15).*

Let now consider the general case again, i.e., let κ, κ^*, τ and τ^* be non-constant. Since the distance between opposite points of C and C_ζ is constant, we can write

$$\| \tilde{\zeta} - \tilde{\phi} \|^2 = | \xi_1 \overline{\gamma} + \xi_2 \overline{\delta} + \xi_3 \overline{\lambda} | = constant. \tag{16}$$

Then, we have the following special cases.

Case 1. $\xi_1 = -1$, $\xi_2 = \xi_3 = 1$. Thus, equation (16) becomes

$$| -\overline{\gamma}^2 + \overline{\delta}^2 + \overline{\lambda}^2 | = constant. \tag{17}$$

By differentiating (17) with respect to $\overline{\varphi}$ it follows

$$-\overline{\gamma}\frac{d\overline{\gamma}}{d\overline{\varphi}} + \overline{\delta}\frac{d\overline{\delta}}{d\overline{\varphi}} + \overline{\lambda}\frac{d\overline{\lambda}}{d\overline{\varphi}} = 0. \tag{18}$$

Solution of the differential equation (18) was investigated by Yılmaz [23]. In this way, he obtained the followings:

By virtue of (5), the differential equation (18) yields

$$\overline{\gamma}\left(\frac{d\overline{\gamma}}{d\overline{\varphi}} + \overline{\delta}\right) = 0. \tag{19}$$

From (19) the following cases are obtained.

(i) $\overline{\gamma} = 0$. Then, the other components are

$$\overline{\delta} = -f, \quad \overline{\lambda} = -\frac{\overline{\kappa}}{\overline{\tau}}\frac{df}{d\overline{\varphi}}, \tag{20}$$

and the relationship between dual position vectors of constant breadth timelike curves is given by

$$\tilde{\zeta} = \tilde{\phi} - f\tilde{n} - \frac{\overline{\kappa}}{\overline{\tau}}\frac{df}{d\overline{\varphi}}\tilde{b}. \tag{21}$$

(ii) $\frac{d\overline{\gamma}}{d\overline{\varphi}} = -\overline{\delta}$. That is $f = 0$. Then followings are obtained,

(a) $\kappa = \tau$ and $\kappa^* = \tau^*$. Thus, the components are

$$\overline{\gamma} = \frac{\overline{s}^2}{2} + c\overline{s}, \quad \overline{\delta} = -\overline{s} - c, \quad \overline{\lambda} = \frac{\kappa}{\tau}\left(\frac{\overline{s}^2}{2} + c\overline{s} + 1\right). \tag{22}$$

(b) $\frac{\kappa}{\tau}=\frac{\kappa^*}{\tau^*}$ and $\tau>\kappa$. Then the components

$$\begin{cases}\overline{\gamma}=A\cos\left(\overline{\varphi}\sqrt{\frac{\tau^2}{\kappa^2}-1}\right)+B\sin\left(\overline{\varphi}\sqrt{\frac{\tau^2}{\kappa^2}-1}\right)\\ \overline{\delta}=-\sqrt{\frac{\tau^2}{\kappa^2}-1}\left[B\cos\left(\overline{\varphi}\sqrt{\frac{\tau^2}{\kappa^2}-1}\right)-A\sin\left(\overline{\varphi}\sqrt{\frac{\tau^2}{\kappa^2}-1}\right)\right]\\ \overline{\lambda}=\frac{\kappa}{\tau}\left(\sqrt{\frac{\tau^2}{\kappa^2}-1}\right)\left[A\cos\left(\overline{\varphi}\sqrt{\frac{\tau^2}{\kappa^2}-1}\right)+B\sin\left(\overline{\varphi}\sqrt{\frac{\tau^2}{\kappa^2}-1}\right)\right].\end{cases} \tag{23}$$

(c) $\frac{\kappa}{\tau}=\frac{\kappa^*}{\tau^*}$ and $\tau<\kappa$. Consequently the components

$$\begin{cases}\overline{\gamma}=K\cosh\left(\overline{\varphi}\sqrt{1-\frac{\tau^2}{\kappa^2}}\right)+L\sinh\left(\overline{\varphi}\sqrt{1-\frac{\tau^2}{\kappa^2}}\right)\\ \overline{\delta}=-\sqrt{1-\frac{\tau^2}{\kappa^2}}\left[L\cosh\left(\overline{\varphi}\sqrt{1-\frac{\tau^2}{\kappa^2}}\right)+K\sinh\left(\overline{\varphi}\sqrt{1-\frac{\tau^2}{\kappa^2}}\right)\right]\\ \overline{\lambda}=\frac{\kappa}{\tau}\left(\sqrt{\frac{\tau^2}{\kappa^2}-1}\right)\left[K\cosh\left(\overline{\varphi}\sqrt{1-\frac{\tau^2}{\kappa^2}}\right)+L\sinh\left(\overline{\varphi}\sqrt{1-\frac{\tau^2}{\kappa^2}}\right)\right]\end{cases} \tag{24}$$

(See [23]).

Theorem 4. *The relationship between dual position vectors of dual timelike curves of constant breadth is determined by the components obtained in (20)-(24).*

Let new consider other case not studied in [23]. Then by considering a dual spacelike curve we can give the followings according to Lorentzian casual character of principal normal and binormal vectors of the curve.

Case 2. Let C be a dual spacelike curve with a timelike principal normal. Then $\xi_1=\xi_3=1$, $\xi_2=-1$. Thus, equation (16) becomes

$$|\,\overline{\gamma}^2-\overline{\delta}^2+\overline{\lambda}^2\,|=constant. \tag{25}$$

By differentiating (25) with respect to $\overline{\varphi}$ we have

$$\overline{\gamma}\frac{d\overline{\gamma}}{d\overline{\varphi}}-\overline{\delta}\frac{d\overline{\delta}}{d\overline{\varphi}}+\overline{\lambda}\frac{d\overline{\lambda}}{d\overline{\varphi}}=0. \tag{26}$$

By using system (5) we get

$$\overline{\gamma}\left(2\frac{d\overline{\gamma}}{d\overline{\varphi}}+f\right)=0. \tag{27}$$

and from (27) we have the following special cases.

(i) $\overline{\gamma}=0$. Thus $\overline{\delta}=f$ and we have the components as follows

$$\overline{\gamma}=0,\quad \overline{\delta}=f,\quad \overline{\lambda}=-\frac{\overline{\kappa}}{\overline{\tau}}\frac{df}{d\overline{\varphi}} \tag{28}$$

(ii) $2\frac{d\overline{\gamma}}{d\overline{\varphi}}+f=0$. Then we get the following components

$$\overline{\gamma}=-\frac{1}{2}\frac{df}{d\overline{\varphi}}+\frac{\overline{\tau}^2}{\overline{\kappa}^2}\overline{\varphi}f,\quad \overline{\delta}=\frac{f}{2},\quad \overline{\lambda}=-\frac{1}{2}\frac{\overline{\tau}}{\overline{\kappa}}\overline{\varphi}f \tag{29}$$

Case 3. Let now C be a dual spacelike curve with a timelike binormal. Then $\xi_1 = \xi_2 = 1$, $\xi_3 = -1$. So equation (16) gives us

$$|\,\overline{\gamma}^2 + \overline{\delta}^2 - \overline{\lambda}^2\,| = constant. \tag{30}$$

By differenting (30) with respect to $\overline{\varphi}$ it follows

$$\overline{\gamma}\frac{d\overline{\gamma}}{d\overline{\varphi}} + \overline{\delta}\frac{d\overline{\delta}}{d\overline{\varphi}} - \overline{\lambda}\frac{d\overline{\lambda}}{d\overline{\varphi}} = 0. \tag{31}$$

By using system (5) from (31) we have $\overline{\gamma}\left(2\frac{d\overline{\gamma}}{d\overline{\varphi}} + f\right) = 0$, which is the same with (27). Then we obtained the special cases given in (28) and (29) again. So that, we can give the following theorem.

Theorem 5. *The relationship between dual position vectors of spacelike curves of constant breadth is determined by the components obtained in (28) and (29).*

So far we have deal with a pair of dual curves having parallel tangents in opposite directions at corresponding points in $\mathbb{D}_1^3$. Now let us consider a simple closed unit speed dual curve C of class C^3 in $\mathbb{D}_1^3$ for which the normal plane of every point P on the curve meets the curve of a single opposite point Q other than P. Let $\tilde{\phi}_P$ and $\tilde{\phi}_Q$ denote the dual position vectors of the points P and Q, respectively. Then, we may give the following theorem concerning the dual space curves of constant breadth.

Theorem 6. *Let C be a dual curve having parallel tangents in opposite directions at the opposite points of the curve in $\mathbb{D}_1^3$. If the chord joining the opposite points of C is a double-normal, then C has constant breadth, and conversely, if C is a dual curve of constant breadth in $\mathbb{D}_1^3$ then every normal of C is a double-normal.*

Proof. Let dual vector $\tilde{\mathbb{D}} = \tilde{\phi}_Q - \tilde{\phi}_P = \overline{\gamma}\tilde{t} + \overline{\delta}\tilde{n} + \overline{\lambda}\tilde{b}$ be a double-normal of C where $\overline{\gamma}$, $\overline{\delta}$ and $\overline{\lambda}$ are arbitrary functions of $\overline{s}$ which is the dual are length parameter of C. Then we get $<\tilde{\mathbb{D}}, \tilde{t}_P> = -<\tilde{\mathbb{D}}, \tilde{t}_Q> = \overline{\gamma} = 0$ where $\tilde{t}_P$ and $\tilde{t}_Q$ are dual tangents at the points P and Q, respectively. Thus from (4) we get

$$\xi_2\overline{\delta}\frac{d\overline{\delta}}{d\overline{s}} + \xi_3\overline{\lambda}\frac{d\overline{\lambda}}{d\overline{s}} = 0.$$

It follows that $\xi_2\overline{\delta}^2 + \xi_3\overline{\lambda}^2 = constant$, i.e., the breadth of (C) is constant. Conversely, if $\|\,\tilde{\mathbb{D}}\,\|^2 = \xi_1\overline{\gamma}^2 + \xi_2\overline{\delta}^2 + \xi_3\overline{\lambda}^2 = constant$, then as shown, $\overline{\gamma} = 0$. This means that $\tilde{\mathbb{D}}$ is perpendicular to $\tilde{t}_P$ and $\tilde{t}_Q$. So $\tilde{\mathbb{D}}$ is the double-normal of (C).

Let now consider the system (5). Assume that dual curve C is a timelike curve. Then from Theorem 6 we have $\overline{\gamma} = 0$. In this case from (5) we obtain

$$\begin{cases} \overline{\delta} = Mcos\int_0^{\overline{\varphi}} \overline{\tau}\overline{\rho}d\overline{\varphi} + Nsin\int_0^{\overline{\varphi}} \overline{\tau}\overline{\rho}d\overline{\varphi} \\ \overline{\lambda} = -Msin\int_0^{\overline{\varphi}} \overline{\tau}\overline{\rho}d\overline{\varphi} + Ncos\int_0^{\overline{\varphi}} \overline{\tau}\overline{\rho}d\overline{\varphi} \end{cases} \tag{32}$$

where M, N are constants. Then (2) gives us

$$\tilde{\zeta} = \tilde{\phi} + \left(Mcos\int_0^{\overline{\varphi}} \overline{\tau}\overline{\rho}d\overline{\varphi} + Nsin\int_0^{\overline{\varphi}} \overline{\tau}\overline{\rho}d\overline{\varphi}\right)\tilde{n} + \left(-Msin\int_0^{\overline{\varphi}} \overline{\tau}\overline{\rho}d\overline{\varphi} + Ncos\int_0^{\overline{\varphi}} \overline{\tau}\overline{\rho}d\overline{\varphi}\right)\tilde{b} \tag{33}$$

A simple closed dual timelike curve having parallel tangents in opposite directions at opposite points may be represented by equation (33). In this case a pair of opposite poinst of the curve is $(\tilde{\phi}_P(\overline{\varphi}), \tilde{\phi}_Q(\overline{\varphi}))$ for $\overline{\varphi}$ where $0 \leq \overline{\varphi} \leq 2\pi$. Since C is a simple closed dual timelike curve we get $\tilde{\phi}_Q(0) = \tilde{\phi}_Q(2\pi)$. Hence from (33) we have

$$\int_0^{2\pi} \overline{\rho}\,\overline{\tau} d\overline{\varphi} = 2n\pi, \quad (n \in \mathbb{Z}).$$

Using the equality $d\overline{s} = \overline{\rho} d\overline{\varphi}$, this formula may be given as $\int_C \overline{\tau} d\overline{s} = 2n\pi$. This says that the total torsion of C is equal to $2n\pi$, $(n \in \mathbb{Z})$. So, we can give the following corollary.

Corollary 3. *The total torsion of a simple closed dual timelike curve C of constant breadth is* $2n\pi$, $(n \in \mathbb{Z})$.

Let now assume that C is a spacelike curve. Then from Theorem 9 we have $\overline{\gamma} = 0$ and from (5) we obtain

$$\begin{cases} \overline{\delta} = R\cosh \int_0^{\overline{\varphi}} \overline{\tau}\,\overline{\rho} d\overline{\varphi} + S\sinh \int_0^{\overline{\varphi}} \overline{\tau}\,\overline{\rho} d\overline{\varphi} \\ \overline{\lambda} = -\left(R\sinh \int_0^{\overline{\varphi}} \overline{\tau}\,\overline{\rho} d\overline{\varphi} + S\cosh \int_0^{\overline{\varphi}} \overline{\tau}\,\overline{\rho} d\overline{\varphi} \right) \end{cases} \tag{34}$$

where R, S are constants. Then (2) gives us

$$\tilde{\zeta} = \tilde{\phi} + \left(R\cosh \int_0^{\overline{\varphi}} \overline{\tau}\,\overline{\rho} d\overline{\varphi} + S\sinh \int_0^{\overline{\varphi}} \overline{\tau}\,\overline{\rho} d\overline{\varphi} \right) \tilde{n} - \left(R\sinh \int_0^{\overline{\varphi}} \overline{\tau}\,\overline{\rho} d\overline{\varphi} + S\cosh \int_0^{\overline{\varphi}} \overline{\tau}\,\overline{\rho} d\overline{\varphi} \right) \tilde{b} \tag{35}$$

A simple closed dual spacelike curve having parallel tangents in opposite directions at opposite points may be represent by equation (35). In this case a pair of opposite points of the curve is $(\tilde{\phi}_P(\overline{\varphi}), \tilde{\phi}_Q(\overline{\varphi}))$ for $\overline{\varphi}$ where $0 \leq \overline{\varphi} \leq 2\pi$. Since C is a simple closed dual spacelike curve we get $\tilde{\phi}_Q(0) = \tilde{\phi}_Q(2\pi)$. Hence from (35) we have

$$\int_0^{2\pi} \overline{\rho}\,\overline{\tau} d\overline{\varphi} = 0.$$

Using the equality $d\overline{s} = \overline{\rho} d\overline{\varphi}$, this formula may be given as $\int_C \overline{\tau} d\overline{s} = 0$. This says that the total torsion of C is equal to zero. So, we can give the following corollary.

Corollary 4. *The total torsion of a simple closed dual spacelike curve C of constant breadth is zero.*

4 Conclusion

Characterizations of dual curves of constant breadth in dual Lorentzian space $\mathbb{D}_1^3$ are investigated in the paper. Some special cases such as the curves are helices are considered and the relationships between dual position vectors are obtained.

References

[1] Ayyıldız, N., Çöken, A.C., Yücesan, A., A Characterization of Dual Lorentzian Spherical Curves in the Dual Lorentzian Space, Taiwanese Journal of Mathematics, 11(4), 999-1018, (2007).

[2] Ball, N. H., On Ovals, American Mathematical Monthly, 37(7), 348-353, (1930).

[3] Barbier, E., Note sur le probleme de l'aiguille et le jeu du joint couvert. J. Math. Pures Appl., II. Ser. 5, 273-286, (1860).

[4] Blaschke, W., Konvexe bereiche gegebener konstanter breite und kleinsten inhalts, Mathematische Annalen, B. 76(4), 504-513, (1915).

[5] Blaschke, W., Einige Bemerkungen über Kurven und Flächen konstanter Breite. Ber. Verh. Sächs. Akad. Leipzig, 67, 290-297, (1915).

[6] Blaschke, W., Differential Geometric and Geometrischke Grundlagen ven Einsteins Relativitasttheorie Dover, New York, (1945).

[7] Hacısalihoğlu, H.H., Hareket Geometrisi ve Kuaterniyonlar Teorisi, Gazi Üniversitesi Fen-Edb. Fakültesi, (1983)

[8] Euler, L., De Curvis Triangularibus, Acta Acad. Petropol., 3-30 (1778-1780).

[9] Fujivara, M., On space curves of constant breadth, Tohoku Math., J., 5, 180-184, (1914).

[10] Kocayiğit, H., Önder, M., Space curves of constant breadth in Minkowski 3-space, Annali di Matematica, 192(5), 805-814, (2013).

[11] Kazaz, M., Önder, M., Kocayiğit, H., Spacelike curves of constant breadth in Minkowski 4-space, Int. Journal of Math. Analysis, 22(2), 1061-1068, (2008).

[12] Köse, Ö., On space curves of constant breadth, Doğa Tr. J. Math., 10(1), 11-14, (1986).

[13] Köse, Ö., Some Properties of Ovals and Curves of Constant Width in a Plane, Doğa Mat., 2(8), 119-126, (1984).

[14] Mağden, A., Köse, Ö., On the curves of constant breadth in E^4 space, Turkish J. Math., 21(3), 277-284, (1997).

[15] Mellish, A.P., Notes of Differential Geometry, Annals of Mathematics, 32(1), 181-190, (1931).

[16] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press, London, (1983).

[17] Önder, M., Kocayiğit, H., Candan, E., Differential Equations Characterizing Timelike and Spacelike Curves of Constant Breadth in Minkowski 3-Space E_1^3, J. Korean Math. Soc. 48(4), 849-866, (2011).

[18] Sezer, M., Differential equations characterizing space curves of constant breadth and a criterion for these curves, Turkish J. of Math. 13(2), 70-78, (1989).

[19] Struik, D. J., Differential Geometry in the Large, Bull. Amer. Math. Soc., 37(2), 49-62, (1931).

[20] Uğurlu, H., Çalışkan, A., The Study Mapping for Directed Space-like and Time-like Lines in Minkowski 3-Space $\mathbb{R}_1^3$, Math. and Comp. Appl. 1(2), 142-148, (1996).

[21] Veldkamp, G.R., On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics, Mech. Mach. Theory, 11(2), 141-156, (1976).

[22] Walrave, J., Curves and surfaces in Minkowski space, Doctoral thesis, K. U. Leuven, Fac. of Science, Leuven, (1995).

[23] Yılmaz, S., Time-like Dual Curves of Constant Breadth in Dual Lorentzian Space, IBSU Scientific Journal, 2(2), 129-136, (2008).

[24] Yılmaz, S., Savcı, Ü.Z., Ünlütürk, Y., On Dual Spacelike Curves of Constant Breadth in Dual Lorentzian Space $\mathbb{D}_1^3$, New Trends in Mathematical Sciences, 3(4), 164-170, (2015).