NEW TRENDS IN MATHEMATICAL SCIENCES
Vol. 2, No. 3, 2014, p.199-205.
ISSN 2147-5520 - www.ntmsci.com

Space Curves of Constant Breadth according to Bishop Frame in Euclidean 3-Space

Hüseyin KOCAYİĞİT, Muhammed ÇETİN

Department of Mathematics, Faculty of Science, Celal Bayar University, Manisa, Turkey
(E-mails : huseyin.kocayigit@cbu.edu.tr; mat.mcetin@hotmail.com)

Abstract: In this paper, space curves of constant breadth according to Bishop frame in Euclidean 3-space are studied. It is shown that in some special cases, space curves of constant breadth are slant helix. Moreover, the differential equations characterizing the space curves of constant breadth in E^3 are given.

Keywords: Curves of constant breadth; Bishop frame; Differential characterizations of curve; Euclidean 3-space .

1 Introduction

Euler introduced the curves of constant breadth [1]. He gave the constant breadth curves in the plane. Then, many geometers were interested in these special curves [2-9]. Furthermore, Reuleaux studied the curves of constant breadth and gave the method related to these curves for the kinematics of machinery [10]. Köse, showed that when a space curve (C) is given, a space curve (C^*) could be determined so that corresponding points the curves have parallel tangents in the opposite directions and the distance between these points is constant and using the concepts related to the space curves of constant breadth which were presented in [11]. The differential equations characterizing these curves were established and a criterion for these curves were given [12]. The concepts related to space curve of constant breadth were extended to E^n-space in [13]. Akdoğan and Mağden obtained an approximate solution of the equation system which belongs to these curves. Using this solution vectorial expression of the curve of constant breadth was gained. Also, Mağden and Köse investigated the curves of constant breadth in E^4-space [14]. After then, Önder and et. al., gave the diferential equations characterizing the timelike and spacelike curves of constant breadth in Minkowski 3-space [15]. Furthermore, they gave a criterion for a timelike or spacelike curve to be curve of constant breadth in E_1^3. Also, Kocayiğit and Önder showed that in E_1^3 spacelike and timelike curves of constant were normal, helices and spherical curves in some special cases [16].

In this paper, we study the space curves of constant breadth according to Bishop frame in E^3. And we give differential characterizations of these kind of curves. In addition, we show that space curves of constant breadth are related to slant helix.

2 Preliminaries

Now, we give some basic concepts on classical differential geometry of space curves. If a space curve in Euclidean 3-space is differentiable at each point of an open interval, a set of mutually orthogonal unit vectors can be constructed. These vectors are called Frenet frame. The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curve. Let $\alpha(s)$ be a space curve, where s is an arc length parameter and let $\left\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\right\}$ be Frenet frame of this curve. Here $\vec{T}$, $\vec{N}$ and $\vec{B}$ are called, the unit tangent vector field, the unit principal normal vector field and the unit binormal vector field of the curve, respectively. $\kappa(s)$ and $\tau(s)$ are called, curvature and torsion of the curve α , respectively. The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{N}} \\ \dot{\vec{B}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

where $\langle \vec{T},\vec{T} \rangle = \langle \vec{N},\vec{N} \rangle = \langle \vec{B},\vec{B} \rangle = 1$ and $\langle \vec{T},\vec{N} \rangle = \langle \vec{N},\vec{B} \rangle = \langle \vec{T},\vec{B} \rangle = 0$.

The parallel transport frame is an alternative approach to defining a moving frame that is well-defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame [17].

Its mathematical properties derive from the observation that, while $\vec{T}(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $\left(\overrightarrow{N_1}(s), \overrightarrow{N_2}(s)\right)$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $\vec{T}(s)$ at each point. If the derivatives of $\left(\overrightarrow{N_1}(s), \overrightarrow{N_2}(s)\right)$ depend only on $\vec{T}(s)$ and not each other, we can make $\overrightarrow{N_1}(s)$ and $\overrightarrow{N_2}(s)$ vary smoothly throughout the path regardless of the curvature. We may therefore choose the alternative frame equations

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\overrightarrow{N_1}} \\ \dot{\overrightarrow{N_2}} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \overrightarrow{N_1} \\ \overrightarrow{N_2} \end{bmatrix} \tag{1}$$

where $\langle \vec{T},\vec{T} \rangle = \langle \overrightarrow{N_1},\overrightarrow{N_1} \rangle = \langle \overrightarrow{N_2},\overrightarrow{N_2} \rangle = 1$ and $\langle \vec{T},\overrightarrow{N_1} \rangle = \langle \overrightarrow{N_1},\overrightarrow{N_2} \rangle = \langle \vec{T},\overrightarrow{N_2} \rangle = 0$. [18,19].

One can show that [18]

$$\kappa(s) = \sqrt{k_1^2 + k_2^2}\ ,\ \theta(s) = \arctan\left(\frac{k_2}{k_1}\right),\ \tau(s) = \frac{d\theta(s)}{ds}$$

$$k_1 = \kappa\cos(\theta)\ ,\ k_2 = \kappa\sin(\theta)$$

and

$$\vec{T} = \vec{T}\ ,\ \overrightarrow{N_1} = \vec{N}\cos(\theta) - \vec{B}\sin(\theta)\ ,\ \overrightarrow{N_2} = \vec{N}\sin(\theta) + \vec{B}\cos(\theta)$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ,θ with $\theta = \int \tau(s)ds$. A fundamental ambiguity in the parallel transport frame compared to the Frenet frame thus arise from the arbitrary choice of an integration constant for θ_0, which disappears from τ due to the differentation [19].

Theorem 1. Let $\alpha : I \to E^3$ be a unit speed curve with non-zero natural curvatures. Then α is a slant helix if and only if $\frac{k_1}{k_2}$ is constant [20].

3 Curves of Constant Breadth

In this section, we study the space curves of constant breadth according to Bishop frame in Euclidean 3-space. We obtain the differential equations characterizing curves of constant breadth according to Bishop frame in Euclidean 3-space and it is shown that space curves of constant breadth are slant helix in some special cases.

Definition 1. Let (C) be a space curve in E^3. If (C) has parallel tangents in opposite directions at the opposite points $\alpha(s)$ and $\alpha^*(s)$ and if the distance between these points is always constant, then (C) is called a space curve of constant breadth [6].

Definition 2. A pair of space curves (C) and (C^*) in E^3 for which the tangents at the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$ are parallel and in opposite directions, and the distance between these points is always constant are called space curve pair of constant breadth [11].

Let (C) and (C^*) be a pair of unit-speed space curves with non-zero Bishop curvatures in E^3 and let those curves have paralel tangents in opposite directions at the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$, repectively. The position vector of the curve (C^*) at the point $\alpha^*(s^*)$ can be expressed as

$$\overrightarrow{\alpha^*}\left(s^*\right)=\vec{\alpha}\left(s\right)+\lambda_1\left(s\right)\vec{T}\left(s\right)+\lambda_2\left(s\right)\overrightarrow{N_1}(s)+\lambda_3\left(s\right)\overrightarrow{N_2}(s) \tag{2}$$

where $\lambda_i(s)$ $(i=1,2,3)$ are differentiable functions of s which is arc lenght of (C). Denote by $\left\{\vec{T},\overrightarrow{N_1},\overrightarrow{N_2}\right\}$, k_1 and k_2 the moving Bishop frame, Bishop curvatures along the curve (C), respectively. And denote by $\left\{\overrightarrow{T^*},\overrightarrow{N_1^*},\overrightarrow{N_2^*}\right\}$, k_1^* and k_2^* the moving Bishop frame, Bishop curvatures along the curve (C^*), respectively.

Differentiating (2) with respect to s and by using the Bishop formulae given by (1), we obtain

$$\frac{d\overrightarrow{\alpha^*}}{ds}=\overrightarrow{T^*}\frac{ds^*}{ds}=\left(1+\frac{d\lambda_1}{ds}-k_1\lambda_2-k_2\lambda_3\right)\vec{T}+\left(k_1\lambda_1+\frac{d\lambda_2}{ds}\right)\overrightarrow{N_1}+\left(k_2\lambda_1+\frac{d\lambda_3}{ds}\right)\overrightarrow{N_2}.$$

Since $\overrightarrow{T^*}=-\vec{T}$ at the corresponding points of the curves (C) and (C^*), we obtain the following differential equation system

$$\begin{cases}\dfrac{d\lambda_1}{ds}=-\dfrac{ds^*}{ds}-1+k_1\lambda_2+k_2\lambda_3\\ \dfrac{d\lambda_2}{ds}=-k_1\lambda_1\\ \dfrac{d\lambda_3}{ds}=-k_2\lambda_1.\end{cases} \tag{3}$$

It is well-known that the curvature $\kappa(s)$ of the curve (C) is

$$\lim_{\Delta s\to 0}\frac{\Delta\varphi}{\Delta s}=\frac{d\varphi}{ds}=\kappa\left(s\right)$$

where φ is the angle between the tangent $\vec{T}$ of the curve (C) and a given fixed direction at the point $\alpha(s)$. Hence, we can rewrite the system (3) as follow.

$$\begin{cases}\dfrac{d\lambda_1}{d\varphi}=\mu_1\lambda_2+\mu_2\lambda_3-f\\ \dfrac{d\lambda_2}{d\varphi}=-\mu_1\lambda_1\\ \dfrac{d\lambda_3}{d\varphi}=-\mu_2\lambda_1\end{cases} \tag{4}$$

where

$$\mu_1=\rho k_1=\frac{k_1}{\kappa}=\cos(\theta),\ \mu_2=\rho k_2=\frac{k_2}{\kappa}=\sin(\theta),\quad \left(\theta=\int\tau ds\right)$$

and

$$f(\varphi)=\rho+\rho^*,\ \rho=\frac{1}{\kappa}\ \text{and}\ \rho^*=\frac{1}{\kappa^*}.$$

Here, ρ and ρ^* indicates the radius of curvatures at the points $\alpha(s)$ and $\alpha^*(s^*)$, respectively.

Eliminating λ_2, λ_3 and their derivatives from the system (4), we obtain the following differential equation of third order with respect to λ_1.

$$a_1\lambda_1''' + b_1\lambda_1'' + c_1\lambda_1' + d_1\lambda_1 = e_1 \tag{5}$$

where

$$\begin{aligned}
a_1 &= -\left(\mu_1\mu_2' - \mu_2\mu_1'\right)^3 - \mu_1''\mu_2^3\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\\
b_1 &= -\mu_2\left[(\mu_1'')^2\mu_2^3 + \mu_1''\left(-\mu_2^2\mu_1\mu_2'' + (\mu_1\mu_2' - \mu_2\mu_1')^2\right) + \mu_2''(\mu_1\mu_2' - \mu_2\mu_1')^2\right]\\
c_1 &= \mu_1''\left[-\mu_2^3\mu_2''\mu_1' - (\mu_1\mu_2' - \mu_2\mu_1')\left(\mu_2^3\mu_1^2 - \mu_1(\mu_2')^2 + \mu_2\mu_1'\mu_2' + \mu_2^5\right)\right]\\
&\quad + (\mu_1'')^2\mu_2^3\mu_2' - (\mu_1\mu_2' - \mu_2\mu_1')^2\left(-\mu_2''\mu_2' + (\mu_1^2 + \mu_2^2)(\mu_1\mu_2' - \mu_2\mu_1')\right)\\
d_1 &= -\mu_2\mu_1''\left[\left(-\mu_2^4\mu_1 - \mu_2^2\mu_1^3\right)\mu_2'' + \left(\left(\mu_1^3 + \mu_1^2\mu_2 + 3\mu_2^3\right)\mu_2' - \mu_1\mu_2\mu_1'(-2\mu_2 + \mu_1)\right)\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\right]\\
&\quad - \left(\mu_2^6 + \mu_2^4\mu_1^2\right)(\mu_1'')^2 - \left[\left(\mu_1^2\mu_2 + \mu_2^3\right)\mu_2'' + 3\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(\mu_1\mu_1' + \mu_2\mu_2'\right)\right]\left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\\
e_1 &= \mu_1''\left[\mu_2^3\left(f\mu_1' - f'\mu_1\right)\mu_2'' - \left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(-\mu_2^3 f'' + \left(\mu_1\mu_2' - \mu_2\mu_1'\right)\left(-\mu_2 f' + \mu_2' f\right)\right)\right]\\
&\quad + (\mu_1'')^2\left(-\mu_2^3\mu_2' f + \mu_2^4 f'\right) - \left(\mu_1\mu_2' - \mu_2\mu_1'\right)^2\left[\left(-\mu_2 f' + \mu_2' f\right)\mu_2'' - f''\left(\mu_1\mu_2' - \mu_2\mu_1'\right)\right].
\end{aligned}$$

Here and later $(')$ denotes the differentiation with respect to "φ". Similarly, eliminating λ_1, λ_3 and their derivatives from the system (4), we obtain the following differential equation of third order with respect to λ_2.

$$a_2\lambda_2''' + b_2\lambda_2'' + c_2\lambda_2' + d_2\lambda_2 = e_2 \tag{6}$$

where

$$\begin{aligned}
a_2 &= -\mu_1^2\mu_2\\
b_2 &= 2\mu_1'\mu_1\mu_2 + \mu_2'\mu_1^2\\
c_2 &= \mu_1''\mu_1\mu_2 - 2(\mu_1')^2\mu_2 - \mu_1^4\mu_2 - \mu_2'\mu_1'\mu_1 - \mu_2^3\mu_1^2\\
d_2 &= -\mu_1^3\left(\mu_1'\mu_2 - \mu_2'\mu_1\right)\\
e_2 &= \mu_1^3\left(\mu_2' f - f'\mu_2\right).
\end{aligned}$$

Furthermore, eliminating λ_1, λ_2 and their derivatives from the system (4), we obtain the following differential equation of third order with respect to λ_3.

$$a_3\lambda_3''' + b_3\lambda_3'' + c_3\lambda_3' + d_3\lambda_3 = e_3 \tag{7}$$

where

$$\begin{aligned}
a_3 &= -\mu_2^2\mu_1\\
b_3 &= 2\mu_2'\mu_2\mu_1 + \mu_1'\mu_2^2\\
c_3 &= \mu_2''\mu_2\mu_1 - 2(\mu_2')^2\mu_1 - \mu_1'\mu_2'\mu_2 - \mu_1^3\mu_2^2 - \mu_2^4\mu_1\\
d_3 &= -\mu_2^3\left(\mu_2'\mu_1 - \mu_1'\mu_2\right)\\
e_3 &= \mu_2^3\left(\mu_1' f - f'\mu_1\right).
\end{aligned}$$

When the curve (C) and the function $f(\varphi)$ are given, from the solving the systems (3), (4) or the equations (5), (6), (7), we can find the values of λ_1 $(i=1,2,3)$. Eqs. (5), (6), (7) express the differential characterizations for the space curves (C) and (C^*) according to λ_i.

As the curves (C) and (C^*) are a space curve pair of constant breadth, then the distance d between the corresponding points $\alpha(s)$ and $\alpha^*(s^*)$ is constant. Hence,

$$d^2=\left\|\vec{d}\right\|^2=\left\|\overrightarrow{\alpha^*}(s^*)-\vec{\alpha}(s)\right\|^2=\lambda_1^2+\lambda_2^2+\lambda_3^2=k^2=const. \quad k\in\mathbb{R} \tag{8}$$

Differentiating (8) with respect to φ, we gain

$$\frac{1}{2}\frac{d}{d\varphi}\left\|\vec{d}\right\|^2=\lambda_1\lambda_1'+\lambda_2\lambda_2'+\lambda_3\lambda_3'=0\,.$$

Substituting the equalities given by (4) into the (8), we obtain the following equality.

$$\lambda_1 f=0\,.$$

This relation express to be curve pair of constant breadth of the space curves (C) and (C^*) in E^3. Here, there are two main cases.

Case 1. Let $f(\varphi)=0$ $\left(\dfrac{ds^*}{ds}+1=0\right)$. This means that the curve (C^*) is a translation of the curve (C) by the constant vector

$$\vec{d}=\lambda_1\vec{T}+\lambda_2\overrightarrow{N_1}+\lambda_3\overrightarrow{N_2}\,. \tag{9}$$

In fact, if $f(\varphi)=0$ then the vector $\vec{d}$ is constant. To verify this fact, differentiate (9) with respect to φ and use the equalities (4) for $f=0$ and Bishop formulae given by (1). Hence, we obtain $\dfrac{d\vec{d}}{d\varphi}=0$.

Consequently, if $\dfrac{d\vec{d}}{d\varphi}=0$ then the vector $\vec{d}$ is constant. In this case we can rewrite the systems (3), (4) and the equations (5), (6), (7) as follows:

$$\begin{cases}\dfrac{d\lambda_1}{ds}=k_1\lambda_2+k_2\lambda_3\\[2mm] \dfrac{d\lambda_2}{ds}=-k_1\lambda_1\\[2mm] \dfrac{d\lambda_3}{ds}=-k_2\lambda_1\end{cases} \tag{10}$$

$$\begin{cases}\dfrac{d\lambda_1}{d\varphi}=\mu_1\lambda_2+\mu_2\lambda_3\\[2mm] \dfrac{d\lambda_2}{d\varphi}=-\mu_1\lambda_1\\[2mm] \dfrac{d\lambda_3}{d\varphi}=-\mu_2\lambda_1\end{cases} \tag{11}$$

and

$$a_1\lambda_1'''+b_1\lambda_1''+c_1\lambda_1'+d_1\lambda_1=0 \tag{12}$$

where

$$a_1=-\left(\mu_1\mu_2'-\mu_2\mu_1'\right)^3-\mu_1''\mu_2^3\left(\mu_1\mu_2'-\mu_2\mu_1'\right)$$

$$b_1=-\mu_2\left[(\mu_1'')^2\mu_2^3+\mu_1''\left(-\mu_2^2\mu_1\mu_2''+(\mu_1\mu_2'-\mu_2\mu_1')^2\right)+\mu_2''(\mu_1\mu_2'-\mu_2\mu_1')^2\right]$$

$$c_1 = \mu_1'' \left[-\mu_2^3 \mu_2'' \mu_1' - (\mu_1\mu_2' - \mu_2\mu_1') \left(\mu_2^3\mu_1^2 - \mu_1(\mu_2')^2 + \mu_2\mu_1'\mu_2' + \mu_2^5 \right) \right]$$
$$+ (\mu_1'')^2 \mu_2^3 \mu_2' - (\mu_1\mu_2' - \mu_2\mu_1')^2 \left(-\mu_2''\mu_2' + (\mu_1^2 + \mu_2^2)(\mu_1\mu_2' - \mu_2\mu_1') \right)$$
$$d_1 = -\mu_2\mu_1'' \left[\left(-\mu_2^4\mu_1 - \mu_2^2\mu_1^3 \right)\mu_2'' + \left(\left(\mu_1^3 + \mu_1^2\mu_2 + 3\mu_2^3 \right)\mu_2' - \mu_1\mu_2\mu_1'(-2\mu_2 + \mu_1) \right)\left(\mu_1\mu_2' - \mu_2\mu_1' \right) \right]$$
$$-\left(\mu_2^6 + \mu_2^4\mu_1^2 \right)(\mu_1'')^2 - \left[\left(\mu_1^2\mu_2 + \mu_2^3 \right)\mu_2'' + 3\left(\mu_1\mu_2' - \mu_2\mu_1' \right)\left(\mu_1\mu_1' + \mu_2\mu_2' \right) \right]\left(\mu_1\mu_2' - \mu_2\mu_1' \right)^2 .$$

$$a_2\lambda_2''' + b_2\lambda_2'' + c_2\lambda_2' + d_2\lambda_2 = 0 \tag{13}$$

where

$$a_2 = -\mu_1^2\mu_2$$
$$b_2 = 2\mu_1'\mu_1\mu_2 + \mu_2'\mu_1^2$$
$$c_2 = \mu_1''\mu_1\mu_2 - 2(\mu_1')^2\mu_2 - \mu_1^4\mu_2 - \mu_2'\mu_1'\mu_1 - \mu_2^3\mu_1^2$$
$$d_2 = -\mu_1^3\left(\mu_1'\mu_2 - \mu_2'\mu_1 \right).$$

$$a_3\lambda_3''' + b_3\lambda_3'' + c_3\lambda_3' + d_3\lambda_3 = 0 \tag{14}$$

where

$$a_3 = -\mu_2^2\mu_1$$
$$b_3 = 2\mu_2'\mu_2\mu_1 + \mu_1'\mu_2^2$$
$$c_3 = \mu_2''\mu_2\mu_1 - 2(\mu_2')^2\mu_1 - \mu_1'\mu_2'\mu_2 - \mu_1^3\mu_2^2 - \mu_2^4\mu_1$$
$$d_3 = -\mu_2^3\left(\mu_2'\mu_1 - \mu_1'\mu_2 \right).$$

Theorem 2. The general differential equations and systems characterizing space curve pair of constant breadth according to Bishop frame in E^3 are given by (10), (11), (12), (13) and (14).

Case 2. Let $\lambda_1 = 0$. Then, there are three cases here.
i) We can take $\lambda_2 = const.$ and $\lambda_3 = 0$ (from (4)). Then, $f(\varphi) = \mu_1\lambda_2$.
ii) We can take $\lambda = 0$ and $\lambda_3 = const.$ (from (4)). Then, $f(\varphi) = \mu_2\lambda_3$.
iii) Now, we consider the third and interesting case $\lambda_2 = const.$ and $\lambda_3 = const.$
If $\lambda_2 = const.$, $\lambda_3 = const.$ and $f(\varphi) = 0$ (from (4)), then we obtain $\dfrac{k_1}{k_2} = \dfrac{-\lambda_3}{\lambda_2} = const.$ This means that the curve (C) is a slant helix according to Bishop frame. Thus we can give following theorem.

Theorem 3. Let consider the curve pair of constant breadth which has the sum of curvature radius at corresponding points is zero according to Bishop frame in E^3. If the first normal component $\lambda_2 = const.$ and the second normal component $\lambda_3 = const.$ given by (2), then the curve (C) is a slant helix.

References

[1]. Euler, L., De Curvis Triangularibus, Acta Acad. Petropol., 3-30, 1778 (1780).
[2]. Ball, N.H., On ovals, American Mathematical Monthly, 37(7) : 348-353, 1930.
[3]. Barbier, E., Note sur le probléme de I'aiguille et le jeu du joint couvert, Journal de mathématiques pures et appliquées, 2(5) : 273-286, 1860.
[4]. Blaschke, W., Leibziger Berichte, 67 : 290, 1917.
[5]. Mellish, A.P., Notes on Differential Geometry, Annals of Mathematics, 32(1) : 181-190, 1931.
[6]. Fujiwara, M., On Space Curves of Constant Breadth, Tohoku Mathematical Journal, 5 : 180-184, 1914.
[7]. Hammer, P.C., Constant Breadth Curves in the Plane, Procedings of the American Mathematical Society, 6(2) : 333-334, 1955.

[8]. Smakal, S., Curves of Constant Breadth, Czechoslovak Mathematical Journal, 23(1) : 86-94, 1973.
[9]. Köse, Ö., Düzlemde Ovaller ve Sabit Genişlikli Eğrilerin Bazı Özellikleri, Doğa Bilim Dergisi, Seri B, 8(2) : 119-126, 1984.
[10]. Reuleaux, F., The Kinematics of Machinery, Translated by A. B. W. Kennedy, Dover Pub. New York, 1963.
[11]. Köse, Ö., On Space Curves of Constant Breadth, Doğa Tr. J. Math, 10(1) : 11-14, 1986.
[12]. Sezer, M., Differential Equations Characterizing Space Curves of Constant Breadth and a Criterion for These Curves, Turkish J. of Math, 13(2) : 70-78, 1989.
[13]. Akdoğan, Z., and Mağden, A., Some Characterization of Curves of Constant Breadth in E^n Space, Turk J Math, 25 : 433-444, 2001.
[14]. Mağden, A., and Köse, Ö., On the Curves of Constant Breadth in E^4 Space, Tr. J. of Mathematics, 21 277-284, 1997.
[15]. Önder, M., Kocayiğit, H. and Candan, E., Differential Equations Characterizing Timelike and Spacelike Curves of Constant Breadth in Minkowski 3-Space E_1^3, J. Korean Math. Soc. 48(4) : 849-866, 2011.
[16]. Kocayiğit, H. and Önder, M., Space Curves of Constant Breadth in Minkowski 3-Space, Annali di Matematica, 192(5) : 805-814, 2013.
[17]. Hanson, A.J. and Ma, H., Parallel Transport Approach to Curve Framing, Indiana University, Technical Report TR425, January 11, 1995.
[18]. Bishop, R.L., There is More Than One Way to Frame a Curve, American Mathematical Monthly, 82(3) : 246-251, 1975.
[19]. Hanson, A.J., and Ma, H., Quaternion Frame Approach to Streamline Visualization, IEEE Transactions on Visulation and Computer Graphics, 1(2) : 164-174, 1995.
[20]. Bükcü, B. and Karacan, M.K., The Slant Helices according to Bishop Frame, World Academy of Science, Engineering and Technology, 3(11) : 853-856, 2009.