BIRUNI UNIVERSITY
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10th (Online) International Conference on Applied Analysis and Mathematical Modeling

ICAAMM22 July 1-3 2022
Istanbul-Turkey

Abstracts and Proceedings Book

Editors
Mustafa Bayram
Aydın Seçer

10th (Online) International Conference on Applied Analysis and Mathematical Modeling ($\mathcal{ICAAMM}$22) July 1-3, 2022, Istanbul-Turkey

Abstracts and Proceedings Book

Prof. Dr. Mustafa Bayram
Prof. Dr. Aydın Seçer

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Participant Statistics

112 participants from 31 different countries attended the conference, 20 of them from Turkey and the others from abroad, so 82% participants are foreigners and 18% participants are Turkish.

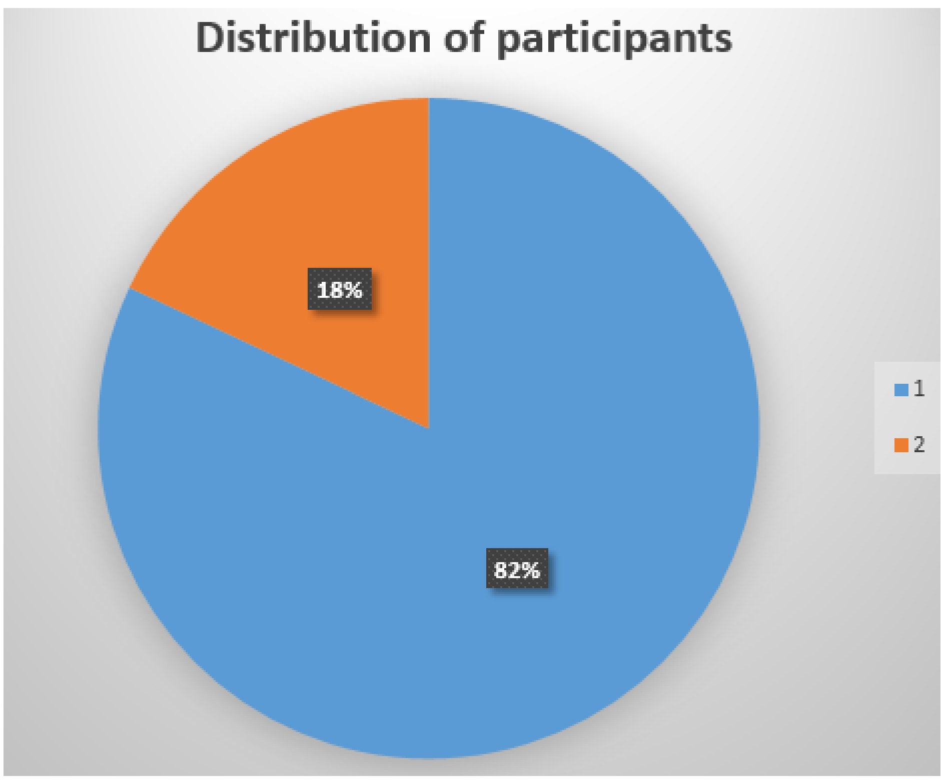

Figure 1: 1. Foreign participants, 2. Turkish participants

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MESSAGE FROM CHAIRMAN

The "10th (Online) International Conference on Applied Analysis and Mathematical Modeling, 2022" organized by Biruni University will be held on 1-3 July 2022 in Istanbul, Turkey. Due to the Covid-19 Pandemic, we could not meet face to face. For this reason, we decided to make it online by technology. The aim of this conference is to bring the Mathematics & Engineering Sciences community working in the new trends of applications of Mathematics together in a wonderful city of the world, Istanbul.

There have been quite a big number of applications from different part of the world and as you know when the number increase task of the organizing committee will increase. Thus it was a very difficult task to select and classify the abstracts for all the participants. We tried to do our best to accommodate many speakers in order to have a better and enjoyable research session which will provide more interactions, exchanges among the participants.

Besides the scientific program, we had some social activities (excursion boat trip, city tour, etc.) where we could continue some informal discussions that would serve the purpose of our meeting in such a short time. We had to cancel due to the pandemic. As we can see from the list of participants, many speeches by young researchers will also serve the purpose of this conference.

The talks will cover a wide range of mathematics and its applications such as analysis, algebra, statistics, computer mathematics, discrete mathematics, geometry, engineering, etc. as well as their use in modeling. We believe that this richness will provide the basis for interdisciplinary collaborations.

We also would very much thank to all presenters and participants for their interests in the conference and believe and hope that each of them will get the maximum benefit in terms of networking and interaction from this meeting.

We would like to thank Dumitru Baleanu, Aydin Secer, Tuğçem Partal, Neslihan Ozdemir, Melih Cinar, Handenur Esen and Ismail Onder all our colleagues who worked for the organization of the conference.

Finally, we also would to thank to chairman of the board of trustees of Biruni University and Prof. Dr. Adnan Yüksel the Rector of Biruni University which is Host University.

Further we thank to all the plenary speakers that kindly accepted our invitation and spend their precious time by sharing their ideas during the conference. We also thank to all members of organizing committee.

We apologize for any shortcomings or might not be mentioned unintentionally or may have been forgotten to be mentioned explicitly here. We really hope their kind understanding, we thank all and each individual that have put their effort to make this occasion possible.

We welcome each and every one of you again to this conference; we wish a enjoyable and productive conference and hope to meet again in future occasions.

Sincerely Yours,
Prof. Dr. Mustafa Bayram,
Conference Chair

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James Davenport, University of Bath, United Kingdom
Konstantinos Lazopoulos, National Technical University of Athens, Greece
Sverre Holm, University of Oslo, Norway
Dumitru Baleanu, Çankaya University, Turkey
Maslina Darus, Universiti Kebangsaan Malaysia, Malaysia
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Xiao Jun Yang, China University of Mining and Technology, China
Abdel Salam Gomaa Abdel Salam, Qatar University, Qatar
Md. Haider Ali Biswas, Khulna University, Bangladesh

.

ORAL PRESENTATIONS

Hom type generalization of cohomology and deformation of n Lie algebras morphisms

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Abstract: The main purpose of this paper is to define cohomology complex of n-Hom-Lie algebra morphisms and consider their deformation theory. In particular, we discuss infinitesimal deformations, equivalent deformations and obstructions. Moreover, we study Morphism of 3-Hom-Lie algebra induced by morphism of Hom-Lie algebra and provide example.

Keywords: n-Hom-Lie algebra, n-Hom-Lie algebra morphism, cohomology, deformation

Mathematics Subject Classification: 17A40, 17A42, 17B56, 17B61

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A New Method for Solving the Conformable Fractional Nonlinear Partial Differential Equations with Proportional Delay

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Abstract: The conformable fractional nonlinear partial differential equations with proportional by a new method, called conformable q-homotopy analysis transform method are analyzed. The suggested method is the combination of q-homotopy analysis transform method and conformable fractional derivative. We have observed that the numerical simulations verify the proposed method is efficient and reliable.

Keywords: Conformable q-homotopy analysis transform method, conformable fractional nonlinear partial differential equation, conformable Laplace transform

Mathematics Subject Classification: 35R11, 35C05, 65R10.

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To investigate numerically a multi term q-differential equations of arbitrary fractional order

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Abstract: In this research, we investigate a fractional boundary value problem of nonlinear q-differential equations with arbitrary orders. New existence and uniqueness results are established using Banach contraction principle, Schaefer and Krasnoselskii fixed point theorems. In order to clarify our results, some illustrative examples are also presented with numerical technique.

Keywords: Multi term, q-differential equations, numerical technique, arbitrary orders

Mathematics Subject Classification: 34A08, 34B16, 39A13

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An antiplane frictional contact problems with viscosity

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Abstract: We study a mathematical model which describes the antiplane shear deformation of a cylinder in frictionless contact with a rigid foundation. First we derive the classical variational formulation of the model with viscosity term, then the problem is a system coupling an evolutionary variational equality for the displacement field with a time-dependent variational equation for the potential field. Then we prove the existence of a unique weak solution to the model.

Keywords: Restriction, Interpolation, error estimate, contacts problem, weak solution, formulation variational.

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Mathematical Modeling of Tumor Growth and the Application of Chemo-immunotherapy and Radiotherapy Treatments

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Abstract: In this research, we proposed the following: a model that represents tumor growth caused by carcinogenic substances from an initially diagnosed level; secondly, we further proposed an expanded tumor growth model with several other therapeutic models such as chemotherapy, immunotherapy, and radiotherapy in an attempt to reduce the growing tumor with a steady supply of dosage. The objectives are to solve the proposed models analytically where independent solutions were obtained for chemotherapy, immunotherapy, and radiotherapy with some pertinent parameters depicting the interaction of the tumor and various treatments. We developed numerical simulation codes using Wolfram Mathematica software, version 12, to simulate and study the various interactions by varying the pertinent parameters' effects on tumor growth and on the therapeutic treatment solutions. The numerical simulation revealed that continuous exposure to radiation could cause pain and normal cell death, thereby supporting the tumor cell proliferation, but the chemotherapeutic and immunotherapeutic drugs help in shrinking the tumor and reducing cell proliferation. The simulated results for the mixed treatment application for all therapies indicate that there is a successful and faster decrease in tumor size by combining the treatment regimes. In conclusion, we have successively proposed and solved mathematical models that represent tumor growth without treatment and with the application of treatment. The models in this article can be used by scientists and oncologists in studying tumor cell proliferation.

Keywords: Modeling, Carcinogenic, Treatment, Cancer, Tumor, Proliferation, Radiotherapy, Chemotherapy, Immunotherapy, Growth

Mathematics Subject Classification: 92C50.

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On a time-dependent second-order sweeping process

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Abstract: In my presentation, I will aim to highlight the existence of solutions of a second-order sweeping process with a single-valued Lipschitz. I proceed without any compactness assumption in the moving set, which is required to be crucial in the proof of the existence of solution for the time and state-dependent first or second-order sweeping processes (see [1,2,3,4,6] and references therein). Actually, dealing without compacity in infinite dimentional space seems to be an open problem. One will be probably need at some point to an alternative way. In such a view setting, following the work of Bounkhel and al [5], dealing with the absolutely continuous solutions for a second-order state-dependent sweeping process, to compensate the lack of compacity, the only outlet was the use of anti-monotonicity of the set in normal cone. Likewise, for the first-order case, Adly-Haddad-Le [1] argue by the property of hypomonotony-like of the normal cone. Recently, Nacry-Sofonea [7], used arguments of monotonicity, convexity and fixed point to get the existence of solution for a first-order type sweeping process.

Keywords: Cauchy criterion, Lipschitz perturbation, normal cone, prox-regularity, sweeping process

Mathematics Subject Classification: 40A30, 34A60, 49J40, 49J52, 49J53.

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An Extension of Functional Projection Pursuit Regression to the Generalized Partially Linear Single Index Models (EFPPR-GPLSIM)

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Abstract: In this paper, we introduce a functional approach to approximate the non-parametric function in the case of multivariate predictors, the single-index coefficient, the non-linear regression function in the case of functional predictors and a scalar response. Following the Fisher-scoring algorithm and the principle of projection pursuit regression, we derive an additive decomposition that exploits the most predictive direction, the most predictive additive component of the functional predictor variable and the single-index component to explain the scalar response. On the one hand, this approach allows us to avoid the well-known problem of the curse of dimensionality in the non-parametric case with the notion of single index and the projection pursuit regression in the functional case, on the other hand, it can be used as an explonatory tool for the analysis of a multivariate and functional random variable belonging to a separable Hilbert space H. The terms of this decomposition are estimated with an iterative Fisher scoring procedure that uses the Quasi-Likelihood function and an approximation of the non parametric function by normalized B-splines. The good behaviour of our procedure is illustrated from a theoretical and practical point of view. Asymptotic results indicate that the nonparametric function, the single index coefficient and the terms of the additive decomposition can be estimated without suffering from curse of dimensionality , while some applications to real and simulated data show the high predictive performance of our method.

Keywords: Additive decomposition, Asymptotic normality, Fisher scoring algorithm, Functional Data analysis (FDA), Polynomial Splines, Predictive directions, Projection pursuit regression, Quasi-likelihood, Single-index model.

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Rbf-Pum solution of magnetoconvection in a triangular cavity exposed to a uniform magnetic field

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Abstract: Numerical simulation of Al_2O_3-Cu/water hybrid nanofluid flow in an isosceles right triangular cavity exposed to either vertical or horizontal uniform magnetic field is numerically investigated. A local method, radial basis function based partition of unity method (Rbf-Pum), is performed to solve steady dimensionless governing equations in stream function-vorticity form numerically. Vertical magnetic field suppresses the fluid flow and heat transfer more than the horizontal one. The rise in magnitude of uniform magnetic field suppresses fluid flow and heat transfer. The dominance of convection is pronounced at large Rayleigh numbers.

Keywords: hybrid nanofluid, uniform magnetic field, Rbf-Pum, triangular cavity.

Mathematics Subject Classification: 76R10, 76W05, 80M22.

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Boundedness in generalized topological groups

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Abstract: The notions of generalized neighborhood systems and generalized topological space were given by Csaszar in 2002 [1]. Later, Hussain and et.al defined the notion of generalized topological group and studied some properties of the generalized topological groups [2]. In this study, we define the concept of bounded subset in a generalized topological group. Also, we obtain some characterizations about the bounded subsets in generalized topological groups.

Keywords: Generalized topology, generalized topological group, bounded subset.

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Some topological properties on Orlicz generalized difference sequence spaces

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Abstract: The main purpose of this work is to generalize the Orlicz sequence space by using generaized difference operators and a sequence of non-zero scalars and investigate some topological structure relevant to this generalized space.

Keywords: Generalized difference sequence space, Orlicz function, Köthe-Toeplitz dual.

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Some results of the generalized difference sequence space $l_p(\widehat{T}^q)_s^r$

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Abstract: In this work, we introduce a new matrix $\widehat{T}^q(r,s) = \left(\widehat{t}^q_{nk}(r,s)\right)$ in which $t_k > 0$ for all $k \in IN$, $(t_k) \in c \backslash c_0$ and $r, s \in IR - \{0\}$. By using the matrix, we introduce a new sequence space $l_p(\widehat{T}^q)_s^r$ for $1 \leq p < \infty$. Moreover, we obtain some theorems on inclusion relations associated with newly defined space and calculate the $\alpha-$, $\beta-$ and $\gamma-$duals of this space.

Keywords: Generalized difference sequence space, $\alpha-$, $\beta-$ and $\gamma-$duals.

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On numerical solutions of nonlinear differential equations with initial conditions by Hermite collocation method

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Abstract: In this study, Hermite collocation method is used for solving a class of nonlinear differential equations with initial conditions. The problem is reduced into a nonlinear algebraic system, later the unknown coefficients of the approximate solution function are calculated. A test problem is presented to show the performance of the proposed method. Additionally, the obtained numerical results are compared with exact solution of the test problem.

Keywords: Hermite collocation method, Nonlinear differential equations, Initial value problems.

Mathematics Subject Classification: 65D15, 65L60, 65L70.

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Approximation of Kantorovich-type generalization of $(p,q)-$Bernstein type Rational Functions Via statistical convergence

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Abstract: Recently, different $q-$generalizations of Balázs-Szabados operators have been studied by several researchers. İn [1], The Kantorovich type $q-$analogue of the Balázs-Szabados operators is defined by Hamal and Sabancigil as follows:

$$R_{n,q}^{*}(f,x)=\sum_{k=0}^{n} r_{n,k}(q,x)\int_{0}^{1} f\left(\frac{[k]_q+q^k t}{b_n}\right) d_q t, \tag{1}$$

where $f:[0,\infty)\to \mathrm{R}$, $q\in(0,1)$,$a_n=[n]_q^{\beta-1}$, $b_n=[n]_q^{\beta}$, $0<\beta\le\frac{2}{3}$, $n\in\mathrm{N}$, $x\ge 0$,
and $r_{n,k}(q,x)=\frac{1}{(1+a_n x)^n}\begin{bmatrix} n \\ k\end{bmatrix}_q (a_n x)^k \prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_n x\right)$.
Latterly, Hamal and Sabancigil introduced a new Kantorovich-type $(p,q)-$analogue of the Balázs–Szabados operators by generalizing the new Kantorovich-type $q-$analogue of Balázs–Szabados operators, given by (1), as follows:

$$R_{n,p,q}^{*}(f,x)=\sum_{k=0}^{n} r_{n,k}^{*}(p,q,x)\int_{0}^{1} f\left(\frac{p^{n-k}\left([k]_{p,q}+q^k t\right)}{b_n}\right) d_{p,q} t\ , \tag{2}$$

where $r_{n,k}^{*}(p,q,x)=\frac{1}{p^{n(n-1)/2}}\begin{bmatrix} n \\ k\end{bmatrix}_{p,q} p^{k(k-1)/2}\left(\frac{a_n x}{1+a_n x}\right)^k \prod_{j=0}^{n-k-1}\left(p^j-q^j\frac{a_n x}{1+a_n x}\right)$
and $0<q<p\le 1$, $a_n=[n]_{p,q}^{\beta-1}$, $b_n=[n]_{p,q}^{\beta}$, $0<\beta\le\frac{2}{3}$, $n\in\mathrm{N}$, $x\ge 0$, $f:[0,\infty)\to\mathrm{R}$. Fast [4] and Fridy [5] provided the following notions.
Suppose that $E\subseteq\mathrm{N}=\{1,\ 2,\ ...\}$ and $E_n=\{k\le n: k\in E\}$. Then $\delta(E)=\lim_{n\to\infty}\frac{1}{n}|E_n|$ is called natural density of E provided that the limit exists.
Definition 1. A sequence $x=(x_n)$ is statistically convergent to the number L if for every $\varepsilon>0$, we have $\delta\{k\in\mathrm{N}:|x_k-L|\ge\varepsilon\}=0$ is denoted by $st_A-\lim_{n\to\infty}x_n=L$. Because all finite subsets of the natural numbers have density zero, any convergent sequence is statistically convergent, but not contrariwise.
In [6], Bohman –Korovkin type statistical approximation theorem was proved by Gadjiev and Orhan.
Now, the main result of this research is to use modulus of continuity to study the rate of A-statistical convergence of Kantorovich type $(p,q)-$analogue of the Balázs–Szabados operators $R_{n,p,q}^{*}(f,x)$.
Theorem 1. Let $q=(q_n)$, $p=(p_n)$, $0<q_n<p_n\le 1$ such that $st_A-\lim_n q_n=1$, $st_A-\lim_n p_n=1$. Then for each compact interval $[0,b]\subset[0,\infty)$, we have $st_A-\lim_n\left\|R_{n,p,q}^{*}(f,x)-f(x)\right\|=0$, $\forall\ f\in C([0,b])$.

Keywords: $(p,q)-$Balazs-Szabados operators; Korovkin theorem; statistical convergence.

Mathematics Subject Classification: Primary 4H6D1, Secondary 4H6R1, 4H6R5.

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Exponential growth for a semi-linear viscoelastic heat equation with $L^p_\rho(\mathbb{R}^n)$-norm in bi-Laplacian type

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Abstract: The problem considered here is a class of semi-linear viscoelastic heat equations in bi-Laplacian type. We introduce a weighted space to overcome the difficulties in the non-compactness of some operators and some useful Sobolev embedding inequalities. Under certain conditions on the parameters p, ρ, η, we prove that the local solutions grow as an exponential function in the $L^p_\rho(\mathbb{R}^n)$-norm, i.e. $\|u\|^p_{L^p_\rho(\mathbb{R}^n)} \longrightarrow \infty$ as t tends to $+\infty$.

Keywords: Generalized Sobolev spaces, heat equation, weighted spaces, exponential growth of solution, initial condition.

Mathematics Subject Classification: 35Kxx, 74Dxx, 35Dxx, 35Jxx.

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Predict model based on Deep learning for the Algerian Stock Marckets

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Abstract: In this article, we proposed predicting the Algerian stock market index using LSTM model (Long Short Term Memory), a sub-class of RNN (Recurrent Neural Network). In order to test the universality of the LSTM model. We have tried different experiments on the parameters of the model: the number of epochs, layers and units. Until we get the optimal ones with a result of loss equal to 1.6704e-04. Then results of our LSTM model were compared to three similar randomly selected projects gets from Kaggle. As a result, the selected projects: Istanbul Stock prediction LSTM by STPETE_ISHII, DJIA Stocks by STPETE_ISHII, Stock Market Analysis + Prediction using LSTM by FARES SAYAH. The LSTM model achieved respectively 0.0165, 0.0352, 7.0498e-04 prediction loss of all three projects, where is clear that the last one was better than our research and that could be due to lack of data. During the process of our research, we found that exists a GRU (Gated Recurrent Units) model which is on trial, GRUs are a little speedier to train than LSTMs. There is not a clear winner which one is better. Researchers and engineers usually try both to determine which one works better for their use case. We decide to try it in our case where we find that GRU uses less training parameter and therefore uses less memory and executes faster than LSTM. The results get from GRU, made us try it on Bitcoin cryptocurrency chares where we got a better result, which conforms to us that our model needs more data to be more accurate. Finally, we concluded that the performance of LSTM is highly dependent on the choice of several parameters that need to be experimented with to get the right ones for optimal values. We optimize the LSTM model by testing different configurations, i.e., the number of epochs, layers and units. The GRU model gives better results but it is still in a trial from the researcher's side.

Keywords: Galerkin approximation, Maple Computer Algebra System, Differential equations.

Mathematics Subject Classification: 12X34, 56Y78.

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Fundemental Analysis of Solutions of Integro-Differential Equations

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Abstract: In this work, we examine some properties of solutions to an integro-differential equation by using a Lyapunov-Krasovkii functional. We prove four theorems about asymptotically stability, exponentially stability, integrability and instability of solutions of considered equation. An example is given to demonstrate the accuracy of the conditions of our main results.

Keywords: Integrability, exponentially stability, instability, asymptotically stability.

Mathematics Subject Classification: 34D05, 34K20, 45J05.

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Numerical approximation of electro-elastic antiplane contact problem with friction

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Abstract: We study the antiplane frictional contact models for electro-elastic materials. The material is assumed to be electro-elastic and the friction is modeled with Tresca's law and the foundation is assumed to be electrically conductive. First we establish the existence of a unique weak solution for the model. Moreover, the Proof is based on arguments of evolutionary inequalities. Some numerical results are presented at the end of this work.

Keywords: Antiplane frictional, electro-elastic materials, Tresca's law, Numerical results.

Mathematics Subject Classification: 74M10, 74F15, 74G25, 49J40.

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Identification of antinomies by complement analysis

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Abstract: It is not just colloquial language that gives rise to misunderstandings and different interpretations. Also, highly formalized languages used by scientists, including mathematicians, can be based on misinterpretations. It was noticed that the antinomian self-referential formulations are accompanied by additional ambiguous formulations, which allowed for a re-examination of Cantor's diagonal method. The conclusions are revolutionary! The problems I have presented concern the foundations of one of the most sophisticated fields of science, which is set theory, and the revealed contradictions should be discussed on a basic, i.e. philosophical, level.

Keywords: Cantor, self-reference, antinomies, complements, a new hypothesis.

Mathematics Subject Classification: 03E65.

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Optical soliton solutions of nonlinear Schrödinger-Hirota model involving nonlinear chromatic dispersion

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Abstract: Nonlinear wave problems in optical fibers have a special field of study and importance among nonlinear evolution equations (NLEs). In this respect, recently, a new research area has emerged in this field and new models and solution techniques are currently being studied. This study deals with the chromatic dispersion problem, which is one of the main problems encountered in soliton transmission in optical fibers. Definitions of this kind of problem and its solution are limited in number, and they have been put forward with some models developed in recent years, and current studies in this area continue. The Schrödinger-Hirota equation, which will be examined within the scope of the article. This is one of the important model among the models such as Sasa-Satsuma, Chen-Lee-Liu, Fokas-Lenells that have been developed recently in the field of optics. The review in the article deals with a current and important issue and also supports and interprets optical soliton solutions with graphic presentations.

Keywords: Group velocity dispersion, Optical soliton solution, Schrödinger-Hirota, Higher order nonlinear equation

Mathematics Subject Classification: 35Qxx, 35C08, 35Q55

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An Extension of Functional Projection Pursuit Regression to the Generalized Partially Linear Single Index Models (EFPPR-GPLSIM)

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Abstract: In this paper, we introduce a functional approach to approximate the non-parametric function in the case of multivariate predictors, the single-index coefficient, the non-linear regression function in the case of functional predictors and a scalar response. Following the Fisher-scoring algorithm and the principle of projection pursuit regression, we derive an additive decomposition that exploits the most predictive direction, the most predictive additive component of the functional predictor variable and the single-index component to explain the scalar response. On the one hand, this approach allows us to avoid the well-known problem of the curse of dimensionality in the non-parametric case with the notion of single index and the projection pursuit regression in the functional case, on the other hand, it can be used as an explonatory tool for the analysis of a multivariate and functional random variable belonging to a separable Hilbert space H. The terms of this decomposition are estimated with an iterative Fisher scoring procedure that uses the Quasi-Likelihood function and an approximation of the non parametric function by normalized B-splines. The good behaviour of our procedure is illustrated from a theoretical and practical point of view. Asymptotic results indicate that the nonparametric function, the single index coefficient and the terms of the additive decomposition can be estimated without suffering from curse of dimensionality , while some applications to real and simulated data show the high predictive performance of our method.

Keywords: Additive decomposition, Asymptotic normality, Fisher scoring algorithm, Functional Data analysis (FDA), Polynomial Splines, Predictive directions, Projection pursuit regression, Quasi-likelihood, Single-index model..

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Numerical Solution of Generalized Fractional Differential Equations

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Abstract: .Most of the analytical methods used to solve fractional differential equations, which are based on truncated series solutions, provide an approximation to the real solution in a very small region. Compared with integer-order differential equations, the multi-step process of such analytical methods is not appropriate for fractional differential equations due to the non-local property of fractional differentiation operators. Therefore, it has become important to expand, develop, and improve stable and robust methods for numerical treatment of fractional differential equations. The predictor-corrector method, which is an extension of the Adams-Bashforth-Moulton method, is one of the most effective and powerful methods that are extensively used for the numerical simulation of IVPs equipped with fractional derivatives of Caputo type. Furthermore, some predictor-corrector techniques have been proposed to numerically solve generalized Caputo-type fractional differential equations. We, mainly, discussed the formulation of the predictor-corrector algorithm for the numerical simulation of IVPs involving generalized Caputo-type fractional derivatives with respect to another function. Numerical solutions of some generalized Caputo-type fractional derivative models have been introduced to demonstrate the applicability and efficiency of the presented algorithm. The proposed algorithm is expected to be widely used and utilized in the field of simulating fractional-order models.

Keywords: Fractional differential equation; generalized fractional derivative; Caputo derivative; predictor-corrector algorithm; numerical solution.

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How can cell-to-cell transmission and HIV viral load drive the HIV/HCV coinfection dynamics?

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Abstract: We study the role of cell-to-cell transmission and HIV viremia in driving the dynamics of HIV/HCV coinfection. We derive the model, study its theoretical properties and perform epidemiological relevant simulations. We will detail useful conclusions for clinical practice.

Keywords: cell-to-cell transmission, viral load, HIV/HCV coinfection.

Mathematics Subject Classification: 34A34, 00A71, 92D30.

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On Λ-Fractional Peridynamic Mechanics

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Abstract: Applying a new Fractional derivative, the Λ- Fractional Derivative, with the corresponding Λ-Fractional space, Λ-Fractional Mechanics has already been established. The introduced mechanics is a non-local mechanics not conforming with Noll's local action postulate. Peridynamic mechanics is a non-local mechanics with interacting points inside a distance called horizon. That Λ-fractional mechanics with horizon is applied in the Λ-fractional space. Transferring the results into the initial space, the non-homogeneous peridynamic mechanics is established. The Λ-fractional peridynamic mechanics is applied to a Cantor rod, where the displacement and stress fields are defined in the initial space. Further, the proposed theory is applied to the deformation of the composite materials. Stresses and displacements as well are defined for non-constant distribution of the fillers into the composite materials.

Keywords: Λ- Fractional Derivative, Λ-Fractional space, Λ-Fractional stress, Λ-fractional strain,Cantor rod, peridynamics, horizon, composite material.

Mathematics Subject Classification: 53Z05.

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Investigating the Role of Mobility between Rural Areas and Forests on the Spread of Zika

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Abstract: A mathematical model of Zika virus transmission incorporating human movement between rural areas and nearby forests is presented to investigate the role of human movement in the spread of Zika virus infections in human and mosquito populations. Proportions of both susceptible and infected humans living in rural areas are assumed to move to nearby forest areas. Direct, indirect and vertical transmission routes are incorporated for all populations. Mathematical analysis of the proposed model has been presented. The analysis starts with normalizing the proposed model. Positivity and boundedness of solutions to the normalized model have been then addressed. The basic reproduction number has been calculated using the next generation matrix method and its relation to the three routes of disease transmission has been presented. The sensitivity analysis of the basic reproduction number to all model parameters has been investigated. The analysis also includes existence and stability of disease free and endemic equilibrium points. Bifurcation analysis has been also carried out. Finally, numerical solutions to the normalized model have been obtained to confirm the theoretical results and to demonstrate the impact of human movement in the disease transmission in human and mosquito populations.

Keywords: Zika; vertical transmission; Basic Reproduction Number; Stability Analysis;Sensitivity Analysis; Bifurcation analysis.

Mathematics Subject Classification: 34C23, 34D23, 92D30, 93A30.

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Analytical solutions to nonhomogeneous fractional differential equations and their applications

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Abstract: Non-homogeneous fractional differential equations containing variable coefficients with Caputo fractional derivative and hyper-Bessel operator, respectively, are considered. General solutions to these equations are obtained using the successive approximation method and are expressed in the integral form. Example solutions with particular choices of the non-homogeneous term are presented.
Direct and inverse source problems of a fractional diffusion equation with these operators are presented. Solutions to these problems are constructed based on appropriate eigenfunction expansions and results on existence are established.

Keywords: Galerkin approximation, Maple Computer Algebra System, Differential equations.

Mathematics Subject Classification: 12X34, 56Y78.

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New piecewise hybrid fractional order derivatives of Coronavirus (2019-nCov) mathematical model; Numerical Treatments

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Abstract: Piecewise fractional differential equation (deterministic-stochastic differential equations or vice versa) has been introduced recently in literature. The purpose of this piecewise approach is to study effectively the model with real data. In this talk, we extended the Coronavirus (2019-nCov) mathematical model by applying the piecewise differential equation system. The new hybrid fractional order operator can be written as a linear combination of the fractional order integral of Riemann-Liouville and the fractional order derivative Caputo is applied to extend the deterministic model and the fractional Brownian motion is applied to extend the stochastic differential equations. A new parameter ξ is presented in order to be consistent with the physical model problem. The positivity, boundedness, existence of the solutions for the model are discussed. New numerical algorithms are improved to solving the proposed model. These method are Nonstandard fractional Euler Maruyama technique to solve the fractional stochastic model and Grünwald-Letnikov nonstandard finite difference to solving the hybrid fractional order deterministic models. We consider the real cases of COVID-19 in Spain.

Keywords: Stochastic-deterministic models, Piecewise numerical method; Hybrid fractional Coronavirus (2019-nCov) mathematical model, Grünwald-Letnikov nonstandard finite difference method; Nonstandard fractional Euler Maruyama technique.

Mathematics Subject Classification: 37N25, 26A33, 65L12.

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Analytical soliton solution of $(2 + 1)$-dimensional Kadomtsev-Petviashivilli-Joseph-Egri Equation via efficient two integral schemes

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Abstract: Nonlinear partial differential equations (NLPDEs) are widely used in understanding and modeling many physical phenomena in real life, and many models and solution methods have been developed in the last 30 years, especially depending on the development of symbolic programs such as Mathematica, Matlab and Maple. Most of the modeling and solution techniques developed for NLPDE equations are based on soliton solutions, and there are hundreds of NLPDEs defined in this way in the literature. The contribution of each equation to the understanding of many physical phenomena in real life is of particular importance. In this study, it is aimed to obtain soliton solutions by using two different efficient methods for analytical soliton solutions of the $(2 + 1)$-dimensional Kadomtsev-Petviashvili-Joseph-Egri equation, and to make physical interpretations of the obtained solutions by supporting them with graphics.

Keywords: New Kudryashov scheme; Unified Riccati equation expansion method; Bell-shape; Soliton.

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Investigation of Hirota's equation under the influence of model parameter in single-mode fiber

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Abstract: This study, investigates one of the nonlinear partial differential equation namely, the $(1+1)-$dimensional Hirota equation by using the two efficient methods, the new Kudryasov and an auxiliary equation. The Hirota equation is an important equation which is generally used to model the optical wave propagation of the femto-second (fs) soliton pulse propagation in the single-mode fibers. By applying both proposed methods effectively, soliton solutions, 3-dimensional, 2-dimensional and contour graphical representations of solutions and necessary comments were made on these representations. In addition, the effect of the model parameter on the soliton behavior was also examined and the obtained results were interpreted by supporting them with graphic presentations.

Keywords: Higher-order nonlinear Schrödinger; Strongly dispersive; Nonlinear wave train; Single-mode fiber; Femto-second pulse; Soliton.

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An investigation of the DNA Peyrard–Bishop equation with beta-derivative

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Abstract: In this paper, the DNA dynamic equation appearing in the oscillator-chain defined as Peyrard–Bishop model is examined to acquire soliton solutions utilizing an efficient analytical technique. The resulted solutions are verified through symbolic soft computations. Necessary comments are made by presenting the obtained results graphically. Some results are also explained that express the novelty of our work as compared to the existing literature about the classical Peyrard–Bishop model.

Keywords: DNA Peyrard–Bishop equation with beta-derivative, optical soliton.

Mathematics Subject Classification: 35C08.

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Hybridized Metaheuristics for the Multi-Objective Quadratic Knapsack Problem.

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Abstract: The knapsack problem is basic in combinatorial optimization and has many variants and expansions. We focus on the quadratic stochastic multi-objective knapsack problem with random weights. We propose a Multi-Objective Memory Algorithm with Local Pareto Neighborhood Selection Search. At each iteration of our algorithm, a crossover, a mutation, and a local search are applied to a population of solutions to generate new solutions that will constitute an offspring population.
Then, we apply, on the combined population of parents and offspring, the best solution selection operator based on the termination of the non-domination rank and the crowding distance obtained respectively by the non-domination sorting algorithm and the crowding distance computation algorithm. To prove the performance of our algorithm, we compare it with both an exact algorithm and the NSGAII algorithm. Our experimental results show that the MASNPL algorithm leads to significant efficiency.

Keywords: Non-dominated Sort Algorithm, Crowding-Distance, Gradient Algorithm, Memetic Algorithm With Selection Neighborhood Pareto Local Search.

Mathematics Subject Classification: 90B50, 90C27.

Mathematical Study of a Class of Reaction-Diffusion System Resulting from Chemical Kinetics : Non-linear parabolic systems

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Abstract: In this work, we define a diffusion reaction system resulting from the modeling of certain chemical reactions coming from quantitative or formal chemical kinetics. The key amount is that of the reaction rate. The objective of this work is to prove the local, global existence, uniqueness and positivity for these reaction-diffusion systems.

Keywords: chemical kinetics, mathematical modelisation, the reaction rate, diffusion reaction system, global existence and et positivity.

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Prony's series and modern fractional calculus

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Abstract: The talk addresses Prony's series approximation of monotonically responses in material viscoelastic rheology and possibilities to implement on this basis modern fractional operator with non-singular kernels, precisely the Caputo-Fabrizio operator. The origins of the Prony's series in time and frequency domains are outlined together with relevant approximation and calculation techniques. Examples in the field of linear viscoelasticity are developed. In general, the content of this chapter expresses the author's work on implementation of new fractional operators with non-singular kernels as well as other published results thus allowing the amalgamated text to be as much as possible well done explanation of Prony's series application to modelling problems emerging in mechanical and chemical engineering, and related disciplines.

Mathematical study of initial flow past a circular cylinder with combined streamwise and transverse oscillations

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Abstract: A numerical investigation of the initial flow past an oscillating circular cylinder with combined streamwise and transverse oscillations. The motion is governed by the two-dimensional unsteady Navier–Stokes equations in non-primitive variables. The perturbation theory is strongly used to find the explicit solutions for stream function and vorticity. The well-known collocation method is implemented to solve certain part of the equations. The development of the physical properties of the flow such as the first time separation, drag and lift forces at early times are captured. Comparisons with existing results verify the accuracy of the present results.

Keywords: Combined streamwise and transverse oscillation, Initial flow, Perturbation theory, Collocation method.

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Mild Solutions of Stochastic integro-differential equations in Hilbert spaces

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Abstract: The main purpose of this is paper to study the existence of mild solutions for a class of fractional neutral stochastic integrodifferential equations with infinite delay in Hilbert spaces. Using fractional calculus, Schaefer fixed point theorem and stochastic analysis techniques, under non-Lipschitz conditions, we obtain a sufficient condition for the existence result. An example is provided to illustrate the application of this result.

Keywords: Infinite Delay, Stochastic Fractional Integrodifferential Equations, Mild Solution, Fixed Point Theorem Method.

Mathematics Subject Classification: 34K50, 60G22, 60H20.

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Some fixed points results for (λ, Ψ)- partial hybrid functions in CAT(0) spaces

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Abstract: In this paper is defined a new class of contractions, (λ, Ψ)- partial hybrid functions in CAT(0) space. The goal of this paper is to present a new convergent fixed point result based on a (λ, Ψ)- partial hybrid contraction on CAT(0) space. We have assured the set of the fixed points for a (λ, Ψ)- partial hybrid on CAT(0) space is nonempty. Furthermore, a Δ-convergent theorem on CAT(0) space is proved.

Introduction: M. Gromov [1], studied for the first time CAT(0) space in 1987. The study of CAT(0) spaces have many applications in Graph Theory, Fixed Point Theory, etc. W. Kirk [2], [3] defined and proved many fixed point results for nonexpansive functions. Dhompongsa and Panyanak[4] studied Δ-convergence in CAT(0) space. Many authors have worked on Theory of Fixed Point in CAT(0) space by assuring the existence of fixed point for various class of functions[5], [6], [7] or by presenting new iterations which obtain approximating fixed point [8], [9]. Inspired by above, we propose a new class of contractions in CAT(0) space called (λ, Ψ)- partial hybrid functions. Related to them, some fixed point theorems and some convergent results are obtained.

Keywords: Fixed point, (λ, Ψ)- partial hybrid contraction, CAT(0) space, Δ-convergence, Mann Iteration.

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Mixtures models for clustering: review and comparison

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Abstract: The concepts of machine learning and artificial intelligence were first mentioned in 1956. Since then, artificial intelligence has constantly been evolving but has only reached its peak in the last decade. Machine learning is applied in medicine, online technology, marketing, sales, logistics, and many others. In clustering, the aim is to find hidden relationships in the data. This type of machine learning can be divided into many different groups of methods. This work aims to briefly discuss the methods of clustering mixtures, compare these methods using different data, and compare them with the currently most popular clustering methods. We present k-means, Gaussian Mixture Model, Bayesian Gaussian Mixture Model, and Modified Inversion Formula clustering in work. In this work, the different clustering methods were briefly reviewed, and a modified inversion formula based on the new clustering method currently being developed was mentioned. The results showed that the best methods are also different for different data sets. Therefore, neither method is universal and unsuitable for all data sets. These results also showed that the newly developed data clustering method has relatively good clustering results. For this reason, the results of this method continue to be validated, and new modifications of this method are being developed.

Keywords: machine learning, artificial intelligence, clustering, mixture models, inversion formula.

Mathematics Subject Classification: 62G05; 62G07; 62G30.

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Global Existence and decay estimates for a Coupled System of Wave Equations with nonlinear Dampings

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Abstract: Our interest in this paper is to analyse the asymptotic behaviour of a coupled system of wave equations with non-linear dampings. We show that the system is well-posed using the semigroup theory. Furthermore, under suitable conditions on functions g(.), we estimate the energy decay rate by using the multiplier method.

Keywords: Coupled systems, Nonlinear damping, Well-posedness, Decay estimates, Multiplier method.

Mathematics Subject Classification: 35D30, 93D15, 74J30.

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Soliton collision in the coupled nonlinear Schrödinger equations

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Abstract: In this work, we have studied the dynamics of soliton collision in the coupled nonlinear Schrödinger equations. We have demonstrated the collision in contour, two and three dimensional plots using Matlab. All computations have been fulfilled via Mathematica. The obtained results and analysis might aid in enhancing the capacity of optical fiber communication.

Keywords: Soliton collision, coupled nonlinear Schrödinger equations

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Highly dispersive optical solitons of the nonlinear Schrödinger' s equation

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Abstract: In this study, the highly dispersive optical solitons of the nonlinear Schrödinger equation have been investigated. We have applied the new Kudryashov method to the considered equation and successfully derived some types of the soliton. Two and three-dimensional plots of the solutions have been demonstrated. The method is an efficient technique that can be applied to nonlinear physical models for extracting highly dispersive solitons.

Keywords: Highly dispersive optical solitons, nonlinear Schrödinger equation, analytical methods

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An investigation on nonlinear higher order Schrödinger equation having refractive indices and chromatic dispersion

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Abstract: With the widespread use of fiber technology in the fields of communication, data transfer, optics and optoelectronics, research in these fields has gained importance and has become the area of interest of many researchers. Although the problems in this field belong to the nonlinear partial differential equations (NLPDEs) class like many nonlinear evolution equations (NLEs), related to optics and optoelectronics have their own importance, some complex difficulties and solution techniques. In this respect, it would not be a wrong approach to consider the equations of optics and optoelectronics as a separate class from other NLEs. In recent years, data transmission in fiber optics, soliton behavior and related refractive indices have been defined and many studies have been carried out and these studies still maintain their importance and continue. Refractive index is an important topic in these research areas. In this study, analytical soliton solution based study was carried out on the higher order nonlinear Schrödinger equation regarding the refractive index and the obtained results were also supported graphically.

Keywords: Optical solution; Nonlinear refractive index; Auxiliary method; Group velocity dispersion.

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Soliton solutions of Fokas system in monomode optical media

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Abstract: With the developing technology, the use of communication and tools in the world is increasing, so the importance of data transfer is also high. With these developments, there has been a huge increase in the number of studies in the optical subfield of physics in recent years. In fiber optic cables, many events such as data transmission, non-linear throw, refractive index of light are modeled with NLPDEs. In this study, analytical solutions of the Fokas system were obtained. The Fokas system models nonlinear pulse transmission in monomode optical fibers. Unified Riccati equation expansion method (UREEM) was used to obtain optical soliton solutions. We have obtained various solitons for the model. Results are depicted via 3D, 2D and contour plots.

Keywords: Optical solution; Nonlinear refractive index; Auxiliary method; Group velocity dispersion.

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Two new insights in fractional calculus that have the potential to make significant changes

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Abstract: Fortunately, many researchers are working on a variety of topics of fractional calculus. Although there are always exceptions, but most works have no basic novelties. This area needs fundamental changes. If we want to guarantee the future of this field, we have no choice but to introduce completely new and creative approaches to this field. In this talk, we wanna to provide two insights for helping future of the field.

Keywords: Continuity, Discontinuity, Discrete fractional differential equation, Estimates, Inclusion modeling, Mathematical softwares.

(2020) Mathematics Subject Classifications: 34A08; 39A13.

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Existence and uniqueness results for a nonlinear integral equation related to infectious disease

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Abstract: A nonlinear integral equation related to infectious disease is investigated. Using a fixed-point theorem for convex-concave and nondecreasing operators defined in a Banach space with a normal solid cone, we derive some existence and uniqueness results of positive solutions to the considered equation. Moreover, an iterative algorithm that converges to the unique solution is provided. Our results are supported by examples.

Keywords: Integral equation; convex–concave nonlinearities; positive solution

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An Application of Fuzzyfied Environment for SAIR Model Using COVID-19 Data in Turkey

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Abstract: In this talk we propose a mathematical model in a Fuzzy environment for COVID-19 in Turkey [1] during two periods, P1 and P2. In our model we extend the SAIR model [2] to the fuzzyfied form [3]. Supposing the vaccination is the highest effect of treating/curing COVID-19, here we fuzzyfy the SAIR model by taking the vaccination parameter in a fuzzy environment. Also in our model asymptomatic individuals [4], or silent spreaders, have dominant roles spreading the disease.

Keywords:COVID-19, SAIR model, Fuzzyfied environment, Vaccination parameter.

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Hidden attractors: new horizons in exploring dynamical systems

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Abstract: Hidden attractors have been known since the 1960s when they were discovered and observed in various nonlinear control systems[1]. It is the last decade where several scientists have intensively studied hidden attractors. Hidden are called the attractors whose basin of attraction does not intersect with small neighborhoods of the unstable equilibrium point, i.e., their basins of attraction do not touch unstable equilibrium points and are located far away from them[2]. They can be found in systems with no equilibrium points[3], with one stable equilibrium[4], or in systems with lines of equilibrium points[5]. Hidden attractors often have small basins of attractions, are strongly chaotic, and have complex dynamics. This property can be helpful or catastrophic, especially in technological applications. In this talk, we will make a short review on this topic and present different systems with Hidden Attractors [6-7].

Keywords: Hidden attractors, chaotic dynamical systems, complex dynamics.

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The applied mathematics and modelling in the path from physics to biology

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Abstract: We present a panoramic view of the evolution associated to the different emerged methods and applied to the different conceptual steps of the Physics in the confluence paths with the Biology. In this context, we have the basic computational methods up to the Data Analysis and related issues. The different kinds of differential equations are a basic stone together with the dynamical competition of scales either in space or time.

Optimization, dynamical systems, & mathematics of networks for Engineering problems

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Keywords:differential; dynamical; networks; optimization; materials; power systems; gas.

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Abstract multivariate algebraic function activated neural network approximations

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Abstract: Here we exhibit multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $R2N$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We study also the case of approximation by iterated operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the algebraic sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer.

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Highly dispersive optical soliton perturbation with complex—Ginzburg Landau model by semi—inverse variation

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Abstract: The dynamics of perturbed highly dispersive optical solitons is studied in this work. The governing model is the complex Ginzburg—Landau equation with six dispersion terms. The perturbation effects appear with maximum allowable intensity or full nonlinearity. Three forms of self—phase modulation are considered. They stem from Kerr effect, parabolic law and finally the polynomial form. The semi—inverse variational principle is implemented to recover bright 1—soliton solutions to the model which is otherwise non—integrable with any of the known integration schemes. The applied principle retrieves analytical, but not exact, bright 1—soliton solutions to the model. The parameter constraints, that guarantee the existence of such solitons, are also identified, and presented.

Keywords:solitons; Cardano; semi—inverse.

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A comparative analysis of an application of Subgradient and Nelder-Mead methods to one layer-case Inverse problem of gravimetry.

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Abstract: We had conducted qualitative and quantitative comparative analysis with computer simulation for one layer-case Inverse problem of gravimetry with conditions on the part of boundary. We use the Poisson equation $\Delta u(x;y) = -4\pi G\psi(x;y),\ (x;y) \in \Omega$ based model (where Ω is whole *search area*), and, given a distribution functions of gravitational potential u and density ψ, we have to restore in-depth c and thickness d parameters of the target deposit layer, minimizing the target function $I(c;d) = \int_0^L \frac{\partial u(x,0)}{\partial y} - \beta(x)^2 dx \rightarrow \min$, where β is *a posteriōri* known real distribution of the gravitational potential vertical derivative, and all associated boundary conditions are met [1],[3]. The problem is real data-based. Within the problem statement, the Subgradient and Nelder-Mead methods were chosen for testing. The reasons for this choice are: 1) a target function property, which does not allow any gradient methods, as is stated during the study [1]; 2) the typicality of both mentioned methods as common non-gradient methods. Our goal was to esteem their behavior and accuracy here. The uniqueness of exact solution is proven [2], but in real calculation, some different data setups sometimes might give the same outputs due to the well-known peculiarities of the applied methods; though, numeric results still are quite good for practice. In comparison, we concluded that, though subgradient method works noticeably slower (due to random search involved), it restores the data better; Nelder-Mead also gives good results, but requires a good initial approximation; and it is possible to use both in joint for more better results. Research was supported by grant project "Development of geographic information system for solving the problem of gravimetric monitoring of the state of the subsoil of oil and gas regions of Kazakhstan based on high-performance computing in conditions of limited experimental data" (Grant N0AP05135158-OT-19, SC MES RK). The results obtained appear to be useful for oil-and-gas field practitioners.

Keywords: Inverse and ill-posed problems, non-gradient algorithms, oil industry.

Subject Classification: 49N45, 86A22, 65M32.

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On Connections and Novelty in Fractional Calculus

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Abstract: Mathematics does not consist of proving unrelated results in a vacuum, and good mathematics does not consist of copy-pasting known proofs to new settings where they work in the same way without modification. Mathematics is a deeply interconnected beast, and the connections should be used to prove new results using old ones as much as possible. On the other hand, if you notice that the same method of proof can be used for many different results, it suggests that reproducing the proof many times is unnecessary, and instead you should look for a general setting where the method can apply to prove a single result with many special cases.
Applying these philosophies to fractional calculus, we see that new fractional operators should always be understood in terms of their connections with old and established ones, if such connections can be found. We also see that it is useful to consider general fractional operators which contain many existing definitions as special cases, to avoid time-wasting and redundancy in mathematical research. This talk will examine some general classes of fractional operators, and their connections with the original Riemann–Liouville fractional calculus that allow many classical results to be easily extended to more general settings.

Keywords:fractional calculus; algebraic conjugation; generalised fractional calculus.

Subject Classification: 26A33.

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Discrete Wolbachia Diffusion in Mosquito Populations with Allee Effects

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Abstract: We study stability analysis of a discrete-time dynamical system of Wolbachia diffusion in mosquito populations with Allee effects on the wild mosquito population. We analyze the competition between released mosquitoes and wild mosquitos. We show local and global stabilities of the fixed points, and type of bifurcations with respect to parameters. The results are verified by numerical simulations.
Applying these philosophies to fractional calculus, we see that new fractional operators should always be understood in terms of their connections with old and established ones, if such connections can be found. We also see that it is useful to consider general fractional operators which contain many existing definitions as special cases, to avoid time-wasting and redundancy in mathematical research. This talk will examine some general classes of fractional operators, and their connections with the original Riemann–Liouville fractional calculus that allow many classical results to be easily extended to more general settings.

Keywords:competition model, discrete dynamical systems, bifurcation, fixed point.

Subject Classification: 92D25, 34C60, 92D30.

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Exponential Stability of the transmission Schrödinger equation with boundary time-varying delay

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Abstract: In this paper, we study stability problems for the transmission Schrödinger equation with a Neumann feedback control that contains a time-varying delay term and that acts on the exterior boundary. Under suitable assumptions, we prove exponential stability of the solution. These results are obtained by introducing suitable energies and suitable Lyapunov functionals.

Keywords: Schrödinger equation; transmission problems; time-varying delay; exponential stability.

Subject Classification: 93D15; 35J10.

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327 years of fractional calculus: theory and applications

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Abstract: Fractional calculus has a huge history and it is an interdisciplinary field with a potential impact in several areas of science and engineering. In my talk I will present some open problems of the fractional calculus.

Keywords: Fractional calculus, Modelling, Caputo operator.

Subject Classification: 26A33.

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Pull-Back Vector Fields

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Abstract: The problem studied in this paper is related to the bienergy of a pull-back vector field from a Riemannian manifold to its tangent bundle equipped with the Sasaki metric. We show that a pull-back vector field on a compact manifold which covers harmonic map. then the pull-back bundle V is biharmonic if and only if V is parallel.

Keywords: Horizontal lift, vertical lift, Pull-Back, biharmonic map.

Subject Classification: Primary 53A45; Secondary 53C20.

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Rbf-Pum solution of magnetoconvection in a triangular cavity exposed to a uniform magnetic field

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Abstract: Numerical simulation of Al_2O_3-Cu/water hybrid nanofluid flow in an isosceles right triangular cavity exposed to either vertical or horizontal uniform magnetic field is numerically investigated. A local method, radial basis function based partition of unity method (Rbf-Pum), is performed to solve steady dimensionless governing equations in stream function-vorticity form numerically. Vertical magnetic field suppresses the fluid flow and heat transfer more than the horizontal one. The rise in magnitude of uniform magnetic field suppresses fluid flow and heat transfer. The dominance of convection is pronounced at large Rayleigh numbers.

Keywords: Polyharmonic spline, radial basis function, hybrid nanofluid, triangular cavity.
Mathematics Subject Classification:

1 Introduction and Problem Definition

Nanofluids are very popular in recent years due to their capability of improvement on heat transfer. Choi et al. [1] experimentally showed that nanaoparticle addition into a base fluid enhances thermal conductivity, so does the heat transfer performance. Many numerical studies involving nanofluids are carried out. Finite element method [2, 3, 4], finite volume method [5] and finite difference method [6] are mostly encountered numerical methods used in these studies. Rbf based methods are rarely used in these type of problems. Therefore, in the current study, Rbf-Pum is presented as an alternative numerical method.
The two-dimensional, steady, laminar, incompressible flow is concerned in an isosceles right triangle involving a hybrid nanofluid. The description of the problem is sketched in Fig. 1. Brownian motion, viscous dissipation, radiation, Hall effect, induced magnetic field and Joule heating effect are ignored.

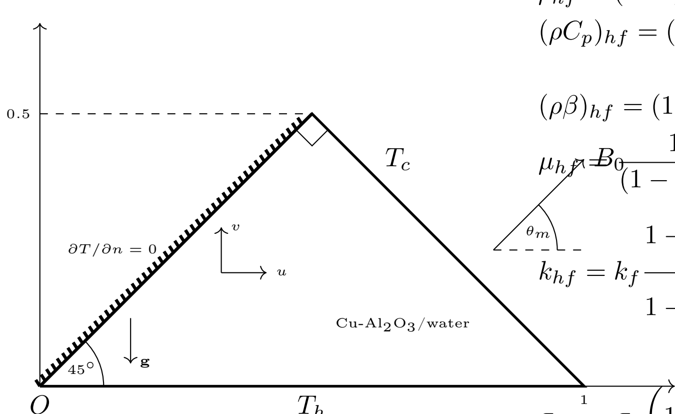

Figure 1: Description of the problem.

$$\rho_{hf} = (1-\phi)\rho_f + \rho_1\phi_1 + \rho_2\phi_2 \tag{1.1a}$$

$$(\rho C_p)_{hf} = (1-\phi)(\rho C_p)_f + (\rho C_p)_1\phi_1 + (\rho C_p)_2\phi_2 \tag{1.1b}$$

$$(\rho\beta)_{hf} = (1-\phi)(\rho\beta)_f + (\rho\beta)_1\phi_1 + (\rho\beta)_2\phi_2 \tag{1.1c}$$

$$\mu_{hf} = \frac{1}{(1-\phi)^{2.5}} \tag{1.1d}$$

$$k_{hf} = k_f \frac{1-\phi+2\phi\dfrac{k_{np}}{k_{np}-k_f}\ln\left(\dfrac{k_{np}+k_f}{2k_f}\right)}{1-\phi+2\phi\dfrac{k_f}{k_{np}-k_f}\ln\left(\dfrac{k_{np}+k_f}{2k_f}\right)} \tag{1.1e}$$

$$\sigma_{hf} = \sigma_f\left(1+\frac{3(\sigma_{np}/\sigma_f-1)\phi}{(\sigma_{np}/\sigma_f+2)-(\sigma_{np}/\sigma_f-1)\phi}\right) \tag{1.1f}$$

Some physical properties of base fluid and nanoparticles are taken as given in [7]. Adopting single phase nanofluid model, physical relations of hybrid nanofluid are given in the right side of Fig. 1, where subindices $1, 2$, f, np, hf refer to the nanoparticles Al_2O_3, Cu, fluid, nanoparticle and hybrid nanofluid, respectively, $\phi = \phi_1 + \phi_2$ is the solid volume concentration of hybrid nanofluid, $k_{np} = (k_1\phi_1 + k_2\phi_2)/\phi$ is the thermal conductivity of nanoparticle, ρ is the density, (ρC_p) is the heat capacitance, $(\rho\beta)$ is the thermal expansion coefficient, $\alpha_{hf} = k_{hf}/(\rho C_p)_{hf}$ is the thermal diffusivity of nanofluid, $\sigma_{np} = (\sigma_1\phi_1 + \sigma_2\phi_2)/\phi$ is the electrical conductivity of nanoparticle, Eq. (1.1d) is the Brinkman's model [8], Eq. (1.1e) is the Xue's model [9] and Eq. (1.1f) is the Maxwell's model [10].
The governing equations in dimensionless stream function-vorticity formulation are derived as follows

$$\nabla^2\psi = -w \tag{1.2a}$$

$$\frac{\alpha_{hf}}{\alpha_f}\nabla^2 T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}, \tag{1.2b}$$

$$\begin{aligned} Pr\nabla^2 w = {} & \frac{\nu_f \rho_{hf}}{\mu_{hf}}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) - \frac{(\rho\beta)_{hf}}{\beta_f}\frac{\nu_f}{\mu_{hf}} Ra Pr \frac{\partial T}{\partial x} \\ & - \frac{\sigma_{hf}\mu_f}{\sigma_f \mu_{hf}} Ha^2 Pr\left(\frac{\partial u}{\partial y}\sin^2\theta_m - \frac{\partial v}{\partial x}\cos^2\theta_m + 2\sin\theta_m\cos\theta_m\frac{\partial u}{\partial x}\right), \end{aligned} \tag{1.2c}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The numbers, Prandtl (Pr), Rayleigh (Ra) and Hartmann (Ha), in these equations are defined as

$$Pr = \frac{\nu_f}{\alpha_f},\ Ra = \frac{g\beta_f(T_h - T_c)L^3}{\nu_f\alpha_f},\ Ha = B_0 L\sqrt{\frac{\sigma_f}{\mu_f}}.$$

No slip boundary conditions for velocity, and in turn for stream function, are inserted on each walls as $u = v = \psi = 0$. For temperature, $T_h = 1$ and $T_c = 0$, and $\partial T/\partial n = 0$ on jagged walls. Vorticity boundary conditions are found by calculating $w = \nabla \times \mathbf{u}$ on each boundaries.

2 Numerical Procedure

Radial basis functions have taken great interest in the last decade. Novel books [11, 12, 13] include many details about Rbfs.
Rbf-Pum is a local method. The localization is constructed by considering subdomains and solution is approximated in each subdomain as in global Rbf. The obtained system matrix becomes sparse. Subdomains here are also referred as open cover or patches. Shapes of these covers may be chosen such as squares, circles or ellipses. In this study, circular patches are used. The radius of patches can be arranged in such a way that the overlap between patches and covering are satisfied. Selection of the radius strongly affect the accuracy of the solution. The following explanations for this method is based on Ref.[14], and additional polynomial terms of degree one are also considered as in Ref. [15].
After getting the differentiation matrices, say D_x, D_y and D_2 for the x-, y- and the Laplacian, respectively, the dimensionless nonlinear governing equations are iteratively solved as follows

$$D_2\psi^{n+1} = -w^n \tag{2.1a}$$

$$u = u^{n+1} = D_y\psi^{n+1},\ v = v^{n+1} = -D_x\psi^{n+1} \tag{2.1b}$$

$$\left(D_2 - \frac{\alpha_f}{\alpha_{hf}}M\right)T^{n+1} = 0 \tag{2.1c}$$

$$\begin{aligned} \left(PrD_2 - \frac{\nu_f\rho_{hf}}{\mu_{hf}}M\right)w^{n+1} = {} & -\frac{(\rho\beta)_{hf}}{\beta_f}\frac{\nu_f}{\mu_{hf}} Ra Pr D_x T^{n+1} \\ & - \frac{\sigma_{hf}\mu_f}{\sigma_f\mu_{hf}} Ha^2 Pr\Big((D_y u)\sin^2\theta_m - (D_x v)\cos^2\theta_m + 2\sin\theta_m\cos\theta_m(D_x u)\Big), \end{aligned} \tag{2.1d}$$

where M is the matrix equal to $diag(u)D_x + diag(v)D_y$ and n is the iteration level.
As a ratio of convective heat transfer to conductive heat transfer, average Nusselt number is also checked as $\overline{Nu} = -\int_0^1 \frac{\partial T}{\partial y} dx$ using the numerical integration presented in [16]. A relaxation parameter τ is used once the vorticity equation is solved as $w^{n+1} \leftarrow \tau w^{n+1} + (1-\tau)w^n$, where $\tau \in (0,1)$. τ is either taken as 0.02 or 0.01 in the current executions.

3 Numerical Computations

All numerical computations are done in 2.3 GHz Quad-Core i7 computer with MATLAB R2020a. Prandtl number ($Pr = (\mu_f (C_p)_f / k_f)$ is computed and fixed at 6.0674. Fluid flow and heat transfer are visualized in streamlines and isotherms in variation of Hartmann number, Rayleigh number and the angle of the uniform magnetic field. Regarding to the single phase model, $\phi_1 = \phi_2$ is kept at 0.01. Polyharmonic spline Rbf, $f = r^3$, is adopted with monomials $1, x, y$.
Node distribution is designed utilizing Gauss-Chebyshev-Lobatto (GCL) grids as in Fig.2 This distribution

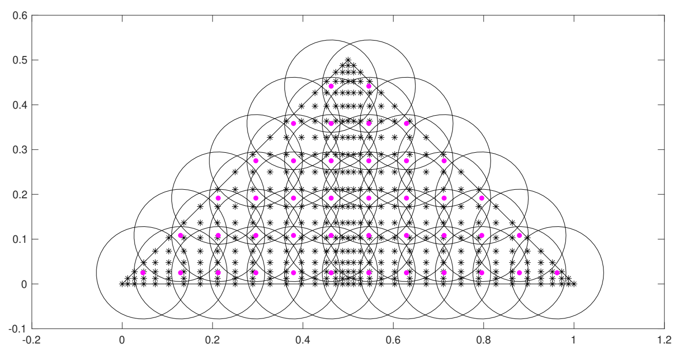

Figure 2: Node setup and circular patches. The number of patches is fixed at 42.

is done concerning the sharp corners in the geometry. On this figure, pink dots point to the center of the patches. $N_b = 192$ number of boundary and $N_i = 2209$ number of interior nodes are used.
In the following figures' illustrations, streamlines are at the top two rows of contours and isotherms are at the bottom two rows. In Fig. 3, the core of streamlines shifts in the direction of the applied magnetic field. At $Ha = 100$, the central fluid velocity becomes smaller at $\theta_m = 90°$ than $\theta_m = 0°$. Almost conductive behavior is exhibited in isotherms in each contours due to the small Rayleigh number.
In Figs. 4-5, fluid flow and heat transfer are examined at $Ra = 10^5$ and $Ra = 10^6$, respectively, in different Ha numbers as well as angles θ_m. At a small Ha number ($Ha = 10$), fluid and temporal behavior are not altering at any angle θ_m in both Rayleigh numbers. On the other hand, the rise in Ha number causes fluid to obey the direction of the magnetic field, and central vortex in streamlines is pushed either horizontally at $\theta_m = 0°$ or vertically at $\theta_m = 90°$. Also, the decrease in central streamline values is noted significantly at a large Ha ($Ha = 100$) comparing angles. In isotherms, stabilization in contours is apparently seen with the augmentation in Ha number at any angle θ_m. That is, convective heat transfer is suppressed by larger Lorentz force. At $Ra = 10^6$, $Ha = 100$, the vertical magnetic field tends to separate the fluid flow, and isotherms are also perturbed vertically.

4 Conclusions

In this study, Rbf-Pum solution of natural convection flow of a hybrid nanofluid in an isosceles right triangle under the effect of either a horizontal or a vertical uniform magnetic field is presented.

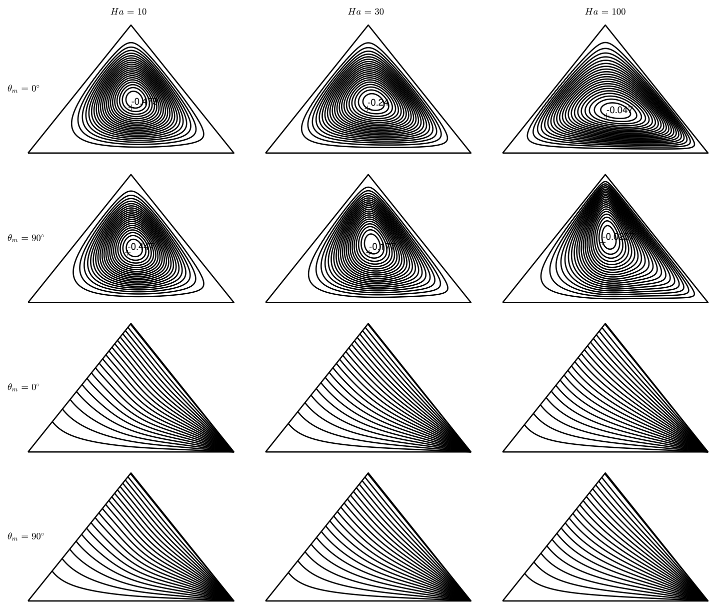

Figure 3: $Ra = 10^4$ in different Ha numbers.

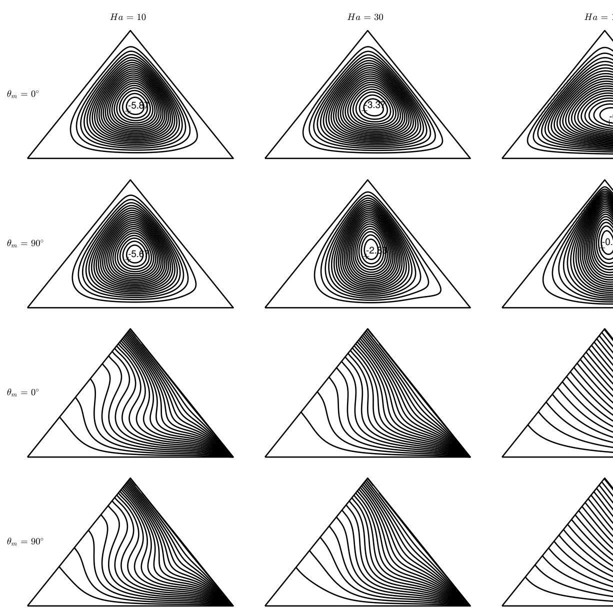

Figure 4: $Ra = 10^5$ in different Ha numbers.

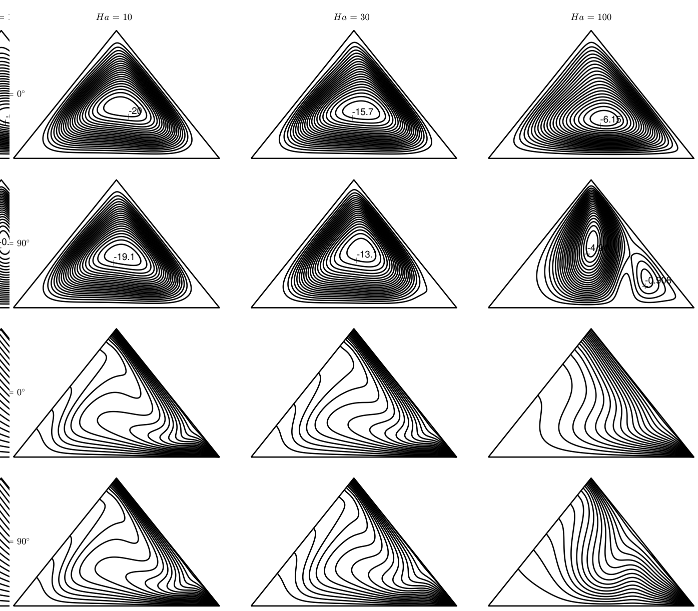

Figure 5: $Ra = 10^6$ in different Ha numbers.

Some results may be listed as follows :

- Buoyancy driven flow is pronounced with the augmentation in Ra number.
- When $\theta_m = 0°$ or $\theta_m = 90°$, from $Ha = 10$ to $Ha = 100$,

  * the most decrease (over 90%) in $|\psi|_{\max}$ occurs at $Ra = 10^4$ and $Ra = 10^5$.
  * On the other hand, while the decrease at $\overline{Nu}$ at $Ra = 10^4$ and $Ra = 10^5$ is almost the same (9%), the decrease at $Ra = 10^6$ is more (19.31%) with $\theta_m = 90°$ than $\theta_m = 0°$ (16.49%). That is, convective heat transfer decreases as Ha rises. This decrease becomes significant as Ra increases.

- At $Ha = 100$, from $\theta_m = 0°$ to $\theta_m = 90°$,
  * 45% at $Ra = 10^4$, 43.49% at $Ra = 10^5$ and 19.54% at $Ra = 10^6$ decrease in $|\psi|_{\max}$ is observed.
  * Although not too much change is noted in $\overline{Nu}$, the most decrease (3.77%) is exhibited at $Ra = 10^6$.

As a consequence, uniform magnetic field coming through the heated part of the cavity is more forcing for fluid flow and convective heat transfer. Further, Rbf-Pum is a good alternative numerical method for solving these type of problems, too.

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Approximation of Kantorovich-type generalization of (p,q) −Bernstein type Rational Functions Via statistical convergence

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Abstract: In this paper we use modulus of continuity to study the rate of A-statistical convergence of Kantorovich type (p,q) −analogue of the Balázs–Szabados operators by using the statistical notion of convergence.

Keywords: (p,q) −calculus, Bernstein operators, (p,q) −Balázs-Szabados operators, Satistical convergence
Mathematics Subject Classification: 4H6D1, 4H6R1, 4H6R5

1 Introduction

Bernstein type rational functions, $R_n(f;x) = \frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\begin{pmatrix} n \\ k \end{pmatrix}(a_nx)^k \quad (n=1,\ 2,\ ...)$ Balázs defined and investigated them in 1975, (see [1]). In this definition, f is a real and single valued function defined on$[0,\infty)$ the interval, a_n andb_n are real numbers that have been appropriately chosen and are independent of$x$. Seven years later, in 1982, Balázs and Szabados cooperates to improve the estimate in [1] by selecting appropriate parameters a_n and b_n under some restrictions for$f(x)$, (see [2]). Recently, different $q-$generalizations of Balázs-Szabados operators have been studied by several researchers, see [3, 4, 5, 6, 7]. In [8], the Kantorovich type $q-$analogue of the Balázs-Szabados operators is defined by Hamal and Sabancigil as follows:

$$R^*_{n,q}(f,x) = \sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 f\left(\frac{[k]_q + q^k t}{b_n}\right) d_q t, \tag{1.1}$$

where $f:[0,\infty)\to \mathrm{R}$, $q\in(0,1)$,$a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$, $0<\beta\le\frac{2}{3}$, $n\in\mathrm{N}$, $x\ge 0$,
and $r_{n,k}(q,x) = \frac{1}{(1+a_nx)^n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a_nx)^k \prod_{s=0}^{n-k-1}\Big(1+(1-q)[s]_q a_nx\Big)$.

In [10], recently, Hamal and Sabancigil introduced a new Kantorovich-type (p,q) −analogue of the Balázs–Szabados operators by generalizing the new Kantorovich-type $q-$analogue of Balázs–Szabados operators, given by 1.1, as follows:

$$R^*_{n,p,q}(f,x) = \sum_{k=0}^{n} r^*_{n,k}(p,q,x)\int_0^1 f\left(\frac{p^{n-k}\left([k]_{p,q}+q^k t\right)}{b_n}\right) d_{p,q}t\ , \tag{1.2}$$

where $r^*_{n,k}(p,q,x) = \frac{1}{p^{n(n-1)/2}}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2}\left(\frac{a_nx}{1+a_nx}\right)^k \ \prod_{j=0}^{n-k-1}\left(p^j - q^j\frac{a_nx}{1+a_nx}\right)$
and $0<q<p\le 1$, $a_n = [n]_{p,q}^{\beta-1}$, $b_n = [n]_{p,q}^{\beta}$, $0<\beta\le\frac{2}{3}$, $n\in\mathrm{N}$, $x\ge 0$, $f:[0,\infty)\to\mathrm{R}$.
Before stating the main result for these operators, we give some notations and definitions of(p,q) −calculus. For any $p>0$, $q>0$, non-negative integer$n$, the (p,q) −integer of the number$n$is defined as follows:

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^2+...+pq^{n-2}+q^{n-1}=\begin{cases}\frac{p^n-q^n}{p-q} & if\ p\neq q\neq 1\\ np^{n-1} & if\ p=q\neq 1\\ [n]_q & if\ p=1\\ n & if\ p=q=1\end{cases},$$

the (p,q) −factorial is defined by

$$[n]_{p,q}!=\prod_{k=1}^{n}[k]_{p,q}\ ,\ n\geq 1\quad\text{and}\quad [0]_{p,q}!=1,$$

and (p,q) −binomial coefficient is defined by

$$\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!}\quad ,0\leq k\leq n.$$

The formula of (p,q) −binomial expansion is defined by

$$\begin{aligned}(ax+by)^n_{p,q}&=\textstyle\sum_{k=0}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}a^{n-k}b^kx^{n-k}y^k\\ &=(ax+by)(pax+qby)\left(p^2ax+q^2by\right)...\left(p^{n-1}ax+q^{n-1}by\right).\end{aligned}$$

Let $f:C[0,a]\to$ R, the (p,q) −integral of is defined by:

$$\int_0^a f(t)d_{p,q}t=(p-q)a\sum_{k=0}^{\infty}f\left(\frac{q^k}{p^{k+1}}a\right)\frac{q^k}{p^{k+1}}\text{ if }\quad\left|\frac{p}{q}\right|>1\ .$$

Fast [11] and Fridy [12] provided the following notions.
Suppose that $E\subseteq$ N $=\{1,\ 2,\ ...\}$and $E_n=\{k\leq n:k\in E\}$.Then $\delta(E)=\lim\limits_{n\to\infty}\frac{1}{n}|E_n|$ is called natural density of E provided that the limit exists.

Definition 1.1. A sequence $x=(x_n)$ is statistically convergent to the number$L$if for every$\varepsilon>0$, we have $\delta\{k\in$ N $:|x_k-L|\geq\varepsilon\}=0$ is denoted by $st_A-\lim\limits_{n\to\infty}x_n=L$.

Because all finite subsets of the natural numbers have density zero, any convergent sequence is statistically convergent, but not contrariwise.
For example , consider the sequence $A=\{a_n,\ n=1,\ 2,\ 3,...\}$ whose terms are

$$a_n=\begin{cases}\sqrt{n} & \text{when }\ n=m^2,\ \forall m=1,\ 2,\ 3,...\\ 1 & \text{otherwise}\end{cases}$$

we can see that the sequence is divergent in ordinary sense, but it is statistically convergent to 1.
Let $C_B[a,b]$ denote the space of all functions f which are continuous in every point of the interval$[a,b]$ and bounded on the entire positive real line, $|f(x)|\leq M_f,\forall x\in(0,\infty)$.

Lemma 1.1. *[10] For all Let $n\in$ N, $x\in[0,\infty)$ and$0<q<p\leq 1$, we have the following equalities:*

$$R^*_{n,p,q}(1,x)=1.$$

$$R^*_{n,p,q}(t,x)=\frac{p^n}{[2]_{p,q}b_n}+\frac{2q}{[2]_{p,q}}\left(\frac{x}{1+a_nx}\right).$$

$$\begin{aligned}R^*_{n,p,q}\left(t^2,x\right)&=\tfrac{p^{2n}}{[3]_{p,q}b_n^2}+\tfrac{\left(4q^3+5q^2p+3qp^2\right)p^{n-1}}{[2]_{p,q}[3]_{p,q}b_n}\left(\tfrac{x}{1+a_nx}\right)\\ &+\tfrac{q[n-1]_{p,q}}{[n]_{p,q}}\tfrac{4q^3+q^2p+qp^2}{[2]_{p,q}[3]_{p,q}}\left(\tfrac{x}{1+a_nx}\right)^2.\end{aligned}$$

Lemma 1.2. *[10] For all $n \in \mathrm{N},\ x \in [0,\infty)$ $0 < q < p \leq 1$, we have the following estimations:*

$$\left(R^*_{n,p,q}\left((t-x),x\right)\right)^2 \leq \frac{1}{b_n}\left\{\frac{1}{b_n} + \frac{\left(p^n - q^n\right)^2}{b_n}\left(\frac{1}{p+q} + \frac{1}{p-q}\left(a_n x\right)\right)^2\right\},\ x \in [0,\infty), \tag{1.3}$$

$$R^*_{n,p,q}\left((t-x)^2, x\right) \leq \frac{A_1}{b_n}\phi_n\left(p,q\right)\left(1+x\right)^2, \quad x \in [0,\infty), \tag{1.4}$$

$$R^*_{n,p,q}\left((t-x)^4, x\right) \ \leq\ \frac{A_2}{b_n^2}\left(1+x\right)^2, \qquad x \in [0,\infty), \tag{1.5}$$

where $A_1 > 0,\ A_2 > 0$ and $\phi_n\left(p,q\right) = \max\left\{p^{n-1},\ b_n - a_n p^{n-1},\ \frac{1}{[3]_{p,q} b_n}\right\}$.
In the following theorem, Bohman-Korovkin type statistical approximation theorem was proved by Gadjiev and Orhan.

Theorem 1.1. *[13] Let $\left(\ell_n\right)_{n\in\mathrm{N}}$ be a sequence of positive linear operators acting from $C_B[a,b]$ to $B[a,b]$ that is, $\ell_n : C_B[a,b] \to B[a,b]$ satisfies the conditions that*

$$st_A - \lim\left\|\ell_n\left(e_i\right) - e_i\right\| = 0 \ \text{ with } e_i\left(t\right) = t^i \text{ and } \forall\ i = 0,\ 1,\ 2. \tag{1.6}$$

Then, we have

$$st_A - \lim_n\left\|\ell_n f - f\right\| = 0 \quad ,\forall\ f \in C_B\left([a,b]\right).$$

Now, we give the main result of this research is to use modulus of continuity to study the rate of A-statistical convergence of Kantorovich type $(p,q)-$analogue of the Balázs–Szabados operators $R^*_{n,p,q}(f,x)$.

Theorem 1.2. *Let $q = \left(q_n\right),\ p = \left(p_n\right),\ 0 < q_n < p_n \leq 1$ such that $st_A - \lim_n q_n = 1,\ st_A - \lim_n p_n = 1$ and $st_A - \lim_n p_n^n = 1$. Then for each compact interval$[0,b] \subset [0,\infty)$, we have $st_A - \lim_n\left\|R^*_{n,p,q}\left(f,x\right) - f\left(x\right)\right\| = 0$, $\forall\ f \in C\left([0,b]\right)$.*

Proof According to Theorem 1, it is sufficient to show that it satisfies (1.6). By using Lemma 1, it is clear that

$$st_A - \lim_n\left\|R^*_{n,p_n,q_n}\left(e_0;x\right) - e_0\right\| = 0\ ,\ \text{since } R^*_{n,p_n,q_n}\left(e_0;x\right) = 1. \tag{1.7}$$

Again by Lemma 1, we have

$$\begin{aligned}\left|R^*_{n,p_n,q_n}\left(e_1;x\right) - e_1\right| &= \left|\frac{p_n^n}{[2]_{p,q} b_n} + \frac{2q_n}{[2]_{p_n,q_n}}\left(\frac{x}{1+a_{n,p_n,q_n}x}\right) - x\right| \\ &= \frac{p_n^n}{[2]_{p_n,q_n} b_n} + \frac{(p_n - q_n)}{[2]_{p_n,q_n}}\frac{x}{1+a_{n,p_n,q_n}x} + \frac{a_{n,p_n,q_n}x^2}{1+a_{n,p_n,q_n}x}.\end{aligned}$$

By taking the maximum of both sides of the last equality on $[0,b]$ with $0 < b < \frac{1}{a_{n,p_n,q_n}}$, we obtain

$$\left\|R^*_{n,p_n,q_n}\left(e_1;x\right) - e_1\right\| \leq \frac{p_n^n}{[2]_{p_n,q_n} b_n} + \frac{\left(p_n - q_n\right)}{[2]_{p_n,q_n}}\frac{b}{1 + a_{n,p_n,q_n}b} + \frac{a_{n,p_n,q_n}b^2}{1 + a_{n,p_n,q_n}b}.$$

By using the limits $st_A - \lim_n q_n = 1,\ st_A - \lim_n p_n = 1$, we have

$$st_A - \lim_n\frac{p_n^n}{[2]_{p_n,q_n} b_{n,p_n,q_n}} = 0, st_A - \lim_n\frac{\left(p_n - q_n\right)}{[2]_{p_n,q_n}} = st_A - \lim_n a_{n,p_n,q_n} = 0,$$

therefore,

$$\left\|R^*_{n,p_n,q_n}\left(e_1;x\right) - e_1\right\| < \varepsilon.$$

For $\varepsilon > 0$, we define the sets

$$A := \left\{ n \in N : \left\| R^*_{n,p_n,q_n} (e_1; .) - e_1 \right\| \geq \varepsilon \right\}, \tag{1.8}$$

$A_1 = \left\{ n \in \mathrm{N} : \frac{p_n^n}{[2]_{p_n,q_n} b_n} \geq \varepsilon \right\}$, $A_2 = \left\{ n \in \mathrm{N} : \frac{(p_n - q_n)}{[2]_{p_n,q_n}} \frac{b}{1+a_{n,p_n,q_n} b} \geq \varepsilon \right\}$, and
$A_3 = \left\{ n \in \mathrm{N} : \frac{a_{n,p_n,q_n} b^2}{1+a_{n,p_n,q_n} b} \geq \varepsilon \right\}$, thus from (1.8) we can see that $A \subseteq A_1 \bigcup A_2 \bigcup A_3$,

$$\begin{aligned} \delta \left\{ n \in N : \left\| R^*_{n,p_n,q_n} (e_1; .) - e_1 \right\| \geq \varepsilon \right\} &\leq \delta \left\{ n \in \mathrm{N} : \frac{p_n^{n-1}}{b_{n,p_n,q_n}} \frac{b}{1+a_{n,p_n,q_n} b} \geq \frac{\varepsilon}{3} \right\} \\ &+ \delta \left\{ n \in \mathrm{N} : \left(1 - \frac{1}{\left(1+a_{n,p_n,q_n} b\right)^2} \right) b^2 \geq \frac{\varepsilon}{3} \right\} \end{aligned}$$

$$+\delta \left\{ n \in \mathrm{N} : \frac{p_n^{n-1`}}{[n]_{p_n,q_n}} \frac{b^2}{\left(1 + a_{n,p_n,q_n} b\right)^2} \geq \frac{\varepsilon}{3} \right\}. \tag{1.9}$$

By taking the limit of both sides of the above inequality (1.9) , It is obvious that

$$st_A - \lim_n \frac{p_n^{n-1}}{b_{n,p_n,q_n}} \frac{b}{1 + a_{n,p_n,q_n} b} = 0,\ st_A - \lim_n \frac{1}{\left(1 + a_{n,p_n,q_n} b\right)^2} = 1, st_A - \lim_n \frac{p_n^{n-1`}}{[n]_{p_n,q_n}} \frac{b^2}{\left(1 + a_{n,p_n,q_n} b\right)^2} = 0.$$

Which implies

$$st_A - \lim_n \left\| R^*_{n,p_n,q_n} (e_1; x) - e_1 \right\| = 0. \tag{1.10}$$

Also, by using Lemma 1, we may write

$$\begin{aligned} \left| R^*_{n,p_n,q_n} (e_2; x) - e_2 \right| &\leq \left| \frac{p_n^{2n}}{[3]_{p_n,q_n} b_n^2} + \frac{\left(4q_n^3+5q_n^2 p_n+3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n} [3]_{p_n,q_n} b_n} \left(\frac{x}{1+a_{n,p_n,q_n} x} \right) \right| \\ &\left| + \frac{q_n [n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} \frac{4q_n^3+q_n^2 p_n+q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \left(\frac{x}{1+a_{n,p_n,q_n} x} \right)^2 - x^2 \right| \end{aligned}$$

$$\leq \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} + \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n} [3]_{p_n,q_n} b_{n,p_n,q_n}} \left(\frac{x}{1 + a_n x} \right)$$

$$. + \left\{ 1 - \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \frac{1}{(1 + a_{n,p_n,q_n})^2} \right\} x^2 + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \left(\frac{x}{1 + a_n x} \right)^2$$

By taking the maximum of both sides of the last equality on $[0, b]$ with $0 < b < \frac{1}{a_{n,p_n,q_n}}$, we get

$$\left\| R^*_{n,p_n,q_n} (e_2; x) - e_2 \right\| \leq \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} + \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n} [3]_{p_n,q_n} b_{n,p_n,q_n}} \left(\frac{b}{1 + a_{n,p_n,q_n} b} \right)$$

$$+ \left\{ 1 - \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \frac{1}{(1 + a_{n,p_n,q_n} b)^2} \right\} b^2 + \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \left(\frac{b}{1 + a_{n,p_n,q_n} b} \right)^2$$

By using the limits $st_A - \lim_n q_n = 1,\ st_A - \lim_n p_n = 1$, we have

$$st_A - \lim_n \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} = 1,\ st_A - \lim_n \frac{p_n^{n-1}}{[n]_{p_n,q_n}} = 0,\ st_A - \lim_n \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} = 0.$$

Therefore,

$$\left\| R^*_{n,p_n,q_n} (e_2; x) - e_2 \right\| < \varepsilon.$$

Now, for given $\varepsilon > 0$, we introduce the following sets;

$$D := \left\{ n \in N : \left\| R^*_{n,p_n,q_n}(e_2;.) - e_2 \right\| \geq \varepsilon \right\},$$

$$D_1 = \left\{ n \in \mathrm{N} : \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} \geq \frac{\varepsilon}{4} \right\},$$

$$D_2 = \left\{ n \in \mathrm{N} : \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n} [3]_{p_n,q_n} b_{n,p_n,q_n}} \left(\frac{b}{1 + a_{n,p_n,q_n} b} \right) \geq \frac{\varepsilon}{4} \right\},$$

$$D_3 = \left\{ n \in \mathrm{N} : \left\{ 1 - \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \frac{1}{\left(1 + a_{n,p_n,q_n} b\right)^2} \right\} b^2 \geq \frac{\varepsilon}{4} \right\},$$

$$D_4 = \left\{ n \in \mathrm{N} : \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \left(\frac{b}{1 + a_{n,p_n,q_n} b} \right)^2 \geq \frac{\varepsilon}{4} \right\}. \tag{1.11}$$

Then, from (1.11) we may write $D \subseteq D_1 \bigcup D_2 \bigcup D_3 \bigcup D_4$,

$$\begin{aligned} \delta \left\{ n \in N : \left\| R^*_{n,p_n,q_n}(e_2;.) - e_2 \right\| \geq \varepsilon \right\} &\leq \delta \left\{ n \in \mathrm{N} : \frac{p_n^{2n}}{[3]_{p_n,q_n} b^2_{n,p_n,q_n}} \geq \frac{\varepsilon}{4} \right\} \\ &+ \delta \left\{ n \in \mathrm{N} : \frac{\left(4q_n^3 + 5q_n^2 p_n + 3q_n p_n^2\right) p_n^{n-1}}{[2]_{p_n,q_n} [3]_{p_n,q_n} b_{n,p_n,q_n}} \left(\frac{b}{1 + a_{n,p_n,q_n} b} \right) \geq \frac{\varepsilon}{4} \right\} \\ &+ \delta \left\{ n \in \mathrm{N} : \left\{ 1 - \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \frac{1}{\left(1 + a_{n,p_n,q_n} b\right)^2} \right\} b^2 \geq \frac{\varepsilon}{4} \right\} \\ &+ \delta \left\{ n \in \mathrm{N} : \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \frac{4q_n^3 + q_n^2 p_n + q_n p_n^2}{[2]_{p_n,q_n} [3]_{p_n,q_n}} \left(\frac{b}{1 + a_{n,p_n,q_n} b} \right)^2 \geq \frac{\varepsilon}{4} \right\}, \end{aligned}$$

by taking the limit of both sides of the above inequality, It is obvious that $\delta(D) \leq \delta(D_1) + \delta(D_2) + \delta(D_3) + \delta(D_4) = 0$, which implies $st_A - \lim_n \left\| R^*_{n,p_n,q_n}(e_2;x) - e_2 \right\| = 0$. As a result, Equation (1.6) is proven, yielding the desired result. ∎

2 Conclusion

In this paper, by using the notion of (p,q) −calculus and statistical convergence, we give the main result of this research is to use modulus of continuity to study the rate of A-statistical convergence of Kantorovich type (p,q) −analogue of the Balázs–Szabados operators.

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A comparative analysis of an application of subgradient and Nelder-Mead methods to one layer-case inverse problem of gravimetry

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1 Introduction

Not with standing the wide variety of freely distributed and, especially, commercial industrial software for mathematical geophysics, there still is a certain number of companies who develop their own GIS software from scratch, aimed for their specific needs. This study was conducted as a part of such software project, developed for specific task, and successfully completed. Involved in cross-disciplinary work group, we had to develop an evaluation and predictive GIS, mainly driven on gravimetric data. The customer was a company running field development over the several deposits of the Caspian shelf. Due to physical specifics of the problem, the model built is two-dimensional, Poisson equation - based and has conditions on the part of boundary. The problem itself is to restore coordinate parameters for an homogeneous gravitational anomaly by the results of gravimetry, which appears to be an inverse problem of gravimetry. During the study, two versions of problem statement were considered. This work is dedicated mainly to the second problem statement, so-called "layer-case", which is explained further. Here we had conducted qualitative and quantitative comparative analysis with computer simulation for this problem.

2 Problem Statement: Problems 1 and 2

Let us have rectangular area Ω with horizontal coordinate x and vertical one y: $\Omega = \{(x;y)|0 < x < L, 0 < y < H\}$, where the area length L and its depth H are set. The boundary of Ω consists of Earth surface $y = 0$, and of inner bound S: $x = 0$ and $x = L$ - sides, $y = H$ - lower side. In Ω we consider Poisson equation

$$\Delta\varphi(x;y) = -4\pi G\rho(x;y); (x;y) \in \Omega,$$

where $\varphi = \varphi(x;y), \rho = \rho(x;y)$ is distribution of density in the area, and G is the gravitational constant. It is known that somewhere in the Ω there is some heterogeneity, or gravitation *anomaly* Ω_0, which is sought for, i.e. the target area. The densities ρ_1 and ρ_2, first for heterogeneity, second for host rock, are supposed to be known. Then, the formula holds:

$$\rho(x;y) = \begin{cases} \rho_1, & (x;y) \in \Omega_0 \\ \rho_2, & (x;y) \notin \Omega_0. \end{cases}$$

Denoting u the difference between potentials of perturbed and unperturbed gravitational field, and with respect to the system linearity, we arrive at equation

$$\Delta u(x;y) = -4\pi G\psi(x;y),\ (x;y) \in \Omega, \tag{2.1}$$

where

$$\psi(x;y) = \begin{cases} \eta, & (x;y) \in \Omega_0 \\ 0, & (x;y) \notin \Omega_0, \end{cases} \tag{2.2}$$

and η is the difference between host rock and anomaly densities. Further, we will mention values u and η just as *potential* and *density* (really, they are differences of values of corresponding values). The Ω is chosen to be such large, that anomaly influence on its inside boundary practically absents, hence, the potentials of perturbed and unperturbed fields there are equal. We obtain the homogeneous boundary condition:

$$u(x;y) = 0,\ (x;y) \in S. \tag{2.3}$$

By the on-surface gravimetry we know function u and its vertical derivative. Thus, we obtain the boundary conditions:

$$u(x;0) = 0,\ 0 < x < L, \tag{2.4}$$

$$\frac{\partial u(x,0)}{\partial y} = \beta(x), 0 < x < L, \tag{2.5}$$

where functions α and β (gravitational potential and its vertical derivative, correspondently) are supposed known. We set Ω_0 as

$$\Omega_0 = \{(x;y) | a \le x \le a+m,\ b \le y \le b+n\} \tag{2.6}$$

for *Problem 1*, or

$$\Omega_0 = \{(x;y) | a \le x \le b,\ c \le y \le d\} \tag{2.7}$$

for *Problem 2* (layer-case), and here values a and b are known, and the coordinates c and d (layer in-depth and thickness, correspondently), are to be found. Thus, we have inverse problems:

Given the data (1)-(6), *to restore the pair of parameters:*

Problem 1: restore the center coordinates $(a;b)$ *of the target area* Ω_0, *where* (1)-2.6 *holds*;
Problem 2: restore the in-depth and thickness of target layer $(c;d)$, *where* (1)-2.7 *holds*;

According to common principles of inverse problem solving, we pass to the functional minimization problem for the target function:

$$I(k;l) = \int_0^L \left[\frac{\partial u(x,0)}{\partial y} - \beta(x)\right]^2 dx \to \min, \tag{2.8}$$

where $(k;l)$ denotes the parametric pair sought, according to *Problem 1* or *Problem 2*, correspondently.

3 Target Function Study

In the solving practice of unconditional extremum problems of optimal control, gradient methods are most often used, Let $\omega = -4\pi G\eta$, and let $p = p(x;y)$ be solution of Laplace's equation

$$\Delta p(x;y) = 0,\ (x;y) \in \Omega, \tag{3.1}$$

with boundary conditions

$$p(x;0) = 2\left(\beta(x) - \frac{\partial u(x,0)}{\partial y}\right), 0 < x < L; p(x;y) = 0, \tag{3.2}$$

$$(x;y) \in S. \tag{3.3}$$

Then, the next two *Theorems* hold:

Theorem 3.1. *(for Problem 1) If the conditions (8-10) hold, then functional I has derivative in any point $(a;b)$ and in any direction $(g;h)$, and this derivative has form*

$$I'(a;b;g;h) = \omega \int_a^{a+m} [p(x;b) - p(x;b+n)]\, dx|h| + \omega \int_a^{a+m} [p(a;y) - p(a+m;y)]\, dy|g|, \tag{3.4}$$

where values $(a;b)$ are center coordinates of Ω_0.

Theorem 1 describes the derivative when target area might vary both in horizontal and in vertical directions via geometric parallel translation. Next, the Theorem 2 describes more simple situation, when variation occurs only inside vertical stripe, and subsequently resulting formula is much simpler.

Theorem 3.2. *(for Problem 2) If the conditions (8-10) hold, then functional I has derivative in any point $(c;d)$ and in any direction $(g;h)$, and this derivative has form*

$$I'(c,d;g,h) = \omega \cdot |g| \int_a^b p(x,c)dx + \omega \cdot |h| \int_a^b p(x,d)dx, \tag{3.5}$$

where values $(c;d)$ are in-depth and thickness of target layer, correspondently.

Proof outline: (steps) 1. To find variation of the target function; 2. to find dependence between the integration limits variation and form of the difference functional; passing to the limit, to obtain the functional derivative form. Proof is complete. Both theorems have similar proof techniques (see [1] for Theorem 1 proof)

4 Remarks

- By virtue of the fact the obtained derivatives (11-12) dependencies on the directions (g;h) are *nonlinear* (as they contain the *modulus function*), the target function I is *non-smooth*;
- Hence, the stated problem can not be solved with gradient methods, and requires only non-smooth methods of solution;
- Modulus is *subdifferentiable function.*
- Within the problem statement, the subgradient and Nelder-Mead methods were chosen for numeric testing.

5 Passing to Inverse Problem

The next theorem allows to solve the inverse problem stated, setting up the subgradient method (it is possible because of Remark 3).

Theorem 5.1. *If the pair (c', d') is the subgradient for function I at the point $(c;d)$, then the inequalities hold:*

$$|c'| \leq |C(p)|\,, |d'| \leq |D(p)|\,, \tag{5.1}$$

where

$$C(p) = \omega \cdot \int_a^b p(x;c)dx, D(p) = \omega \cdot \int_a^b p(x;d)dx,$$

and p is the solution of adjoint system (3.1-3.3).

6 Algorithms

6.1 The iterational formulae for *subgradient method* have form:

$$a_{k+1} = a_k - \alpha_k \frac{a'_k}{|(a'_k, b'_k)|},\ b_{k+1} = b_k - \alpha_k \frac{b'_k}{|(b'_k, b'_k)|},$$

$$\alpha_k = \frac{\alpha}{\sqrt{k+1}};\ \ \alpha > 0$$

$$I_k = \min[I(a_{k-1}, b_{k-1}), I(a_k, b_k)]$$

6.2 Iterational formulae for *Nelder-Mead method* have their standard form.

7 Numeric results. Comparison. Corollary.

7.1 Numeric Results

7.1.1 The reasons for choice the mentioned two methods for testing here are: 1) a target function property, which does not allow any gradient methods, as is stated during the study [1]; 2) the typicality of both mentioned methods as common non-gradient methods. Our goal was to esteem their behavior and accuracy here; **7.1.2** Numeric results for *Problem 1* were discussed in [1], and here we discuss results for *Problem 2*. ***7.1.3.*** Here are results of computer simulation of two methods for some parameter setups:

Table 1, results for:	c=0.5	d=0.6		
Method	c	d	c, relative error	d, relative error
Subgradient	0.435	0.519	0.13	0.135
Nelder-Mead	0.458	0.534	0.084	0.11

Table 2, results for:	c=0.7	d=0.9		
Method	c	d	c, relative error	d, relative error
Subgradien	0,721	0,802	0,03	0,109
Nelder-Mead	0,697	0,866	0,004286	0,038

Table 3, results for:	c=0.3	d=0.7		
Method	c	d	c, relative error	d, relative error
Subgradient	0.363	0.725	0.210	0.036
Nelder-Mead	0.337	0.745	0.123	0.064

7.2 Comparison. Corollary

We had conducted qualitative and quantitative comparative analysis with computer simulation for the given problem. The problem is real data-based. The uniqueness of exact solution is proven [2], but in real calculation, some different data setups sometimes might give the same outputs due to the well-known peculiarities of the applied methods; though, numeric results still are quite good for practice. In comparison, we concluded that, though subgradient method works noticeably slower (due to random search involved), it restores the data better; Nelder-Mead also gives good results, but requires a good initial approximation; and it is possible to use both in joint for more better results. The results obtained appear to be useful for oil-and-gas field practitioners.

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Hybridized Metaheuristics for the Multi-Objective Quadratic Knapsack Problem.

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Abstract: This study aims to give information about Karci's new fractional derivative. Its properties are given by comparison with popular fractional derivative approaches. In addition, some examples illustrating the concept of Karci's fractional derivative are given. Finally, some new properties are obtained for Karci's fractional derivative.

Keywords: Non-dominated Sort Algorithm, Crowding-Distance, Gradient Algorithm, Memetic Algorithm With Selection Neighborhood Pareto Local Search.
Mathematics Subject Classification: 90B50, 90C27

Abstract: The knapsack problem is basic in combinatorial optimization and has many variants and expansions. We focus on the quadratic stochastic multi-objective knapsack problem with random weights. We propose a Multi-Objective Memory Algorithm with Local Pareto Neighborhood Selection Search. At each iteration of our algorithm, a crossover, a mutation, and a local search are applied to a population of solutions to generate new solutions that will constitute an offspring population.
Then, we apply, on the combined population of parents and offspring, the best solution selection operator based on the termination of the non-domination rank and the crowding distance obtained respectively by the non-domination sorting algorithm and the crowding distance computation algorithm. To prove the performance of our algorithm, we compare it with both an exact algorithm and the NSGAII algorithm. Our experimental results show that the MASNPL algorithm leads to significant efficiency.

1 Mathematical Formulation

We consider a stochastic quadratic multi-objective knapsack problem of the following form: given a knapsack with a fixed weight capacity $c > 0$ as well as a set of n items, $i = 1, \ldots, n$, each item has a weight that is not known in advance, i.e. the decision of which items to choose must be made without the exact knowledge of their weights. Therefore, we treat the weights as random variables and assume that the weights χ_i, $i = 1, \ldots, n$, are independently normally distributed with means $\mu_i > 0$, and standard deviations $\sigma_i, i = 1, \ldots, n$. Moreover, each item $i = 1, \ldots, n$ has a fixed m-vector reward per weight unit $r_i = (r_i^1, \ldots, r_i^m)^T$, $r_i^1 \in \mathbb{Z}^+, k = 1, \ldots, m$, and to each pair of items i and j, $1 \leq i \neq j \leq n$, is associated a m-vector of joint reward per unit of weight $r_{ij} = (r_{ij}^1, \ldots, r_{ij}^m)^T$, for each objective $k, k = 1, \ldots, m$. The choice of a reward per unit weight can be justified by the fact that the value of an item often depends on its weight, which is not known in advance.
In case of overweight, items must be removed and a penalty d must be paid for each unit of weight unwrapped. Our goal is therefore to minimize the total penalty.
The selection of an item is defined by a binary decision variable x_i which takes the value 1 if item i is included in the selection and 0 otherwise.
The Multi-objective Quadratic Stochastic Knapsack Problem can be mathematically formulated as follows:

The Quadratic Stochastic Knapsack Problem with simple recourse

In the formulation of the multi-objective quadratic stochastic knapsack problem with simple recourse, the capacity constraint has been included in the objective function by using the penalty function $[.]^+$ and a penalty factor $d > 0$, in the case of an overload, items have to be removed and a penalty d has to be paid for each unit of weight that is unpacked.

$$\max_{x\in\{0,1\}^n} \mathbf{E}\Big[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} r_{ij}^{(k)} x_i x_j(\chi_i+\chi_j)\Big] + \mathbf{E}\Big[\sum_{i=1}^{n} r_i^{(k)} x_i\chi_i\Big] - d\mathbf{E}\Big[\sum_{i=1}^{n} x_i\chi_i - c\Big]^+, k=1,\ldots,m. \tag{1.1}$$

Equation (1.1) aims to maximize the total profit of all assigned objects.
The special case when $k = 2$ is called bi-objective binary knapsack problem and denoted by 0-1 BOKP. where:

- $\mathbf{E}[.]$ denotes the expectation,
- $g(x,\chi) = \sum_{i=1}^{n} x_i\chi_i$,
- $d \in \mathbb{R}^+$,

We write the objective function of the Quadratic Stochastic Knapsack Problem with simple recourse as follows:

$$\mathbf{J}^{(k)}(x,\chi) = \mathbf{E}\Big[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} r_{ij}^{(k)} x_i x_j(\chi_i+\chi_j)\Big] + \mathbf{E}\Big[\sum_{i=1}^{n} r_i^{(k)} x_i\chi_i\Big] - d\mathbf{E}\Big[\sum_{i=1}^{n} x_i\chi_i - c\Big]^+,\ k=1,\ldots,m. \tag{1.2}$$

Since the function $\mathbf{J}$ is not differentiable, we present an approximation to its gradient, named approximation by convolution. This is one of the two methods presented by Andrieu and al. (for further details on this method see [1]).
Based on *"Approximation By Convolution Method"*, we can approximate the $\nabla(J_t)_x$ gradient of the function J as follows:

$$\begin{aligned}\nabla(\mathbf{J}^{(k_t)})(x,\chi) &= \Big[(r_1^{(k)}\chi_1,\ldots,r_n^{(k)}\chi_n)^T + \Big(\sum_{\substack{j=1\\j\neq1}}^{n} r_{1j}^{(k)}(\chi_1+\chi_j)x_j,\ldots,\sum_{\substack{j=1\\j\neq n}}^{n} r_{nj}^{(k)}(\chi_n+\chi_j)x_j\Big)^T\Big]\\ &-d\Big(-\frac{1}{t}h\Big(\frac{g(x,\chi)-c}{t}\Big).\chi.\big(g(x,\chi)-c\big)+\mathbf{1}_{\mathbb{R}^+}\big(g(x,\chi)-c\big).\chi\Big), k=1,\ldots,m.\end{aligned} \tag{1.3}$$

In [1] and [12], the authors proposed various function may be chosen for h. They compute for each function a reference value for the mean square error of the obtained approximated gradient and compare them. It turns out that, the function $h = \frac{3}{4}\left(1-x^2\right)\left(\mathbf{1}_{\mathbb{R}^+}\right))$, offers the smallest of this value (here the indicator function $(\mathbf{1}_1)$ is defined as:

$$(\mathbf{1}_1) = \begin{cases} 1 & \text{if } \ 1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Based on the result of [12], we give the following estimation of the gradient of J:

$$\begin{aligned}\nabla(\mathbf{J}^{(k_t)})(x,\chi) &= \Big[(r_1^{(k)}\chi_1,\ldots,r_n^{(k)}\chi_n)^T + \Big(\sum_{\substack{j=1\\j\neq1}}^{n} r_{1j}^{(k)}(\chi_1+\chi_j)x_j,\ldots,\sum_{\substack{j=1\\j\neq n}}^{n} r_{nj}^{(k)}(\chi_n+\chi_j)x_j\Big)^T\Big]\\ &-d.\Bigg(-\frac{3}{4t}\Big(1-\Big(\frac{g(x,\chi)-c}{t}\Big)^2\Big)\cdot\mathbf{1}_1\cdot\Big(\frac{g(x,\chi)-c}{t}\Big).\chi.\big(g(x,\chi)-c\big)-\mathbf{1}_{\mathbb{R}^+}\big(g(x,\chi)-c\big).\chi\Bigg),\\ &k=1,\ldots,m.\end{aligned} \tag{1.4}$$

1.1 Deterministic Equivalent Problem

The new random variable is defined as follows: $X := \sum_{i=1}^{n} x_i \chi_i$, which is normally distributed with mean $\widehat{\mu}_i := \sum_{i=1}^{n} \mu_i x_i$,standard deviation $\widehat{\sigma} := \sqrt{\sum_{i=1}^{n} \sigma_i^2 x_i^2}$, density function $\varphi(x) = \frac{1}{\widehat{\sigma}} f\left(\frac{x-\widehat{\mu}}{\widehat{\sigma}}\right)$ and cumulative distribution function $\Phi(x) = F\left(\frac{x-\widehat{\mu}}{\widehat{\sigma}}\right)$.

We write the deterministic equivalent objective function as follows:

$$J_{det}^{k}(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} r_{ij}^{(k)} x_i x_j (\mu_i + \mu_j) + \sum_{i=1}^{n} r_i^{(k)} \mu_i x_i - d \cdot \left[\widehat{\sigma} \cdot f\left(\frac{c-\widehat{\mu}}{\widehat{\sigma}}\right) - (c-\widehat{\mu}) \cdot \left[1 - F\left(\frac{c-\widehat{\mu}}{\widehat{\sigma}}\right)\right]\right], k = 1, ..., m. \tag{1.5}$$

2 Resolution Method

In the decomposition framework, the original $MO-QSKP$ is first decomposed into many $SO-QSKP$. To be specific, given the objective vector $F(x) = \left(f^1(x), f^2(x), \ldots, f^m(x)\right)^T$ and weight vector $\lambda = (\lambda^1, \ldots, \lambda^m)^T$, where the sum of weights vector should be equal to 1, and an approximation of the gradient of the function $J(x, \chi)$ is given by:

$$\begin{aligned}\nabla(\mathbf{J}_t^k)(x,\chi) = \sum_{k=1}^{m} \lambda^{(k)} . \Bigg[\Bigg[&(r_1^{(k)}\chi_1, \ldots, r_n^{(k)}\chi_n)^T + \Bigg(\sum_{\substack{j=1\\ j\neq 1}}^{n} r_{1j}^{(k)}(\chi_1+\chi_j)x_j, \ldots, \sum_{\substack{j=1\\ j\neq n}}^{n} r_{nj}^{(k)}(\chi_n+\chi_j)x_j\Bigg)^T\Bigg] \\ -d.\Bigg(-\frac{3}{4t}\Big(1-\Big(\frac{g(x,\chi)-c}{t}\Big)^2\Big) &\cdot \mathbf{1}_1 \cdot \Big(\frac{g(x,\chi)-c}{t}\Big).\chi.\big(g(x,\chi)-c\big) - \mathbf{1}_{\mathbb{R}^+}\big(g(x,\chi)-c\big).\chi\Bigg)\Bigg], \\ &k = 1, \ldots, m.\end{aligned} \tag{2.1}$$

In this section, we detail our Memetic Algorithm With Selection Neighborhood Pareto Local Search Algorithm $MASNPL$ For $MO-QSKP$, including initialization, crossover, and local search.
Then, the fitness of each individual is evaluated and Non-dominated Sorting is applied to assign a non-domination rank i_{rank} equal to its non-domination level, and Crowding-Distance is computed for each individual i in the population i_{dist}.
These two parameters, i_{rank} and i_{dist}, are used to select individuals in the most crowded region and maintain the diversity of solutions on the Pareto front. Then, binary tournament selection is applied to choose the parents, after whic, crossover and mutation operators are performed to generate new candidate solutions, i.e., the offspring population of size N.
Non-dominated sorting and Crowding distance are applied to a combined population. Then, the population R_t is sorted according to non-domination. Now, solutions belonging to the best non-dominated set, $\mathcal{F}_1$ are of the best solutions in the combined population and must be emphasized more than any other solution in the combined population.
If the size of $\mathcal{F}_1$ equal to N, we definitely add all members of the set $\mathcal{F}_1$ for the new population P_{t+1}.
If the size of $\mathcal{F}_1$ is smaller then N, we definitely put all members of the set $\mathcal{F}_1$ for the new population P_{t+1}. The remaining members of the population P_{t+1} are chosen from subsequent non-dominated fronts in the order of their ranking.
Thus, solutions from the set $\mathcal{F}_2$ are chosen next, followed by solutions from the set $\mathcal{F}_3$, and so on.
The above process (Lines 17-20 of Algorithm 4) is continued until a final set of non-dominated solutions of size N is obtained. The new population P_{t+1} of size N is now used for selection, a binary tournament selection operator, crossover, and mutation to create a new population Q_{t+1} of size N The above process (Lines 6-23 of Algorithm 4) is repeated until $NG_{\max}$ is obtained.

Algorithm 1 Memetic Algorithm With Selection Neighborhood Pareto Local Search ($MASNPL$)

Input: P: the current Population of size N;
    N: the size of population;
    Q: the set of new offspring;
    N: the set of new offspring.
Initialize the population to size N;
Evaluate Fitness for every individual in population of size N;
Apply Non-dominated Sort Algorithm (P, N);
Compute Crowding-Distance Computation Algorithm (P, N);
$NG = 1$;
while $NG \leq NG_{\max}$ **do**
    $Q = \emptyset$;
    for $k = 1$ **to** $\frac{P}{2}$ **do**
        Select Parents using Binary Tournament;
        Apply the crossover operator to generate two new offsprings (Q_1, Q_2);
        Apply Mutation operator on both (Q_1, Q_2) with probability $P_m = \frac{1}{n}$;
        for $j = 1$ **to** 2 **do**
            $Q_j \longleftarrow$ the Selection Neighborhood Pareto Local Search (Q_j);
            $Q = Q \cup \{Q_j\}$;
        end for
    end for
    $R_t = P_t \cup Q_t$;
    Apply Non-dominated Sort Algorithm (R_t)**;**
    $P_{t+1} = \emptyset$ and $i = 1$;
    while $(|P_{t+1}| + |_i|) \leq N$ **do** ▷ until the parent population is filled
        Crowding-Distance Computation (F_i); ▷ calculate crowding-distance in F_i
        $P_{t+1} = P_{t+1} \cup F_i$; ▷ include i-th non-dominated front in the parent population
        $i = i + 1$ ▷ check the next front for inclusion
    end while
    Sort F_i; ▷ sort in descending order using crowded comparison operator
    $P_{t+1} = P_{t+1} \cup F_i[1, N - |P_{t+1}|]$; ▷ choose the first $(N - |P_{t+1}|)$ elements of F_i
    $Q_{t+1} =$ Make New Population (P_{t+1}); ▷ use selection, crossover and mutation to create a new population (Q_{t+1})
    $NG = NG + 1$;
end while

3 Conclusion

In this paper, we detailed the model for the Multi-objective Quadratic Stochastic Knapsack Problem with simple recourse and random weight. As the objective functions were not differentiable, we approximated their gradients by using approximation by convolution method, which is used for numerical resolution. We apply a greedy heuristic for $MO-QSKP$ to obtain an initial population of size N. Then we apply the Non-dominated Sort Algorithm to this population. This algorithm aims to sort each individual of the population into different non-domination levels, after that, we determine the crowding distance value of a solution by applying the Crowding-Distance Computation Algorithm. Those algorithms gave us a population sorted by non-domination levels and the crowding distance for each individual of the population, then, the series of mutations, crossovers, and local search are applying to this population to generate an offspring. To improve the offspring we apply the Selection Neighborhood Pareto Local Search $SNPLS$ algorithm based on the comparison between the old solution (offspring) and the new solution obtained by the Gradient algorithm, then, the Non-dominated Sort Algorithm and the Crowding-Distance Computation Algorithm are applied to offspring improved to select our first final best individuals of population id of size N. Finally, the experimental results for the comparison between the $MASNPL$ algorithm and $NSGAII$ algorithm show the using of the gradient algorithm with $NSGAII$ algorithm is significantly performance and efficient than the $NSGAII$ algorithm

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Some fixed points results for (λ, ψ)- partial hybrid functions in CAT(0) spaces

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Abstract: In this paper is defined a new class of contractions, (λ, ψ)- partial hybrid functions in CAT(0) space. The goal of this paper is to present a new convergent fixed point result based on a (λ, ψ)- partial hybrid contraction on CAT(0) space. We have assured the set of the fixed points for a (λ, ψ)- partial hybrid on CAT(0) space is nonempty. Furthermore, a Δ-convergent theorem on CAT(0) space is proved.

Keywords: Fixed point, (λ, ψ)- partial hybrid contraction, CAT(0) space, Δ-convergence, Mann Iteration

1 Introduction

M. Gromov [1], studied for the first time CAT(0) space in 1987. The study of CAT(0) spaces have many applications in Graph Theory, Fixed Point Theory, etc. W. Kirk [2], [3] defined and proved many fixed point results for nonexpansive functions. Dhompongsa and Panyanak [4] studied Δ-convergence in CAT(0) space. Many authors have worked on Theory of Fixed Point in CAT(0) space by assuring the existence of fixed point for various class of functions [5, 6, 7] or by presenting new iterations which obtain approximating fixed point [8, 9]. Inspired by above, we propose a new class of contractions in CAT(0) space called (λ, ψ)- partial hybrid functions. Related to them, some fixed point theorems and some convergent results are obtained.

2 Preliminaries

Definition 2.1. [1] Let (X, d) be metric space and $x, y \in X$. The map $\gamma : [0, l] \to X$ is called a geodesic curve if it completes the following conditions:

1. $\gamma(0) = x$;
2. $\gamma(l) = y$;
3. $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$, for every $t_1, t_2 \in [0, l]$.

The image of γ is called geodesic segment that joins the point x, y. The couple (X, d) is called (unique) geodesic metric space if for every two points in X there exists a (unique) geodesic curve that joins them. It is denoted $\gamma(t0 + (1-t)l) = tx \bigoplus (1-t)y,\ t \in (0, 1)$.
A subset $Y \subseteq X$ is called convex if for every geodesic segment that joins two points is included in Y.

Definition 2.2. [1] Let (X, d) be a geodesic space. A geodesic triangle consits of three points $x_1, x_2,\ x_3 \in X$ and three geodesic segments. It is denoted $\Delta(x_1, x_2, x_3)$.

Definition 2.3. Let (X, d) be a geodesic space and $\Delta(x_1, x_2, x_3)$ be a geodesic triangle. A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ is the triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x}_2, \overline{x}_3)$ in Euclidian plane E^2 such that $d_{E^2}(\overline{x}_i,\ \overline{x}_j) = d(x_i, x_j)$, for $i, j \in \{1, 2, 3\}$.

Definition 2.4. [1] The geodesic space (X, d) is called CAT(0) space if for every triangle $\Delta(x_1, x_2, x_3)$ and $x, y \in \Delta$, the inequality $d(x, y) \leq d_{E^2}(\overline{x}, \overline{y})$, for $\overline{x}, \overline{y} \in \overline{\Delta}$, holds.

Proposition 2.1. *[10] Let (X, d) be a CAT(0) space. Then*

$$d\left((1-t)\,x \bigoplus ty, z\right) \leq (1-t)\,d\,(x,z) + td(y,z)$$

where $t \in [0,1]\,,\ \ x, y, z \in X$.

Proposition 2.2. *[13]Let (X, d) be a CAT(0) space. Then the following inequality is true*

$$d^2\left((1-t)\,x \bigoplus ty, z\right) \leq (1-t)\,d^2\,(x,z) + td^2\,(y,z) - t(1-t)d^2\,(x,y)$$

for every $t \in [0,1]\,,\ x, y, z \in X$.
The concept of Δ-convergence was introduced by Lim in 1976 [12] in generalized metric space. In 2008, Kirk and Panyanak [11] used the Δ-convergence concept in CAT(0).

Definition 2.5. [14] Let X be a complete CAT(0) space, K be a subset of X and $\{x_n\}$ be a bounded sequence in it.
Asymptotic radius $r\,((x_n))$ of the sequence (x_n) is called $r\,((x_n)) = inf\ \ \{r\,(x,(x_n)) : x \in X\}\,,$ where $r\,(x,(x_n)) = \lim_{n\to+\infty} sup\ \ d(x_n, x)$.
Asymptotic center $A((x_n))$ of the sequence (x_n), $A\,((x_n)) = \{x \in X : r\,(x,(x_n)) = r\,((x_n))\}\ .$

Definition 2.6. [12] Let X be a complete CAT(0) space. The sequence $\{x_n\}$ in X is called Δ-convergent to a point $x \in X$ if x is the only asymptotic center for each subsequence (u_n) of (x_n).

Theorem 2.1. *[13] Let X be a complete CAT(0) space. Every bounded sequence in X, has a Δ-convergent subsequence.*

3 Main Results

Definition 3.1. Let $\mathcal{K}$ a nonempty, closed convex subset of a CAT(0) space (X, d). The function $T : K \to K$ is called (λ, ψ)- partial hybrid if it satisfies the following condition: $\lambda\,(x,y)\,d^2\,(Tx,Ty) \leq \psi\left(\max\left\{d^2\,(x,y)\,,\ d^2\,(x,Ty)\,,\ d^2\,(Tx,y)\right\}\ \right)$, for every $(x,y) \in X \times X$, where $\psi : [0,\ +\infty[\to [0,\ +\infty[$ is a continuous comparison function and $\lambda : X \times X \to [1, +\infty[$.

Lemma 3.1. *Let (X, d) be a complete CAT(0) space and $\mathcal{K}$ a nonempty, closed convex subset of X $T : X \to X$ be a (λ, ψ)- partial hybrid function. If $F(T) \neq \phi$ then $F(T)$is a closed and convex subset of K.*

Proof Since $F\,(T) \neq \phi$ we can take a sequence $\{x_n\}_{n\in N}$ in $F\,(T)$. Suppose that the sequence $\{x_n\}_{n\in N}$ converges to a point $x \in K$. We have that

$$\begin{aligned} d^2\,(Tx,x_n) &\leq \lambda\,(x,x_n)\,d^2\,(Tx,x_n) \leq \psi\left(\max\left\{d^2\,(x,x_n)\,,\ d^2\,(Tx,x_n)\,,\ d^2\,(x,x_n)\right\}\ \right) \\ &< \max\left\{d^2\,(x,x_n)\,,\ d^2\,(Tx,x_n)\,,\ d^2\,(x,x_n)\right\}\ = \max\left\{d^2\,(x,x_n)\,,\ d^2\,(Tx,x_n)\right\} \end{aligned}$$

Case 1. If $\max\left\{d^2\,(x,x_n)\,,\ d^2\,(Tx,x_n)\,,\ d^2\,(x,x_n)\right\}\ = d^2\,(x,x_n)$ then $d^2\,(Tx,x_n) < d^2\,(x,x_n)$. Consequently, $d^2\,(Tx,x) \leq 0$ and $Tx = x$.
Case 2. If $\max\left\{d^2\,(x,x_n)\,,\ d^2\,(Tx,x_n)\,,\ d^2\,(x,x_n)\right\}\ = d^2\,(Tx,x_n)$ then $d^2\,(Tx,x_n) < d^2\,(Tx,x_n)$. This case is trivial.
The next step is to prove that $F(T)$is a convex set. Taking $z = (1-t)\,x \bigoplus ty$ where $x, y \in F(T)$ and $t \in (0,1)$, we have:

$$\begin{aligned} d^2\,(Tz,z) = d^2\left(Tz,(1-t)\,x \bigoplus ty\ \right) &\leq (1-t)\,d^2\,(Tz,x) + td^2\,(Tz,y) - (1-t)\,td^2\,(x,y) \\ &\leq (1-t)\,d^2\,(z,x) + td^2\,(z,y) - (1-t)\,td^2\,(x,y)\,. \end{aligned}$$

We see that

$$d^2(z,x) = d^2\left((1-t)\,x \bigoplus ty, x\right) \le (1-t)\,d^2(x,x) + td^2(y,x) - \mathrm{t}\,(1-\mathrm{t})\,d^2(y,x) = t^2 d^2(x,y).$$

Similarly, we have $d^2(z,y) \le (1-t)^2 d^2(x,y)$.
Using the obtained inequalities, we take

$$d^2(Tz,z) \le (1-t)\,t^2 d^2(x,y) + t(1-t)^2 d^2(x,y) - (1-t)\,td^2(x,y) = 0.$$

Consequently, $d(Tz,z) = 0$ and $Tz = z$. As a result, z is a fixed point of T and $z \in F(T)$. ■

Theorem 3.1. *Let (X,d) be a complete CAT(0) space and $\mathcal{K}$ a nonempty, closed convex subset of X and $T : X \to X$ be a (λ,ψ)- partial hybrid function. The following propositions are equivalent:*

1. *$F(T) \ne \phi$.*
2. *The sequence $\{T^n x\}$ is bounded for some $x \in X$.*

Proof Let suppose that $F(T) \ne \phi$. Consequently, there exists a point $x \in X$ such that $Tx = x$. As a result, the sequence $\{T^n x\}$is bounded in X.
Let see conversely. The sequence $\{T^n x\}$ is bounded for some $x \in X$. Denote $x_n = T^n x$. Since $\{x_n\}_{n\in N}$ be a bounded sequence, there exists a point $x^* \in X$, such that $A(\{x_n\}) = \{x^*\}$.
Using Lemma 3.2 it yields $x^* \in K$. Now, we see that

$$\begin{aligned} d^2(Tx^*, x_n) \le \lambda(x^*, x_{n-1})\, d^2(x^*, Tx_{n-1}) &\le \psi\left(\max\left\{d^2(x^*, x_{n-1}),\ d^2(Tx^*, x_{n-1}),\ d^2(x^*, Tx_{n-1})\right\}\right) \\ &< \max\left\{d^2(x^*, x_{n-1}),\ d^2(Tx^*, x_{n-1}),\ d^2(x^*, Tx_{n-1})\right\}. \end{aligned}$$

This yields $\lim_{n\to\infty}\sup_n d^2(Tx^*, x_n) \le \lim_{n\to\infty}\sup_n \max\left\{d^2(x^*, x_{n-1}),\ d^2(Tx^*, x_{n-1}),\ d^2(x^*, Tx_{n-1})\right\}$
Case 1. If $\max\left\{d^2(x^*, x_{n-1}),\ d^2(Tx^*, x_{n-1}),\ d^2(x^*, Tx_{n-1})\right\} = d^2(x^*, x_{n-1})$ then $d^2(Tx^*, x_n) \le d^2(x^*, x_{n-1})$. In this case we have that $Tx^* = x^*$.
Case 2. If $\max\left\{d^2(x^*, x_{n-1}),\ d^2(Tx^*, x_{n-1}),\ d^2(x^*, x_n)\right\} = d^2(Tx^*, x_{n-1})$ then $d^2(Tx^*, x_n) \le d^2(Tx^*, x_{n-1})$. This case is trivial
Case 3. If $\max\left\{d^2(x^*, x_{n-1}),\ d^2(Tx^*, x_{n-1}),\ d^2(x^*, x_n)\right\} = d^2(x^*, x_n)$ then $d^2(Tx^*, x_n) \le d^2(x^*, x_n)$.
It yields that $\lim_{n\to\infty}\sup_n d^2(Tx^*, x_n) \le \lim_{n\to\infty}\sup_n d^2(x^*, x_n)$ and $(\Phi(Tx^*))^2 = (\Phi(x^*))^2$. Since $A(\{x_n\}) = \{x^*\}$, $Tx^* = x^*$.
In 1953, Mann [15] presented an iteration as follows $y_{n+1} = (1-\alpha_n)y_n \bigoplus \alpha_n T y_n$, where $\alpha_n \in [0,1]$.
Below we assure a convergence theorem for a sequence constructed by Mann iteration. ■

Theorem 3.2. *Let (X,d) be a complete CAT(0) space and $\mathcal{K}$ a nonempty, closed convex subset of X and $T : K \to K$ be a (λ,ψ)- partial hybrid function. Suppose that there exists a point $x \in X$ such that $\{T^n x\}$ is bounded. Let $y_0 \in X$, and define the sequence $\{y_n\}_{n\in N}$ which satisfies the Mann's iteration $y_{n+1} = (1-\alpha_n)y_n \bigoplus \alpha_n T y_n$, where $\alpha_n \in [0,1]$. Then$\lim_{n\to+\infty} d(y_n, p) = 0$, where $p \in F(T)$ and $\lim_{n\to+\infty} d(y_n, Ty_n) = 0$.*

Proof Firstly, the existence of a fixed point p of the (λ,ψ)- partial hybrid function T is guaranteed by Theorem 3.3.
Now we see that

$$\begin{aligned} d^2(y_{n+1}, p) &= d^2\left((1-\alpha_n)\,y_n \bigoplus \alpha_n T y_n, p\right) \\ &\le (1-\alpha_n)\,d^2(y_n,p) + \alpha_n d^2(Ty_n,p) - \alpha_n(1-\alpha_n)\,d^2(y_n, Ty_n) \\ &\le (1-\alpha_n)\,d^2(y_n,p) + \alpha_n d^2(Ty_n,p) \\ &\le (1-\alpha_n)\,d^2(y_n,p) + \alpha_n \frac{\psi\left(\max\left\{\ d^2(y_n,p), d^2(y_n,p), d^2(y_{n+1},p)\right\}\ \right)}{\lambda(y_n,p)} \\ &\le (1-\alpha_n)\,d^2(y_n,p) + \alpha_n \psi\left(\max\left\{\ d^2(y_n,p), d^2(y_{n+1},p)\right\}\ \right) \end{aligned}$$

Case 1. If $\max\left\{ d^2(y_n,p), d^2(y_{n+1},p)\right\} = d^2(y_n,p)$ we have that

$$d^2(y_{n+1},p) \le (1-\alpha_n)\, d^2(y_n,p) + \alpha_n \psi\left(d^2(y_n,p)\right) < (1-\alpha_n)\, d^2(y_n,p) + \alpha_n d^2(y_n,p) = d^2(y_n,p)$$

and this case is trivial.
Case 2. If $\max\left\{ d^2(y_n,p), d^2(y_{n+1},p)\right\} = d^2(y_{n+1},p)$, we obtain a contradiction.
This because

$$d^2(y_{n+1},p) \le (1-\alpha_n)\, d^2(y_n,p) + \alpha_n \psi\left(d^2(y_{n+1},p)\right) < (1-\alpha_n)\, d^2(y_n,p) + \alpha_n d^2(y_{n+1},p)$$

and $d^2(y_{n+1},p) < d^2(y_n,p)$.
Consequently, Case 2 could not happen.
It remains that $d^2(y_{n+1},p) \le d^2(y_n,p)$ for each $n \in N$.
As a result, the sequence $\{d^2(y_{n+1},p)\}_{n\in N}$ is monoton nonincreasing and bounded below by zero. So, it converges to $r \ge 0$.
Suppose that $r > 0$ and we have:
$r \le d^2(y_{n+1},p) \le \frac{\psi\left(d^2(y_n,p)\right\}}{\lambda(y_n,p)} \le \psi\left(d^2(y_n,p)\right)$ and $r \le \psi(d^2(y_n,p))$.
Taking the limit of both sides, we take $r \le \psi(r) < r$, which is a contradiction. It yields that $r = 0$.
In other hand, we see that $d^2(Ty_n,p) \le \frac{\psi\left(d^2(y_n,p)\right\}}{\lambda(y_n,p)} \le \psi\left(d^2(y_n,p)\right) < d^2(y_n,p)$.
So $d(p,Ty_n) \le d(p,y_n)$ and $d(y_n,Ty_n) \le d(y_n,p) + d(p,Ty_n) \le 2d(y_n,p)$.
Consequently, $\lim_{n\to+\infty} d(y_n,Ty_n) = 0$. ■

Theorem 3.3. *Let (X,d) be a complete CAT(0) space and $\mathcal{K}$ a nonempty, closed convex subset of X and $T : K \to K$ be a (λ,ψ)- partial hybrid function. Suppose that there exists a point $x \in X$ such that $\{T^n x\}$ is bounded. Let $y_0 \in K$, and define the sequence $\{y_n\}_{n\in N}$ which satisfies the Mann's iteration $y_{n+1} = (1-\alpha_n)y_n \bigoplus \alpha_n Ty_n$, where $\alpha_n \in [0,1]$. Then the sequence $\{y_n\}_{n\in N}$ is Δ-convergent to $p \in F(T)$.*

Remark 3.1. The proof of Theorem 3.5 is analogous to the proof of Lemma 4.9 and Theorem 4.10 in [16] for (λ,ψ)- partial hybrid functions.

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Mixtures models for clustering: review and comparison

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Abstract: The concepts of machine learning and artificial intelligence were first mentioned back in 1956. Since then, artificial intelligence has constantly been evolving but has only reached its peak in the last decade. Machine learning is applied in medicine, online technology, marketing, sales, logistics, and many others. In clustering, the aim is to find hidden relationships in the data. This type of machine learning can be divided into many different groups of methods. The aim of this work is to briefly discuss the methods of clustering mixtures, provide a comparison of these methods using different data, and compare them with the currently most popular clustering methods. We present k-means, Gaussian Mixture Model, Bayesian Gaussian Mixture Model, and Modified Inversion Formula clustering in work. In this work, the different clustering methods were briefly reviewed, and a modified inversion formula based on the new clustering method currently being developed was mentioned. The results showed that the best methods are also different for different data sets. Therefore, neither method is universal and unsuitable for all data sets. These results also showed that the newly developed data clustering method has relatively good clustering results. For this reason, the results of this method continue to be validated, and new modifications of this method are being developed.

Keywords: Machine learning, artificial intelligence, clustering, mixture models, inversion formula.

1 Introduction

The concepts of machine learning and artificial intelligence were first mentioned back in 1956. Since then, artificial intelligence has constantly been evolving but has only reached its peak in the last decade. Artificial intelligence and its individual fields are now evolving more than ever due to the wide range of options available, such as freely available sources, which are far more computing. They are receiving a great deal of attention not only in science but also in practice. Artificial intelligence's growing popularity is significant because various processes can be performed faster and often with higher quality using these technologies. Using these technologies eliminates the potential human error from the process. Also, using these methods is much easier to automate various monotonous processes, which allows you to avoid a lot of tedious work. Artificial intelligence, and if more precisely one of its fields, machine learning, is widely used in various practice fields.

Machine learning is applied in areas such as medicine [1, 2], online technology [3, 4], marketing [5, 6], sales [7], logistics [8] and many others. Machine learning can be used to solve various tasks: image analysis, text analysis, analysis of structured data, reference systems, and other tasks. Different types of machine learning are used to address these challenges: supervised, unsupervised, or motivational learning. The fourth type of machine learning is also distinguished - semi-supervised machine learning. Still, this type is much less common compared to others. In the case of supervised learning, we have a pair of data in the initial data set that is the predictive variable and the predictive variables. The main task, in this case, is to find a function that matches this data as much as possible and allows predictions to be made later. In the case of stimulating machine learning, the agent receives information about the goal accomplished and possessed and seeks to maximize the function of the goal. To achieve this, the agent must learn about the environment and how to respond to different situations. The application of these models is most often seen in autonomous cars, robots, and other similar fields. Finally, the third type is unsupervised learning. In the case of this learning, the purpose is not known in advance, which means we have no labels in the data. In the case of this learning, the aim is to find hidden relationships in the data. This type of machine learning can be divided into many different groups of methods, the, data reduction methods, and clustering. In this work, the last type is the most important. The aim of this work is to briefly discuss the methods of clustering mixtures, provide a comparison of these methods using different data, and compare them with the currently most popular clustering methods.

The first part of this work briefly reviews the main clustering methods used in work, and their operation. Then, in the second part of the work, a part of the clustering results obtained in the research is reviewed, and a discussion of these results is presented.

2 Clustering Methods Review

As mentioned earlier, finding hidden connections between different observations is sometimes significant. Dividing all observations into certain groups/clusters makes it much easier to interpret such results, as it is possible to create cluster profiles. According to Jain, the main clustering of everything is to group different observations into groups. Dividing observations into groups aims to form groups that are homogeneous and as distinct as possible from other observations. Clustering can, in fact, be divided into separate groups with different functioning. One of the most commonly used clustering methods in practice today is the k-means method [9, 10]. The results of the K-means method are calculated as follows. Suppose we have a data matrix $X = [X_1, X_2, X_3, \ldots, X_n]$ and aim to assign these points to clusters $C = [C_1, C_2, \ldots, C_c]$. Cluster centers are then randomly selected. In the second step of the clustering algorithm, the distance from the cluster centers to each point is calculated using the selected distance. Different distances are used, such as Euclidean, Manhattan, Chebyshev, and other distances [11]. Depending on the distance selected, clusters are also used differently. Each point is then assigned according to the smallest distance to the cluster's center that was calculated in the second step of the algorithm. Finally, new cluster centers are calculated based on the assigned points for each cluster, and the entire cycle is repeated until stable clusters are established. In this work, more attention is paid to mixture clustering methods; one of the mixture clustering methods is the Gaussian mixture method (**GMM**) [12, 13]. This method uses normal distributions and aims to cluster the data based on a mixture of different normal distributions. Suppose we have a C cluster. Many practical application algorithms use a certain number of clusters determined experimentally. An average matrix is estimated for each cluster μ_c and covariance matrices Σ_c. It would be estimated using the maximum likelihood method for only one distribution. However, since the number of clusters is larger and the number of clusters is C, the density is defined as a linear function of the density of all these C distributions.

$$p(X) = \sum_{c=1}^{C} \pi_c N \left(X | \mu_c, \Sigma_c \right) \tag{2.1}$$

where π_c is the coefficient of the different distributions. In the case of the Gaussian mixture method, the EM algorithm is usually used. The EM algorithm is an iterative method that estimates the parameters of the model being constructed. In the case of the EM algorithm, the density of the data is calculated. Next, these data are divided into different clusters, and finally, the averages and covariance matrices are calculated. This method, for example, has different modifications in the scikit-learn Python library, and the Bayesian Gaussian Mixture Method (**BGMM**) is presented in addition to the usual one [14]. Also, these methods use different initialization procedures, which may allow different clusters to be obtained. The aforementioned k-means method is usually used for initialization. A new data clustering method based on a modified inversion formula (**MIDE**) is currently being developed [15]. This method also uses an EM algorithm that allows parameters to be estimated. The method initializes a T matrix consisting of points on a unit sphere. This matrix is then used for further density calculation.

$$\widehat{\psi}_\tau (u) = \sum_{k=1}^{\hat{q}_\tau} \hat{p}_{k,\tau} e^{iu\hat{m}_{k,\tau} - u^2 \widehat{\sigma}^2_{k,\tau}/2} + \hat{p}_{0,\tau} \frac{2}{(b-a)\,u} \sin \frac{(b-a)\,u}{2} \ .e^{\frac{iu(a+b)}{2}} \tag{2.2}$$

3 Results

This section provides a brief overview of the results obtained experimentally. The evaluation of the results of data clustering is reviewed first. Different data sets are used in the work, and finally, the main results. This work uses accuracy metrics to evaluate different clustering methods, indicating how many data points have been assigned to the correct cluster. This metric can be used because the data used has predefined clusters. Other metrics should be used in other cases where clusters are not known in advance. Different data sets are used in this work to assess the quality of clustering of different methods. The following is a table showing the main results using different data sets.

Table *1.* Results based on the 10 000 runs for different datasets

Dataset	K-means (Euclidean)	K-means (Manhattan)	GMM	BGMM	MIDEv1	MIDEv2
Aggregation	0.857	0.789	0.835	**0.907**	0.889	0.895
Gaussians1	1.000	1.000	1.000	1.000	1.000	**1.000**
CPU	0.738	0.698	0.574	0.590	0.808	**0.828**
Diabetes	0.356	0.314	0.419	0.439	0.420	**0.448**
Iris	0.831	0.745	0.953	0.838	0.933	**0.955**
Wine	0.966	0.923	0.953	**0.977**	0.943	0.953

4 Conclusion

In this work, the different clustering methods were briefly reviewed, and a modified inversion formula based on the new clustering method currently being developed was mentioned. The results showed that the best methods are also different for different data sets. Neither method is universal and is not suitable for all data sets. These results also showed that the newly developed data clustering method has relatively good clustering results. For this reason, the results of this method continue to be validated, and new modifications of this method are being developed.

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Dynamical analysis and solutions of nonlinear difference equations of twenty-fourth order

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Abstract: This paper discusses the behaviors and solutions of some rational recursive relations of twenty-fourth order using the iteration technique and the modulus operator. The stability of the equilibrium points are comprehensively analyzed. We also present other properties such as periodicity, oscillation and bounded solutions. Some numerical examples are obviously given to ensure the validity of the theoretical work. These examples are plotted using MATLAB. The proposed techniques can be utilized to be applied on other nonlinear equations.
Keywords: equilibrium, asymptotic stability, periodicity, exact and numerical solutions.

1 Introduction

The evolution of a certain natural phenomena is often described over a course of time using differential equations. However, many real life problems can be sometimes modeled using discrete time steps which lead to difference equations. Hence, recursive equations have an effective and powerful role in mathematics. They are successfully utilized to investigate some applications in engineering, physics, biology, economic and others. For instance, recursive equations have been well used in modeling some natural phenomena such as the size of a population, the Fibonacci sequence, the drug in the blood system, the transmission of information, the pricing of a certain commodity, the propagation of annual plants, and others [1]. In addition, some scholars have used difference equations to find the numerical solutions of some differential equations. More specifically, discretizing a given differential equation gives a difference equation. For example, Runge-Kutta scheme is obtained from discretizing a first order differential equation. The development of technology has motivated the use of recurrence equations as approximations to partial differential equations. It is worth mentioning that fractional order difference equations are often utilized to investigate some real life phenomena emerging in nonlinear sciences.
Most properties of recursive expressions have been widely discussed by some researchers. For example, researchers have explored the stability, periodicity, boundedness and solutions of some recursive equations. We here present some published works. Alayachi et al. [2] analyzed the local and global attractivity, periodicity and the solutions of a sixth order difference equation. Some numerical examples have been also presented in [2]. In [3], Sanbo and Elsayed presented the periodicity, stability and some solutions of a fifth order recursive equation. Almatrafi and Alzubaidi [4] discussed the dynamical behaviors of an eighth order difference relation and showed some 2D figures for the obtained results. Moreover, Ahmed et al. [5], found new solutions and investigated the dynamical analysis for some nonlinear difference relations of fifteenth order. The authors in [6] obtained novel structures for the solutions of a rational recursive relation. The local and global stability, boundedness, periodicity and solutions of a second order difference equation were investigated in [7]. The authors proved that the local asymptotic stability of the equilibrium point implies global asymptotic stability. In [8], Kara and Yazlik expressed the solutions of a $(k + l)$-order recursive equation and investigated the asymptotic stability of the constructed results of the problem when $k = 3$, and $l = k$. Finally, Elsayed [9] analyzed the qualitative behaviors of a nonlinear recursive equation. More discussions about nonlinear recursive problems can be seen in refs. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

The motivation of writing this article arises from the investigation of fifteenth order difference equations given in [5]. We consider more sophisticated rational difference equations of twenty-fourth order. Therefore, this work aims to analyze some dynamical properties such as equilibrium points, local and global behaviors, boundedness, and analytic solutions of the nonlinear recursive equations

$$\chi_{r+1} = \frac{\chi_{r-23}}{\pm 1 \pm \prod_{i=0}^{5} \chi_{r-(4i+3)}}, \qquad r = 0, 1, 2, ...$$

Here, the initial values χ_{-23}, χ_{-22}, ..., χ_0 are arbitrary non-zero real numbers. In this work, we also illustrate some 2D figures with the help of MATLAB to validate the obtained results.

Definition 1.1. We consider $mod\,(\kappa, 4) = \kappa - 4\lfloor\frac{\kappa}{4}\rfloor$, where $\lfloor\Lambda\rfloor$ is the greatest integer less than or equal to the real number Λ.

2 Dynamical analysis of $\chi_{r+1} = \frac{\chi_{r-23}}{1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}$

In this section, we implement specific strategies to investigate the dynamical behaviors of the difference equation

$$\chi_{r+1} = \frac{\chi_{r-23}}{1 + \chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}, \qquad r = 0, 1, 2, ..., \tag{2.1}$$

with the initial conditions $\chi_{-\iota},\ \iota = 0, 1, 2, ..., 23$. More specifically, the attractivity and boundedness of the solutions of Eq. (2.1) are investigated.

Theorem 2.1. *Assume that* $\{\chi_r\}_{r=-23}^{\infty}$ *is a solution to Eq.* (2.1)*. Then, for* $r = 0, 1, 2, ...$, *we have*

$$\chi_{24r-\kappa} = \varepsilon_\kappa \prod_{i=0}^{r-1} \left(\frac{1 + (6i + \eta_\kappa - 1)\,\mu_\kappa}{1 + (6i + \eta_\kappa)\,\mu_\kappa} \right), \tag{2.2}$$

where $\mu_\kappa = \prod_{j=0}^{5} \varepsilon_{mod(\kappa,4)+4j}$, $\eta_\kappa = 6 - \left[\frac{\kappa}{4}\right]$, $\chi_{-k} = \varepsilon_\kappa,\ r\mu_\kappa \neq -1$, $r \in \{1, 2, 3, ...\}$, *and* $\kappa = 0, 1, 2, ..., 23$.

Proof The solutions are true at $r = 0$. Let $r > 0$ and suppose that the results are true at $r-1$, as follows:

$$\chi_{24r-24-\kappa} = \varepsilon_\kappa \prod_{i=0}^{r-2} \left(\frac{1 + (6i + \eta_\kappa - 1)\,\mu_\kappa}{1 + (6i + \eta_\kappa)\,\mu_\kappa} \right). \tag{2.3}$$

Using Eq. (2.1) and Eq. (2.3) gives

$$\begin{aligned}
\chi_{24r-23} &= \frac{\chi_{24r-47}}{1 + \chi_{24r-27}\chi_{24r-31}\chi_{24r-35}\chi_{24r-39}\chi_{24r-43}\chi_{24r-47}} \\
&= \frac{\varepsilon_{23} \prod_{i=0}^{r-2} \left(\frac{1+(6i+\eta_{23}-1)\mu_{23}}{1+(6i+\eta_{23})\mu_{23}} \right)}{1 + \prod_{j=0}^{5} \left(\varepsilon_{4j+3} \prod_{i=0}^{r-2} \left(\frac{1+(6i+\eta_{4j+3}-1)\mu_{4j+3}}{1+(6i+\eta_{4j+3})\mu_{4j+3}} \right) \right)} \\
&= \frac{\varepsilon_{23} \prod_{i=0}^{r-2} \left(\frac{1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{1+(6i+1)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}} \right)}{1 + \varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23} \prod_{i=0}^{r-2} \left(\frac{1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{1+(6i+6)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}} \right)} \\
&= \varepsilon_{23} \prod_{i=0}^{r-1} \frac{1 + (6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{1 + (6i+1)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}.
\end{aligned}$$

Furthermore, utilizing Eq. (2.1) and Eq. (2.3), we have

$$\begin{aligned}
\chi_{24r-22} &= \frac{\chi_{24r-46}}{1+\chi_{24r-26}\chi_{24r-30}\chi_{24r-34}\chi_{24r-38}\chi_{24r-42}\chi_{24r-46}} \\
&= \frac{\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{22}-1)\mu_{22}}{1+(6i+\eta_{22})\mu_{22}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{4j+2}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{4j+2}-1)\mu_{4j+2}}{1+(6i+\eta_{4j+2})\mu_{4j+2}}\right)\right)} \\
&= \frac{\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{1+(6i+1)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right)}{1+\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{1+(6i+6)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right)} \\
&= \varepsilon_{22}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{1+(6i+1)\,\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right).
\end{aligned}$$

Also, from Eq. (2.1) and Eq. (2.3), we have

$$\begin{aligned}
\chi_{24r-21} &= \frac{\chi_{24r-45}}{1+\chi_{24r-25}\chi_{24r-29}\chi_{24r-33}\chi_{24r-37}\chi_{24r-41}\chi_{24r-45}} \\
&= \frac{\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{21}-1)\mu_{21}}{1+(6i+\eta_{21})\mu_{21}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{4j+1}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{4j+1}-1)\mu_{4j+1}}{1+(6i+\eta_{4j+1})\mu_{4j+1}}\right)\right)} \\
&= \frac{\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{1+(6i+1)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right)}{1+\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{1+(6i+6)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right)} \\
&= \varepsilon_{21}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{1+(6i+1)\,\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right).
\end{aligned}$$

Finally, using Eq. (2.1) and Eq. (2.3) gives

$$\begin{aligned}
\chi_{24r-20} &= \frac{\chi_{24r-44}}{1+\chi_{24r-24}\chi_{24r-28}\chi_{24r-32}\chi_{24r-36}\chi_{24r-40}\chi_{24r-44}} \\
&= \frac{\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{20}-1)\mu_{20}}{1+(6i+\eta_{20})\mu_{20}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{4j}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{4j}-1)\mu_{4j}}{1+(6i+\eta_{4j})\mu_{4j}}\right)\right)} \\
&= \frac{\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{1+(6i+1)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right)}{1+\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{1+(6i+6)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right)} \\
&= \varepsilon_{20}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{1+(6i+1)\,\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right).
\end{aligned}$$

We similarly can obtain other relations of Eq. (2.2).

Theorem 2.2. *Suppose that* $\chi_{-23}, \chi_{-22}, ..., \chi_0 \in [0, \infty)$. *Then, every solution of Eq.* (2.1) *is bounded.*

Proof: Let $\{\chi_r\}_{r=-23}^{\infty}$ be a solution to Eq. (2.1). Then, from Eq. (2.1), one obtains

$$0 \leq \chi_{r+1} = \frac{\chi_{r-23}}{1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}} \leq \chi_{r-23}, \qquad \forall\ r \geq 0.$$

Thus, the sequence $\{\chi_{24r-i}\}_{r=0}^{\infty}$, $i = 0, 1, ..., 23$ is decreasing and is bounded from above by $\tau = \max\{\chi_{-23}, \chi_{-22}, ..., \chi_0\}$.

Theorem 2.3. *Equation* (2.1) *has a unique fixed point which is* $\overline{\chi} = 0$.

Proof: Using Eq. (2.1) gives

$$\overline{\chi} = \frac{\overline{\chi}}{1 + \overline{\chi}^6},$$

which leads to

$$\overline{\chi} + \overline{\chi}^7 = \overline{\chi}.$$

Hence, $\overline{\chi}^7 = 0$. This gives that $\overline{\chi} = 0$.

Theorem 2.4. *Let* $\chi_{-23}, \chi_{-22}, ..., \chi_0 \in [0, \infty)$. *Then, the fixed point* $\overline{\chi} = 0$ *of Eq.* (2.1) *is locally stable.*

Proof: Assume that $\epsilon > 0$, and let $\{\chi_\Omega\}_{\Omega=-23}^{\infty}$ be a solution to Eq. (2.1) with

$$\sum_{j=0}^{23} |\chi_{-j}| < \epsilon.$$

Now, it is sufficient to show that $|\chi_1| < \epsilon$. Note that

$$0 < \chi_1 = \frac{\chi_{-23}}{1 + \chi_{-3}\chi_{-7}\chi_{-11}\chi_{-15}\chi_{-19}\chi_{-23}} \leq \chi_{-23} < \epsilon.$$

The proof is done.

Theorem 2.5. *Let* $\chi_{-23}, \chi_{-22}, ..., \chi_0 \in [0, \infty)$. *Then, the fixed point* $\overline{\chi} = 0$ *of Eq.* (2.1) *is globally asymptotically stable.*

Proof: In Theorem 2.4, we showed that the fixed point $\overline{\chi} = 0$ is locally stable. Let $\{\chi_r\}_{r=-23}^{\infty}$ be a positive solution to Eq. (2.1). Then, it is needed to prove that $\lim_{r\to\infty} \chi_r = \overline{\chi} = 0$. Note that Theorem 2.2 gives $\chi_{r+1} < \chi_{r-23},\ \forall\, r \geq 0$. The sequences $\{\chi_{24r-i}\}_{r=0}^{\infty}$, $i = 0, 1, ..., 23$ are decreasing and bounded which means that the sequences $\{\chi_{24r-i}\}_{r=0}^{\infty}, i = 0, 1, ..., 23$ approach to a limit $Z_i \geq 0$. Hence,

$$Z_{23} = \frac{Z_{23}}{1 + Z_3 Z_7 Z_{11} Z_{15} Z_{19} Z_{23}}, Z_{22} = \frac{Z_{22}}{1 + Z_2 Z_6 Z_{10} Z_{14} Z_{18} Z_{22}}, ..., Z_0 = \frac{Z_0}{1 + Z_0 Z_4 Z_8 Z_{12} Z_{16} Z_{20}}.$$

This leads to $Z_0 = Z_1 = ... = Z_{23} = 0$.

3 Dynamical analysis of $\chi_{r+1} = \frac{\chi_{r-23}}{1-\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}$

This part investigates the qualitative behavior of the recursive equation

$$\chi_{r+1} = \frac{\chi_{r-23}}{1 - \chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}, \quad r = 0, 1, 2, ... \tag{3.1}$$

where the initial conditions $\chi_{-\iota},\ \iota = 0, 1, 2, ..., 23$, are real numbers.

Theorem 3.1. *Assume that* $\{\chi_r\}_{r=-23}^{\infty}$ *is a solution to Eq.* (3.1). *Then, for* $r = 0, 1, 2, ...$

$$\chi_{24r-\kappa} = \varepsilon_\kappa \prod_{i=0}^{r-1} \left(\frac{-1 + (6i + \eta_\kappa - 1)\, \mu_\kappa}{-1 + (6i + \eta_\kappa)\, \mu_\kappa} \right). \tag{3.2}$$

Here, $\mu_\kappa = \prod_{j=0}^{5} \varepsilon_{mod(\kappa,4)+4j}$, $\eta_\kappa = 6 - \left[\frac{\kappa}{4}\right]$, $\chi_{-\kappa} = \varepsilon_{k\kappa},\ \ r\mu_\kappa \neq 1,\ r \in \{1, 2, 3, ...\},$ *and* $\kappa = 0, 1, 2, ..., 23$.

Proof: Equation (3.2) is true for $r=0$. We let $r>0$ and suppose that the solutions are true at $r-1$. Hence,

$$\chi_{24r-24-\kappa}=\varepsilon_{\kappa}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{\kappa}-1)\,\mu_{\kappa}}{-1+(6i+\eta_{\kappa})\,\mu_{\kappa}}\right). \tag{3.3}$$

Utilizing Eq. (3.1) and Eq. (3.3) yields

$$\begin{aligned}\chi_{24r-23}&=\frac{\chi_{24r-47}}{1-\chi_{24r-27}\chi_{24r-31}\chi_{24r-35}\chi_{24r-39}\chi_{24r-43}\chi_{24r-47}}\\&=\frac{\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{23}-1)\mu_{23}}{-1+(6i+\eta_{23})\mu_{23}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{4j+3}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{4j+3}-1)\mu_{4j+3}}{-1+(6i+\eta_{4j+3})\mu_{4j+3}}\right)\right)}\\&=\frac{\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{-1+(6i+1)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}\right)}{1-\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{-1+(6i+6)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}\right)}\\&=\varepsilon_{23}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{-1+(6i+1)\,\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}\right).\end{aligned}$$

Furthermore, from Eq. (3.1) and Eq. (3.3), we have

$$\begin{aligned}\chi_{24r-22}&=\frac{\chi_{24r-46}}{1-\chi_{24r-26}\chi_{24r-30}\chi_{24r-34}\chi_{24r-38}\chi_{24r-42}\chi_{24r-46}}\\&=\frac{\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{22}-1)\mu_{22}}{-1+(6i+\eta_{22})\mu_{22}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{4j+2}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{4j+2}-1)\mu_{4j+2}}{-1+(6i+\eta_{4j+2})\mu_{4j+2}}\right)\right)}\\&=\frac{\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{-1+(6i+1)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right)}{1-\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{-1+(6i+6)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right)}\\&=\varepsilon_{22}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{-1+(6i+1)\,\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right).\end{aligned}$$

Moreover, from Eq. (3.1) and Eq. (3.3), we have

$$\begin{aligned}\chi_{24r-21}&=\frac{\chi_{24r-45}}{1-\chi_{24r-25}\chi_{24r-29}\chi_{24r-33}\chi_{24r-37}\chi_{24r-41}\chi_{24r-45}}\\&=\frac{\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{21}-1)\mu_{21}}{-1+(6i+\eta_{21})\mu_{21}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{4j+1}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{4j+1}-1)\mu_{4j+1}}{-1+(6i+\eta_{4j+1})\mu_{4j+1}}\right)\right)}\\&=\frac{\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{-1+(6i+1)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right)}{1-\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{-1+(6i+6)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right)}\\&=\varepsilon_{21}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{-1+(6i+1)\,\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right).\end{aligned}$$

Finally, Eq. (3.1) and Eq. (3.3) lead to

$$\begin{aligned}
\chi_{24r-20} &= \frac{\chi_{24r-44}}{1-\chi_{24r-24}\chi_{24r-28}\chi_{24r-32}\chi_{24r-36}\chi_{24r-40}\chi_{24r-44}} \\
&= \frac{\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{20}-1)\mu_{20}}{-1+(6i+\eta_{20})\mu_{20}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{4j}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{4j}-1)\mu_{4j}}{-1+(6i+\eta_{4j})\mu_{4j}}\right)\right)} \\
&= \frac{\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{-1+(6i+1)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right)}{1-\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{-1+(6i+6)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right)} \\
&= \varepsilon_{20}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{-1+(6i+1)\,\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right).
\end{aligned}$$

Other relations can be similarly done.

Theorem 3.2. *Equation* (3.1) *has a unique equilibrium point* $\overline{\chi}=0$, *which is non-hyperbolic.*

Proof: Equation (3.1) leads to

$$\overline{\chi}=\frac{\overline{\chi}}{1-\overline{\chi}^6},$$

from which we have

$$\overline{\chi}-\overline{\chi}^7=\overline{\chi}.$$

Hence, $\overline{\chi}^7=0$. As a result, $\overline{\chi}=0$. Next, we define a function

$$h\ (x_1,x_2,x_3,x_4,x_5,x_6)=\frac{x_1}{1-x_1x_2x_3x_4x_5x_6},$$

on I^6 where I is a subset of $\mathbb{R}$ such that $0\in I$ and $f(I^6)\subseteq I$. It is obvious that h is continuously differentiable on I^6. Thus,

$$\begin{aligned}
h_{x_1}(x_1,x_2,x_3,x_4,x_5,x_6) &= \frac{1}{(1-x_1x_2x_3x_4x_5x_6)^2}, \quad h_{x_2}(x_1,x_2,x_3,x_4,x_5,x_6)=\frac{x_1^2x_3x_4x_5x_6}{(1-x_1x_2x_3x_4x_5x_6)^2},\\
h_{x_3}(x_1,x_2,x_3,x_4,x_5,x_6) &= \frac{x_1^2x_2x_4x_5x_6}{(1-x_1x_2x_3x_4x_5x_6)^2}, \quad h_{x_4}(x_1,x_2,x_3,x_4,x_5,x_6)=\frac{x^2x_2x_3x_5x_6}{(1-x_1x_2x_3x_4x_5x_6)^2},\\
h_{x_5}(x_1,x_2,x_3,x_4,x_5,x_6) &= \frac{x_1^2x_2x_3x_4x_6}{(1-x_1x_2x_3x_4x_5x_6)^2}, \quad h_{x_6}(x_1,x_2,x_3,x_4,x_5,x_6)=\frac{x_1^2x_2x_3x_4x_5}{(1-x_1x_2x_3x_4x_5x_6)^2}.
\end{aligned}$$

Therefore,
$h_{x_1}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi})=1$, $h_{x_2}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi})=h_{x_3}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi})=h_{x_4}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi})=h_{x_5}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi})=h_{x_6}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi})=0$.
We now obtain the linearized equation of Eq. (3.1) about $\overline{\chi}=0$, which is given by

$$\chi_{\Omega+1}=\chi_{\Omega-23}, \tag{3.4}$$

whose characteristic equation is

$$\lambda^{24}-1=0.$$

This means that

$$|\lambda_i|=1,\quad i=1,2,...,24.$$

As a result, $\overline{\chi}$ is a non hyperbolic equilibrium point.

4 Dynamical analysis of $\chi_{r+1} = \frac{\chi_{r-23}}{-1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}$

This section is assigned to show the solutions of the difference problem

$$\chi_{r+1} = \frac{\chi_{r-23}}{-1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}, \quad r = 0, 1, 2, ..., \tag{4.1}$$

where $\chi_{-\iota},\ \iota = 0, 1, 2, ..., 23$ are real numbers. We also present the periodicity and the oscillation of the solutions.

Theorem 4.1. *Assume that $\{\chi_r\}_{r=-23}^{\infty}$ is a solution to Eq.* (4.1). *Then, for $r = 0, 1, 2, ...,$ we have*

$$\chi_{24r-\kappa} = \frac{\varepsilon_\kappa}{(-1+\mu_\kappa)^{r\alpha_\kappa}}. \tag{4.2}$$

Here, $\mu_\kappa = \prod_{j=0}^{5} \varepsilon_{mod(\kappa,4)+4j}$, $\alpha_\kappa = (-1)^{\left[\frac{\kappa}{4}\right]+1}$, $\chi_{-\kappa} = \varepsilon_\kappa$, $\mu_\kappa \neq 1$, and $\kappa = 0, 1, 2, ..., 23$.

Proof: Equation (4.2) is true for $r = 0$. Let $r > 0$ and assume that the results are true at $r-1$, as follows:

$$\chi_{24r-24-\kappa} = \frac{\varepsilon_\kappa}{(-1+\mu_\kappa)^{(r-1)\alpha_\kappa}}. \tag{4.3}$$

From Eq. (4.1) and Eq. (4.3), we have

$$\begin{aligned}
\chi_{24r-23} &= \frac{\chi_{24r-47}}{-1+\chi_{24r-27}\chi_{24r-31}\chi_{24r-35}\chi_{24r-39}\chi_{24r-43}\chi_{24r-47}} \\
&= \frac{\frac{\varepsilon_{23}}{(-1+\mu_{23})^{r-1}}}{-1+\varepsilon_3(-1+\mu_3)^{r-1}\frac{\varepsilon_7}{(-1+\mu_7)^{r-1}}\varepsilon_{11}(-1+\mu_{11})^{r-1}\frac{\varepsilon_{15}}{(-1+\mu_{15})^{r-1}}\varepsilon_{19}(-1+\mu_{19})^{r-1}\frac{\varepsilon_{23}}{(-1+\mu_{23})^{r-1}}} \\
&= \frac{\varepsilon_{23}}{(-1+\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})^{r-1}(-1+\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})} \\
&= \frac{\varepsilon_{23}}{(-1+\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})^{r}}.
\end{aligned}$$

We also obtain from Eq. (4.1) and Eq. (4.3) that

$$\begin{aligned}
\chi_{24r-22} &= \frac{\chi_{24r-46}}{-1+\chi_{24r-26}\chi_{24r-30}\chi_{24r-34}\chi_{24r-38}\chi_{24r-42}\chi_{24r-46}} \\
&= \frac{\frac{\varepsilon_{22}}{(-1+\mu_{22})^{r-1}}}{-1+\varepsilon_2(-1+\mu_2)^{r-1}\frac{\varepsilon_6}{(-1+\mu_6)^{r-1}}\varepsilon_{10}(-1+\mu_{10})^{r-1}\frac{\varepsilon_{14}}{(-1+\mu_{14})^{r-1}}\varepsilon_{18}(-1+\mu_{18})^{r-1}\frac{\varepsilon_{22}}{(-1+\mu_{22})^{r-1}}} \\
&= \frac{\varepsilon_{22}}{(-1+\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})^{r-1}(-1+\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})} \\
&= \frac{\varepsilon_{22}}{(-1+\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})^{r}}.
\end{aligned}$$

Furthermore, Eq. (4.1) and Eq. (4.3) give

$$\begin{aligned}
\chi_{24r-21} &= \frac{\chi_{24r-45}}{-1+\chi_{24r-25}\chi_{24r-29}\chi_{24r-33}\chi_{24r-37}\chi_{24r-41}\chi_{24r-45}} \\
&= \frac{\frac{\varepsilon_{21}}{(-1+\mu_{21})^{r-1}}}{-1+\varepsilon_1(-1+\mu_1)^{r-1}\frac{\varepsilon_5}{(-1+\mu_5)^{r-1}}\varepsilon_9(-1+\mu_9)^{r-1}\frac{\varepsilon_{13}}{(-1+\mu_{13})^{r-1}}\varepsilon_{17}(-1+\mu_{17})^{r-1}\frac{\varepsilon_{21}}{(-1+\mu_{21})^{r-1}}} \\
&= \frac{\varepsilon_{21}}{(-1+\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})^{r-1}(-1+\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})} \\
&= \frac{\varepsilon_{21}}{(-1+\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})^{r}}.
\end{aligned}$$

Finally, we use Eq. (4.1) and Eq. (4.3) to have

$$\begin{aligned}
\chi_{24r-20} &= \frac{\chi_{24r-44}}{-1+\chi_{24r-24}\chi_{24r-28}\chi_{24r-32}\chi_{24r-36}\chi_{24r-40}\chi_{24r-44}} \\
&= \frac{\frac{\varepsilon_{20}}{(-1+\mu_{20})^{r-1}}}{-1+\varepsilon_0\left(-1+\mu_0\right)^{r-1}\frac{\varepsilon_4}{(-1+\mu_4)^{r-1}}\varepsilon_8\left(-1+\mu_8\right)^{r-1}\frac{\varepsilon_{12}}{(-1+\mu_{12})^{r-1}}\varepsilon_{16}\left(-1+\mu_{16}\right)^{r-1}\frac{\varepsilon_{20}}{(-1+\mu_{20})^{r-1}}} \\
&= \frac{\varepsilon_{20}}{\left(-1+\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}\right)^{r-1}\left(-1+\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}\right)} \\
&= \frac{\varepsilon_{20}}{\left(-1+\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}\right)^{r}}.
\end{aligned}$$

One can similarly proved other relations.

Theorem 4.2. *Equation* (4.1) *has three equilibrium points* 0 *and* $\pm\sqrt[6]{2}$, *which are non-hyperbolic.*

Proof: It can be similarly done as the proof of Theorem 3.2.

Theorem 4.3. *Equation* (4.1) *is periodic of period* 24 *if and only if* $\mu_\kappa = 2$, *for* $\kappa = 0, 1, ..., 23$, *which take the following form:*

$$\chi_{24r-\kappa} = \varepsilon_\kappa,\ \ \kappa = 0, 1, ..., 23,\ \ \textit{and}\ \ r = 0, 1, 2, ...$$

Proof: The proof is done by using Theorem 4.1.

Theorem 4.4. *Let* $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{23} \in (0, 1)$. *Then, the solution* $\{\chi_\Omega\}_{\Omega=-23}^{\infty}$ *oscillates about the equilibrium point* $\overline{\chi} = 0$, *with positive semicycles of length* 24, *and negative semicycles of length* 24.

Proof: Using Theorem 4.1, we find $\chi_1, \chi_2, ..., \chi_{24} < 0$ and $\chi_{25}, \chi_{26}, ..., \chi_{48} > 0$. Thus, the result follows by induction.

5 Dynamical analysis of $\chi_{r+1} = \frac{\chi_{r-23}}{-1-\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}$

This section presents the periodicity, oscillation, and the solutions of the recursive problem

$$\chi_{r+1} = \frac{\chi_{r-23}}{-1-\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}},\ \ r = 0, 1, 2, ..., \tag{5.1}$$

where $\chi_{-\iota},\ \iota = 0, 1, 2, ..., 23$, are real numbers.

Theorem 5.1. *Let* $\{\chi_r\}_{r=-23}^{\infty}$ *be a solution to Eq.* (5.1). *Then, for* $r = 0, 1, 2, ...,$ *we have*

$$\chi_{24r-\kappa} = \frac{\varepsilon_\kappa}{\left(-1-\mu_\kappa\right)^{r\alpha_\kappa}}. \tag{5.2}$$

Here, $\mu_\kappa = \prod_{j=0}^{5}\varepsilon_{mod(\kappa,4)+4j}$, $\alpha_\kappa = (-1)^{\left[\frac{\kappa}{4}\right]+1}$ *and* $\chi_{-\kappa} = \varepsilon_\kappa$, *with* $\mu_\kappa \neq -1,\ \kappa = 0, 1, 2, ..., 23$.

Proof: The results are true at $r = 0$. We let $r > 0$ and suppose that the solutions are true at $r-1$ as follows:

$$\chi_{24r-24-\kappa} = \frac{\varepsilon_\kappa}{\left(-1-\mu_\kappa\right)^{(r-1)\alpha_\kappa}}. \tag{5.3}$$

Next, from Eq. (5.1) and Eq. (5.3), we obtain

$$\begin{aligned}
\chi_{24r-23} &= \frac{\chi_{24r-47}}{-1-\chi_{24r-27}\chi_{24r-31}\chi_{24r-35}\chi_{24r-39}\chi_{24r-43}\chi_{24r-47}} \\
&= \frac{\frac{\varepsilon_{23}}{(-1-\mu_{23})^{r-1}}}{-1-\varepsilon_3(-1-\mu_3)^{r-1}\frac{\varepsilon_7}{(-1-\mu_7)^{r-1}}\varepsilon_{11}(-1-\mu_{11})^{r-1}\frac{\varepsilon_{15}}{(-1-\mu_{15})^{r-1}}\varepsilon_{19}(-1-\mu_{19})^{r-1}\frac{\varepsilon_{23}}{(-1-\mu_{23})^{r-1}}} \\
&= \frac{\varepsilon_{23}}{(-1-\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})^{r-1}(-1-\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})} \\
&= \frac{\varepsilon_{23}}{(-1-\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})^{r}}.
\end{aligned}$$

Moreover, utilizing Eq. (5.1) and Eq. (5.3) leads to

$$\begin{aligned}
\chi_{24r-22} &= \frac{\chi_{24r-46}}{-1-\chi_{24r-26}\chi_{24r-30}\chi_{24r-34}\chi_{24r-38}\chi_{24r-42}\chi_{24r-46}} \\
&= \frac{\frac{\varepsilon_{22}}{(-1-\mu_{22})^{r-1}}}{-1-\varepsilon_2(-1-\mu_2)^{r-1}\frac{\varepsilon_6}{(-1-\mu_6)^{r-1}}\varepsilon_{10}(-1-\mu_{10})^{r-1}\frac{\varepsilon_{14}}{(-1-\mu_{14})^{r-1}}\varepsilon_{18}(-1-\mu_{18})^{r-1}\frac{\varepsilon_{22}}{(-1-\mu_{22})^{r-1}}} \\
&= \frac{\varepsilon_{22}}{(-1-\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})^{r-1}(-1-\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})} \\
&= \frac{\varepsilon_{22}}{(-1-\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})^{r}}.
\end{aligned}$$

We also use Eq. (5.1) and Eq. (5.3) to have

$$\begin{aligned}
\chi_{24r-21} &= \frac{\chi_{24r-45}}{-1-\chi_{24r-25}\chi_{24r-29}\chi_{24r-33}\chi_{24r-37}\chi_{24r-41}\chi_{24r-45}} \\
&= \frac{\frac{\varepsilon_{21}}{(-1-\mu_{21})^{r-1}}}{-1-\varepsilon_1(-1-\mu_1)^{r-1}\frac{\varepsilon_5}{(-1-\mu_5)^{r-1}}\varepsilon_{9}(-1-\mu_{9})^{r-1}\frac{\varepsilon_{13}}{(-1-\mu_{13})^{r-1}}\varepsilon_{17}(-1-\mu_{17})^{r-1}\frac{\varepsilon_{21}}{(-1-\mu_{21})^{r-1}}} \\
&= \frac{\varepsilon_{21}}{(-1-\varepsilon_1\varepsilon_5\varepsilon_{9}\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})^{r-1}(-1-\varepsilon_1\varepsilon_5\varepsilon_{9}\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})} \\
&= \frac{\varepsilon_{21}}{(-1-\varepsilon_1\varepsilon_5\varepsilon_{9}\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})^{r}}.
\end{aligned}$$

Finally, Eq. (5.1) and Eq. (5.3) give

$$\begin{aligned}
\chi_{24r-20} &= \frac{\chi_{24r-44}}{-1-\chi_{24r-24}\chi_{24r-28}\chi_{24r-32}\chi_{24r-36}\chi_{24r-40}\chi_{24r-44}} \\
&= \frac{\frac{\varepsilon_{20}}{(-1-\mu_{20})^{r-1}}}{-1-\varepsilon_0(-1-\mu_0)^{r-1}\frac{\varepsilon_4}{(-1-\mu_4)^{r-1}}\varepsilon_{8}(-1-\mu_{8})^{r-1}\frac{\varepsilon_{12}}{(-1-\mu_{12})^{r-1}}\varepsilon_{16}(-1-\mu_{16})^{r-1}\frac{\varepsilon_{20}}{(-1-\mu_{20})^{r-1}}} \\
&= \frac{\varepsilon_{20}}{(-1-\varepsilon_0\varepsilon_4\varepsilon_{8}\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})^{r-1}(-1-\varepsilon_0\varepsilon_4\varepsilon_{8}\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})} \\
&= \frac{\varepsilon_{20}}{(-1-\varepsilon_0\varepsilon_4\varepsilon_{8}\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})^{r}}.
\end{aligned}$$

Similarly, we can show other formulas.

Theorem 5.2. *Equation* (5.1) *has a unique equilibrium point* $\overline{\chi} = 0$*, which is non-hyperbolic.*

Proof: It can be similarly done as the proof of Theorem 3.2.

Theorem 5.3. *Equation* (5.1) *is periodic of period* 24 *if and only if* $\mu_\kappa = -2$, $\kappa = 0, 1, ..., 23$, *which have the form*

$$\chi_{24r-k} = \varepsilon_\kappa \qquad \kappa = 0, 1, ..., 23 \ \ \text{and} \ \ r = 0, 1, 2, ...$$

Proof: It can be easily done by using Theorem 5.1.

Theorem 5.4. *Let $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{23} \in (0, \infty)$. Then, the solution $\{\chi_r\}_{r=-23}^{\infty}$ oscillates about the equilibrium point $\overline{\chi} = 0$, with positive semicycles of length 24, and negative semicycles of length 24.*

Proof: Using Theorem 5.1, we obtain that $\chi_1, \chi_2, ..., \chi_{24} < 0$, and $\chi_{25}, \chi_{26}, ..., \chi_{48} > 0$. Using induction achieves the proof.

6 Numerical investigation

This part is added to guarantee that the constructed results are correct. We present some 2D example under specific initial conditions.

Example 6.1. *This example presents the behavior of Eq.* (2.1) *under the initial conditions $\chi_{-23} = 0.1$, $\chi_{-22} = 0.5$, $\chi_{-21} = 5.3$, $\chi_{-20} = 2$, $\chi_{-19} = 0.54$, $\chi_{-18} = 1.6$, $\chi_{-17} = 0.07$, $\chi_{-16} = 0.9$, $\chi_{-15} = 0.10$, $\chi_{-14} = 3$, $\chi_{-13} = 5$, $\chi_{-12} = 3$, $\chi_{-11} = 4$, $\chi_{-10} = 5$, $\chi_{-9} = 6.8$, $\chi_{-8} = 7.8$, $\chi_{-7} = 2.8$, $\chi_{-6} = 9$, $\chi_{-5} = 2.9$, $\chi_{-4} = 1.8$, $\chi_{-3} = 6.8$, $\chi_{-2} = 9.3$, $\chi_{-1} = 1.9$ and $\chi_0 = 9.8$, as shown in Figure 1 (left).*

Example 6.2. *Figure 1 (right) illustrates the dynamical behavior of Eq.* (3.1) *when $\chi_{-23} = 1$, $\chi_{-22} = 5$, $\chi_{-21} = 5.3$, $\chi_{-20} = 2$, $\chi_{-19} = 0.54$, $\chi_{-18} = 6$, $\chi_{-17} = 0.07$, $\chi_{-16} = 9$, $\chi_{-15} = 0.10$, $\chi_{-14} = 0.3$, $\chi_{-13} = 5$, $\chi_{-12} = 0.3$, $\chi_{-11} = 0.4$, $\chi_{-10} = 5$, $\chi_{-9} = 6.8$, $\chi_{-8} = 7$, $\chi_{-7} = 2$, $\chi_{-6} = 9.5$, $\chi_{-5} = 2.9$, $\chi_{-4} = 1$, $\chi_{-3} = 6.8$, $\chi_{-2} = 9$, $\chi_{-1} = 1$ and $\chi_0 = 9$.*

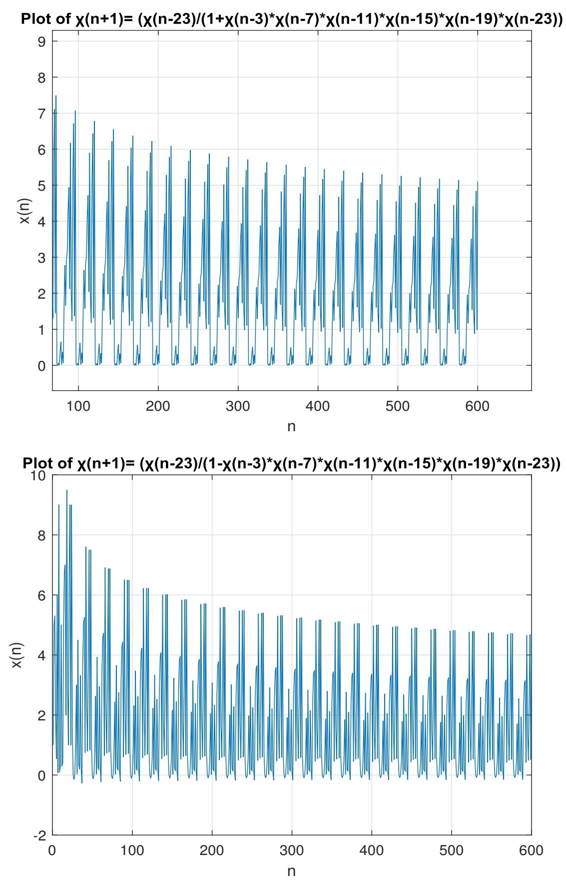

Figure 1: The left plot presents the behavior of Eq. (2.1) while the right picture depicts the behavior of Eq. (3.1).

Example 6.3. *The graph of the Eq.* (4.1) *is plotted in Figure 2 (left) under the random values* $\chi_{-23} = 0.1$, $\chi_{-22} = 0.5$, $\chi_{-21} = 0.15$, $\chi_{-20} = 0.2$, $\chi_{-19} = 0.3$, $\chi_{-18} = 0.2$, $\chi_{-17} = 0.9$, $\chi_{-16} = 0.11$, $\chi_{-15} = 0.6$, $\chi_{-14} = 0.51$, $\chi_{-13} = 0.14$, $\chi_{-12} = 0.5$, $\chi_{-11} = 0.2$, $\chi_{-10} = 0.8$, $\chi_{-9} = 0.5$, $\chi_{-8} = 0.1$, $\chi_{-7} = 0.3$, $\chi_{-6} = 0.7$, $\chi_{-5} = 0.1$, $\chi_{-4} = 0.88$, $\chi_{-3} = 0.12$, $\chi_{-2} = 0.89$, $\chi_{-1} = 0.33$ *and* $\chi_0 = 0.2$.

Example 6.4. *The dynamical behavior of Eq.* (5.1) *is shown in Figure 2 (right) when* $\chi_{-23} = 0.15$, $\chi_{-22} = 0.13$, $\chi_{-21} = 0.55$, $\chi_{-20} = 0.22$, $\chi_{-19} = 0.33$, $\chi_{-18} = 0.62$, $\chi_{-17} = 0.29$, $\chi_{-16} = 0.91$, $\chi_{-15} = 0.56$, $\chi_{-14} = 0.11$, $\chi_{-13} = 0.44$, $\chi_{-12} = 0.25$, $\chi_{-11} = 0.02$, $\chi_{-10} = 0.08$, $\chi_{-9} = 0.85$, $\chi_{-8} = 0.01$, $\chi_{-7} = 0.03$, $\chi_{-6} = 0.17$, $\chi_{-5} = 0.01$, $\chi_{-4} = 0.8$, $\chi_{-3} = 0.02$, $\chi_{-2} = 0.09$, $\chi_{-1} = 0.03$ *and* $\chi_0 = 0.62$.

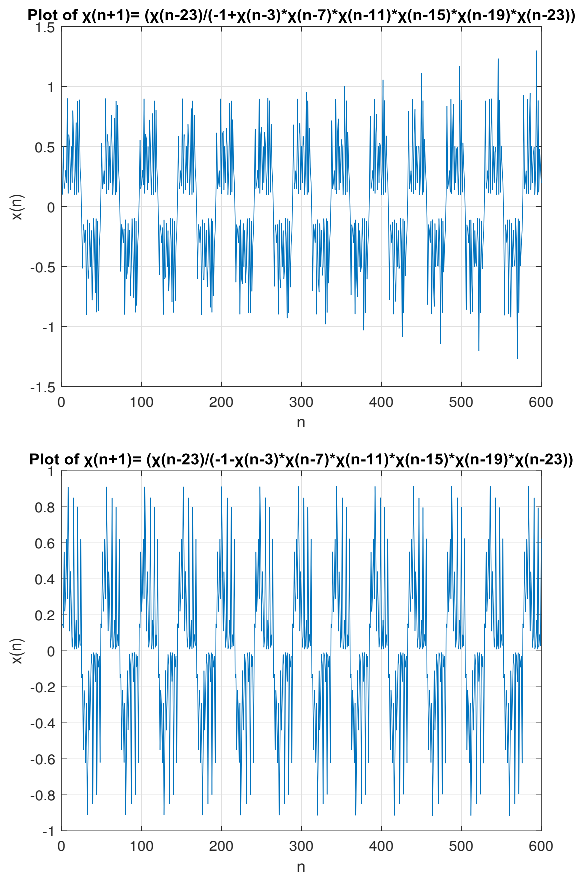

Figure 2: The left plot illustrates the periodicity of Eq. (4.1) while the right picture depicts the periodicity of Eq. (5.1).

7 Conclusion

To sum up, this paper has investigated four main rational difference equations of twenty-fourth order. We have introduced the solutions of the considered equations using modulus operator. In Theorem 2.1, we have presented and proved the solutions of Eq. (2.1), while Theorem 2.2 has shown the boundedness of the solutions of Eq. (2.1). It has been proved that the fixed point of Eq. (2.1) is globally stable. Theorem 4.3 has presented that Eq. (4.1) is periodic of period 24 if and only if $\mu_\kappa = 2$. Furthermore, in Theorem 5.1, we have explored the solutions of Eq. (5.1) which are periodic of period 24 if and only if $\mu_\kappa = -2$. We have also plotted the periodicity of Eq. (4.1) and Eq. (5.1) in Figures 2 (left) and 2 (right), respectively. Finally, the used approaches can be simply applied for other nonlinear equations.

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Solution of a Class of Nonlinear Pantograph Differential Equations

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Abstract: In this paper, we dealt with a class of nonlinear Pantograph Differential Equations with some initial condition. Since, there is not any analytic solution, we have applied the Pell Collocation Method as a numerical method.
Keywords: Nonlinear differential equations, Pantograph equations, Pell collocation method, Delay equations.

1 Introduction

The pantograph differential equations are a special class of the functional differential equations with proportional delay. These equations appear in the modeling of the problems encountered in many fields from economics to quantum mechanics, from nonlinear dynamical systems to electrodynamics. Unfortunately, it is not possible to find analytical solutions in many models presented with these equations. In that case, we need to use various numerical methods. In our work, we consider the numerical method "Pell Collocation Method(PCM)" for the approximate solution of a class of nonlinear Pantograph Differential Equations. Many researchers used this method for the approximations. In this paper, we consider a class of nonlinear Pantograph differential equation given by

$$\begin{aligned}
& \sum_{k=0}^{m}\sum_{r=0}^{n} R_{kr}(x)u^{r}\left(\alpha_{kr}x+\beta_{kr}\left(x\right)\right)u^{(k)}(\lambda_{kr}x+\gamma_{kr}\left(x\right)) \\
& +\sum_{k=1}^{m}\sum_{r=1}^{n} Q_{kr}(x)u^{(r)}\left(\alpha_{kr}x+\beta_{kr}\left(x\right)\right)u^{(k)}(\lambda_{kr}x+\gamma_{kr}\left(x\right)) \\
= \; & g(x), \text{ for } a\leq x\leq b
\end{aligned} \tag{1.1}$$

according to the following initial conditions

$$\sum_{k=0}^{m}\left[a_{jk}u^{(k)}(0)+b_{jk}u^{(k)}(0)\right]=\delta_{j}, \qquad j=0,1 \tag{1.2}$$

where $u^{(0)}(x)=u(x),\ u^{0}(x)=1$ and $u(x)$ is an unknown function. $R_{kr}(x)$, $Q_{kr}(x)$ and $g(x)$ are given continuous functions on interval $[0,1]$, a_{jk}, $a_{jk},\alpha_{kr},\ \lambda_{kr}$ and δ_j are suitable constants. Also $\beta_{kr}\left(x\right)$ and $\gamma_{kr}\left(x\right)$ are suitable constants or arbitrary variables. The propose of our work is to determine the approximate solution as the truncated Pell series given by

$$u\left(x\right)=\sum_{n=1}^{N+1} c_{n}P_{n}(x) \tag{1.3}$$

where $P_n(x)$ denotes the Pell polynomials; c_n $(1\leq n\leq N+1)$ are the unknown coefficients for Pell polynomial, and N is any positive integer which possess $N\geq m$.

2 Properties of Pell polynomials

The recurrence relation of those polynomials is defined by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x) \tag{2.1}$$

For $n \geqslant 3.$, $P_1(x) = 1,\ P_2(x) = 2x$. The properties were further investigated by Horadam, A. F. and Mahon, J. M.[2]. The first few Pell polynomials are

$$\begin{aligned} P_1(x) &= \mathbf{1}, \\ P_2(x) &= 2x, \\ P_3(x) &= \mathbf{4}x^2 + \mathbf{1}, \\ &\vdots \end{aligned} \tag{2.2}$$

3 Fundamental relations

Let us assume that linear combination of Pell polynomials (1.3) is an approximate solution of Eq (1.1). Our purpose is to determine the matrix forms of Eq (1.1) by using (1.3). Firstly,
i) In case of $\alpha_{kr} = \lambda_{kr} = 1$ and $\beta_{kr}(x) = \gamma_{kr}(x) = 0$, we can write Pell polynomials (2.2) in the matrix form

$$\mathbf{P}(x) = \mathbf{T}(x)\mathbf{M} \tag{3.1}$$

where $P(x) = [P_1(x)\ \ P_2(x) \cdots P_{N+1}(x)]$, $\mathbf{T}(x) = \left(1\ x\ x^2\ x^3 ... x^N\right)$, $\mathbf{C} = (c_1\ c_2\ \cdots c_{N+1})^T$ and

$$\mathbf{M} = \begin{bmatrix} \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \cdots \\ 0 & 2 & 0 & 4 & 0 & 6 & 0 & 8 & 0 & \cdots \\ 0 & 0 & \mathbf{4} & 0 & \mathbf{12} & 0 & \mathbf{24} & 0 & \mathbf{40} & \cdots \\ 0 & 0 & 0 & 8 & 0 & 32 & 0 & 80 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mathbf{16} & 0 & \mathbf{80} & 0 & \mathbf{240} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 32 & 0 & 192 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{64} & 0 & \mathbf{448} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 128 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{256} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The matrix form of (1.3) by a truncated Pell series is given by

$$u(x) = \mathbf{P}(x)\mathbf{C}. \tag{3.2}$$

By using (3.1) and (3.2), the matrix relation is expressed as

$$\begin{aligned} u(x) &\cong u_N(x) = \mathbf{T}(x)\mathbf{MC} \\ u'(x) &\cong u'_N(x) = \mathbf{TBMC} \\ u''(x) &\cong u''_N(x) = \mathbf{T}(x)\mathbf{B}^2\mathbf{MC} \\ &\quad ... \\ u^{(k)}(x) &\cong u_N^{(k)}(x) = \mathbf{T}(x)\mathbf{B}^k\mathbf{MC} \end{aligned} \tag{3.3}$$

Also, the relations between the matrix $\mathbf{T}(x)$ and its derivatives $\mathbf{T}'(x)$, $\mathbf{T}''(x)$,...,$\mathbf{T}^{(k)}(x)$ are

$$\begin{aligned} \mathbf{T}'(x) &= \mathbf{T}(x)\mathbf{B},\ \mathbf{T}''(x) = \mathbf{T}(x)\mathbf{B}^2 \\ \mathbf{T}'''(x) &= \mathbf{T}(x)\mathbf{B}^3, ..., \mathbf{T}^{(k)}(x) = \mathbf{T}(x)\mathbf{B}^k \end{aligned} \tag{3.4}$$

where

$$\mathbf{B}=\begin{bmatrix}0&1&0&0&0&0&\cdots&0\\0&0&2&0&0&0&\cdots&0\\0&0&0&3&0&0&\cdots&0\\0&0&0&0&4&0&\cdots&0\\0&0&0&0&0&5&\cdots&0\\0&0&0&0&0&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\ddots&N\\0&0&0&0&0&0&\cdots&0\end{bmatrix},\ \mathbf{B}^0=\begin{bmatrix}1&0&0&0&0&0&\cdots&0\\0&1&0&0&0&0&\cdots&0\\0&0&1&0&0&0&\cdots&0\\0&0&0&1&0&0&\cdots&0\\0&0&0&0&1&0&\cdots&0\\0&0&0&0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\ddots&0\\0&0&0&0&0&0&\cdots&1\end{bmatrix},\ \mathbf{T}_{1,0}=\begin{bmatrix}1&x_0&...&x_0^N\\1&x_1&...&x_1^N\\1&\vdots&...&\vdots\\1&x_N&...&x_N^N\end{bmatrix}$$

ii) In case of α_{kr} , λ_{kr} , $\beta_{kr}(x)$ and $\gamma_{kr}(x)$ are arbitrary constants or variables. Then we set the approximate solution defined by a truncated Pell series (1.3) in the matrix form

$$u(\lambda_{kr}x+\gamma_{kr}(x))\cong u_N(\lambda_{kr}x+\gamma_{kr}(x))=\mathbf{P}(\lambda_{kr}x+\gamma_{kr}(x))\,\mathbf{C}.$$

By using the relations (3.1) and (3.2), the matrix relation is expressed as

$$\begin{aligned}u(\lambda_{kr}x+\gamma_{kr}(x))&\cong u_N(\lambda_{kr}x+\gamma_{kr}(x))=\mathbf{P}(\lambda_{kr}x+\gamma_{kr}(x))\,\mathbf{C}=\mathbf{T}(\lambda_{kr}x+\gamma_{kr}(x))\,\mathbf{MC}\\u'(\lambda_{kr}x+\gamma_{kr}(x))&\cong u_N'(\lambda_{kr}x+\gamma_{kr}(x))=\mathbf{T}_{(\lambda,\ \gamma)}(x)\,\mathbf{BMC}\\u''(\lambda_{kr}x+\gamma_{kr}(x))&\cong u_N''(\lambda_{kr}x+\gamma_{kr}(x))=\mathbf{T}_{(\lambda,\ \gamma)}(x)\,\mathbf{B}^2\mathbf{MC}\\&\ \ \vdots\\u^{(k)}(\lambda_{kr}x+\gamma_{kr}(x))&\cong u_N^{(k)}(\lambda_{kr}x+\gamma_{kr}(x))=\mathbf{T}_{(\lambda,\ \gamma)}(x)\,\mathbf{B}^k\mathbf{MC}\end{aligned}\tag{3.5}$$

Also, the relations between the matrix $\mathbf{T}(\lambda_{kr}x+\gamma_{kr}(x))$ and its derivatives $\mathbf{T}'(\lambda_{kr}x+\gamma_{kr}(x))$, $\mathbf{T}''(\lambda_{kr}x+\gamma_{kr}(x))$,...,$\mathbf{T}^{(k)}(\lambda_{kr}x+\gamma_{kr}(x))$ are

$$\begin{aligned}\mathbf{T}'(\lambda_{kr}x+\gamma_{kr}(x))&=\mathbf{T}(\lambda_{kr}x+\gamma_{kr}(x))\,\mathbf{B},\ \mathbf{T}''(\lambda_{kr}x+\gamma_{kr}(x))=\mathbf{T}(\lambda_{kr}x+\gamma_{kr}(x))\,\mathbf{B}^2\\\mathbf{T}'''(\lambda_{kr}x+\gamma_{kr}(x))&=\mathbf{T}(\lambda_{kr}x+\gamma_{kr}(x))\,\mathbf{B}^3,...,\mathbf{T}^{(k)}(\lambda_{kr}x+\gamma_{kr}(x))=\mathbf{T}(\lambda_{kr}x+\gamma_{kr}(x))\,\mathbf{B}^k\end{aligned}\tag{3.6}$$

where

$$\mathbf{T}_{\lambda,\ \gamma}=\begin{bmatrix}\mathbf{T}(\lambda_{kr}x_0+\gamma_{kr}(x_0))\\\mathbf{T}(\lambda_{kr}x_1+\gamma_{kr}(x_1))\\\vdots\\\mathbf{T}(\lambda_{kr}x_N+\gamma_{kr}(x_N))\end{bmatrix}=\begin{bmatrix}1&\lambda_{kr}x_0+\gamma_{kr}(x_0)&...&(\lambda_{kr}x_0+\gamma_{kr}(x_0))^N\\1&\lambda_{kr}x_1+\gamma_{kr}(x_1)&...&(\lambda_{kr}x_1+\gamma_{kr}(x_1))^N\\1&\vdots&...&\vdots\\1&\lambda_{kr}x_N+\gamma_{kr}(x_N)&...&(\lambda_{kr}x_N+\gamma_{kr}(x_N))^N\end{bmatrix}$$

By using (3.5) and (3.6), we have the matrix relation

$$u^{(k)}(\lambda_{kr}x+\gamma_{kr}(x))=\mathbf{T}(\lambda_{kr}x+\gamma_{kr}(x))\,\mathbf{B}^k\mathbf{MC}.\tag{3.7}$$

By substituting the Pell collocation points given by

$$x_i=a+\frac{(b-a)\,i}{N},\ i=0,1,...N\tag{3.8}$$

into Eq(3.7), we obtain

$$u^{(k)}(\lambda_{kr}x_i+\gamma_{kr}(x_i))=\mathbf{T}_{\lambda,\ \gamma}(x_i)\,\mathbf{B}^k\mathbf{MC},\ \ k=0,1,...,m\tag{3.9}$$

and the compact form of the relation (3.9) becomes

$$\mathbf{U}^{(k)}=\mathbf{T}_{\lambda,\ \gamma}\mathbf{B}^k\mathbf{MC},\ \ k=0,1,...,m\tag{3.10}$$

where

$$\mathbf{U}^{(k)}=\begin{bmatrix} u^{(k)}\left(\lambda_{kr}x_0+\gamma_{kr}\left(x_0\right)\right)\\ u^{(k)}\left(\lambda_{kr}x_1+\gamma_{kr}\left(x_1\right)\right)\\ \vdots\\ u^{(k)}\left(\lambda_{kr}x_N+\gamma_{kr}\left(x_N\right)\right)\end{bmatrix}.$$

In addition, we can obtain the matrix forms $\left(\hat{\mathbf{U}}\right)^r\ \mathbf{U}^{(k)}$and $\left(\hat{\mathbf{U}}\right)^{(r)}\ \mathbf{U}^{(k)}$ which appears in the nonlinear part of Eq. (1.1), by using Eq. (3.3) as

$$\begin{aligned}\left(\hat{\mathbf{U}}\right)^r\ \mathbf{U}^{(k)} &= \begin{bmatrix} u^r\left(\alpha_{kr}x_0+\beta_{kr}\left(x_0\right)\right)u^{(k)}\left(\lambda_{kr}x_0+\gamma_{kr}\left(x_0\right)\right)\\ u^r\left(\alpha_{kr}x_1+\beta_{kr}\left(x_1\right)\right)u^{(k)}\left(\lambda_{kr}x_1+\gamma_{kr}\left(x_1\right)\right)\\ \vdots\\ u^r\left(\alpha_{kr}x_N+\beta_{kr}\left(x_N\right)\right)u^{(k)}\left(\lambda_{kr}x_N+\gamma_{kr}\left(x_N\right)\right)\end{bmatrix} \qquad (3.11)\\ &= \begin{bmatrix} u^r\left(\alpha_{kr}x_0+\beta_{kr}\left(x_0\right)\right) & 0 & \dots & 0\\ 0 & u^r\left(\alpha_{kr}x_1+\beta_{kr}\left(x_1\right)\right) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u^r\left(\alpha_{kr}x_N+\beta_{kr}\left(x_N\right)\right)\end{bmatrix}\\ &\times\begin{bmatrix} u^{(k)}\left(\lambda_{kr}x_0+\gamma_{kr}\left(x_0\right)\right)\\ u^{(k)}\left(\lambda_{kr}x_1+\gamma_{kr}\left(x_1\right)\right)\\ \vdots\\ u^{(k)}\left(\lambda_{kr}x_N+\gamma_{kr}\left(x_N\right)\right)\end{bmatrix}\\ \left(\hat{\mathbf{U}}\right)^{(r)}\ \mathbf{U}^{(k)} &= \begin{bmatrix} u^{(r)}\left(\alpha_{kr}x_0+\beta_{kr}\left(x_0\right)\right)u^{(k)}\left(\lambda_{kr}x_0+\gamma_{kr}\left(x_0\right)\right)\\ u^{(r)}\left(\alpha_{kr}x_1+\beta_{kr}\left(x_1\right)\right)u^{(k)}\left(\lambda_{kr}x_1+\gamma_{kr}\left(x_1\right)\right)\\ \vdots\\ u^{(r)}\left(\alpha_{kr}x_N+\beta_{kr}\left(x_N\right)\right)u^{(k)}\left(\lambda_{kr}x_N+\gamma_{kr}\left(x_N\right)\right)\end{bmatrix}\\ &= \begin{bmatrix} u^{(r)}\left(\alpha_{kr}x_0+\beta_{kr}\left(x_0\right)\right) & 0 & \dots & 0\\ 0 & u^{(r)}\left(\alpha_{kr}x_1+\beta_{kr}\left(x_1\right)\right) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u^{(r)}\left(\alpha_{kr}x_N+\beta_{kr}\left(x_N\right)\right)\end{bmatrix}\\ &\times\begin{bmatrix} u^{(k)}\left(\lambda_{kr}x_0+\gamma_{kr}\left(x_0\right)\right)\\ u^{(k)}\left(\lambda_{kr}x_1+\gamma_{kr}\left(x_1\right)\right)\\ \vdots\\ u^{(k)}\left(\lambda_{kr}x_N+\gamma_{kr}\left(x_N\right)\right)\end{bmatrix}\end{aligned}$$

where

$$\hat{\mathbf{U}}=\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}} \text{ and } \left(\hat{\mathbf{U}}\right)^{(r)}=\hat{\mathbf{T}}\left(\hat{\mathbf{B}}\right)^r\ \hat{\mathbf{M}}\ \hat{\mathbf{C}} \qquad (3.12)$$

$$\hat{\mathbf{T}}_{\lambda,\gamma} = \begin{bmatrix} \mathbf{T}\left(\lambda_{kr}x_0+\gamma_{kr}\left(x_0\right)\right) & 0 & \dots & 0\\ 0 & \mathbf{T}\left(\lambda_{kr}x_1+\gamma_{kr}\left(x_1\right)\right) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{T}\left(\lambda_{kr}x_N+\gamma_{kr}\left(x_N\right)\right)\end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \dots & 0\\ 0 & \mathbf{B} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{B}\end{bmatrix},$$

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & 0 & ... & 0 \\ 0 & \mathbf{M} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \mathbf{M} \end{bmatrix}, \quad \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & 0 & ... & 0 \\ 0 & \mathbf{C} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \mathbf{C} \end{bmatrix}.$$

Substituting the collocation points ($x_i = a + (b-a)i/N, i = 0, 1, \cdots, N$) into Eq. (3.11), gives the system of equations

$$\begin{aligned} & \sum_{k=0}^{m}\sum_{r=0}^{n} R_{kr}(x_i)u^r\left(\alpha_{kr}x_i + \beta_{kr}(x_i)\right)u^{(k)}(\lambda_{kr}x_i + \gamma_{kr}(x_i)) \\ & + \sum_{k=1}^{m}\sum_{r=1}^{n} Q_{kr}(x_i)u^{(r)}\left(\alpha_{kr}x_i + \beta_{kr}(x_i)\right)u^{(k)}(\lambda_{kr}x_i + \gamma_{kr}(x_i)) \\ = \; & g(x_i), \end{aligned}$$

which can be expressed with the aid of Eqs. (3.9) and (3.11) as

$$\sum_{k=0}^{m}\sum_{r=0}^{n}\mathbf{R}_{kr}\left(\hat{\mathbf{U}}\right)^{r}\ \mathbf{U}^{(k)} + \sum_{k=1}^{m}\sum_{r=1}^{n}\mathbf{Q}_{kr}\left(\hat{\mathbf{U}}\right)^{(r)}\ \mathbf{U}^{(k)} = \mathbf{G} \tag{3.13}$$

where

$$\begin{aligned} \mathbf{R}_{kr} &= diag\left[R_{kr}(x_0)\ \ R_{kr}(x_1)\ \ ...\ \ R_{kr}(x_N)\right], \\ \mathbf{Q}_{kr} &= diag\left[Q_{kr}(x_0)\ \ Q_{kr}(x_1)\ \ ...\ \ Q_{kr}(x_N)\right] \\ \text{and}\ \ \mathbf{G} &= \left[\ g(x_0)\quad g(x_1)\quad ...\quad g(x_N)\ \right]^T. \end{aligned}$$

By substituting the relations (3.10) and (3.12) into Eq. (3.13), the fundamental matrix equation is attained as

$$\left\{\sum_{k=0}^{m}\sum_{r=0}^{n}\mathbf{R}_{kr}\left(\hat{\mathbf{T}}_{\alpha,\beta}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}\ \right)^{r}\ \mathbf{T}_{\lambda,\gamma}\mathbf{B}^{k}\mathbf{M} + \sum_{k=1}^{m}\sum_{r=1}^{n}\mathbf{Q}_{kr}\hat{\mathbf{T}}_{\alpha,\beta}\ \left(\hat{\mathbf{B}}\right)^{r}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}\ \ \mathbf{T}_{\lambda,\gamma}\mathbf{B}^{k}\mathbf{M}\right\}\mathbf{C} = \mathbf{G} \tag{3.14}$$

Briefly, Eq. (3.14) can also be shown as,

$$\mathbf{WC} = \mathbf{G} \quad \text{or}\ \ [\mathbf{W};\mathbf{G}] \tag{3.15}$$

where

$$\mathbf{W} = \sum_{k=0}^{m}\sum_{r=0}^{n}\mathbf{R}_{kr}\left(\hat{\mathbf{T}}_{\alpha,\beta}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}\ \right)^{r}\ \mathbf{T}_{\lambda,\gamma}\mathbf{B}^{k}\mathbf{M} + \sum_{k=1}^{m}\sum_{r=1}^{n}\mathbf{Q}_{kr}\hat{\mathbf{T}}_{\alpha,\beta}\ \left(\hat{\mathbf{B}}\right)^{r}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}\ \ \mathbf{T}_{\lambda,\gamma}\mathbf{B}^{k}\mathbf{M}.$$

Here, Eq. (3.15) is a system containing $(N+1)$ nonlinear algebraic equations with the $(N+1)$ unknown Pell coefficients. Using Eq. (3.10) at the points a and b, the matrix representation of the conditions in Eq. (1.2) is given by

$$\left\{\sum_{k=0}^{m-1}\left[a_{jk}\mathbf{T}(0) + b_{jk}\mathbf{T}(0)\right](\mathbf{B})^{(k)}\mathbf{M}\right\}\mathbf{C} = \delta_j,\ \ j = 0, 1, 2, ..., m-1$$

or we can write as

$$\mathbf{V}_j\ \mathbf{C} = [\delta_j] \qquad \text{or} \qquad [\mathbf{V}_j;\delta_j]\,; \quad j = 0, 1, 2, ..., m-1 \tag{3.16}$$

Here

$$\mathbf{V}_j = \sum_{k=0}^{m-1}\left[a_{jk}\mathbf{T}(0) + b_{jk}\mathbf{T}(0)\right](\mathbf{B})^{(k)}\mathbf{M} = \left[v_{j0}\ \ v_{j1}\ \ v_{j2}\ ...\ v_{jN}\right].$$

Table 1: Numerical results of the error function E_N at the different values of N for *Example 2*

x	$E_8(MDTM)$	$E_{11}(MDTM)$	E_8	E_{11}
0.1	3.56812×10^{-12}	3.59265×10^{-12}	2.21233×10^{-9}	3.60961×10^{-13}
0.3	4.67857×10^{-10}	4.52835×10^{-12}	6.10001×10^{-9}	9.75359×10^{-13}
0.5	4.58254×10^{-8}	5.85598×10^{-11}	7.06047×10^{-9}	1.14569×10^{-12}
0.7	9.28161×10^{-7}	2.78563×10^{-10}	5.60762×10^{-9}	1.11228×10^{-12}
0.9	8.72761×10^{-6}	6.64594×10^{-9}	8.19878×10^{-10}	1.40254×10^{-12}

Therefore, by replacing the condition matrices in (3.16) by the m rows of the augmented matrix (3.15), the new augmented matrix will be

$$\left[\hat{\mathbf{W}};\hat{\mathbf{G}}\right] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & ; & g(x_1) \\ w_{20} & w_{21} & w_{22} & \cdots & w_{2N} & ; & g(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{(N-m)0} & w_{(N-m)1} & w_{(N-m)2} & \cdots & w_{(N-m)N} & ; & g(x_{N-m}) \\ v_{00} & v_{01} & v_{02} & \cdots & v_{0N} & ; & \delta_0 \\ v_{10} & v_{11} & v_{12} & \cdots & v_{1N} & ; & \delta_1 \\ v_{20} & v_{21} & v_{22} & \cdots & v_{2N} & ; & \delta_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ v_{(m-1)0} & v_{(m-1)1} & v_{(m-1)2} & \cdots & v_{(m-1)N} & ; & \delta_{m-1} \end{bmatrix} \tag{3.17}$$

In this way, the unknown Pell coefficients $c_n,\ n = 1, 2, ..., N + 1$ are obtained by solving the system in (3.17). Then, these coefficients are substituted into (1.3), and the approximate solution is obtained.

4 Illustrative example

In this section, a numerical example is presented to illustrate the efficient of the proposed method. On this problem, the method is tested by using the error function. The obtained numerical results are presented with table and graphic.

Example 1. Assume that the following differential equation

$$u''(x) - u(x) + \frac{8}{x^2}u^2\left(\frac{x}{2}\right) = 0; \quad u(0) = 0, u'(0) = 1 \tag{4.1}$$

The exact solution of Eq.(4.1) is given by $u(x) = xe^{-x}$. Table 1 presents values of error function and a numerical comparison of proposed method with modified differential transform method (MDTM) when $N = 8, 11$. In Figure 1, it is presented that graphical comparison of approximate and exact solutions obtained by the proposed method for $N = 3$.

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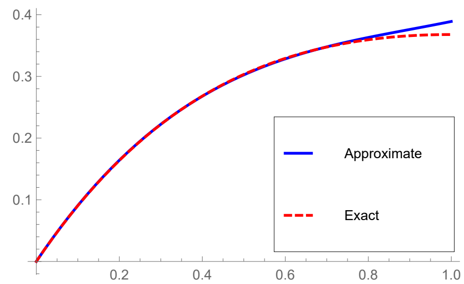

(a) $N = 3$

Figure 1: Graphical comparison of the exact and approximate solutions when $N = 3$ for Example 1

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An extension of Functional Projection Pursuit Regression to the Generalized Partially Linear Single Index Models. (FPPR-GPLSIM)

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Abstract: In this paper, we introduce a generalized functional approach to approximate the non-parametric function in the case of multivariate predictors, the single-index coefficient, and the non-linear regression behavior in the case of functional predictors and a scalar response. Following the Fisher-scoring algorithm and the principle of projection pursuit regression, we derive an additi, ve decomposition that exploits the most predictive direction, the most predictive add the active component of the functional predictor variable and the single-index component to explain the scalar response. On the one hand, this approach allows us to avoid the well-known problem of the curse of dimensionality in the non-parametric case with the notion of single index and the projection pursuit regression in the functional case, on the other hand, it can be used as an explonatory tool for the analysis of a multivariate and functional random variable belonging to a separable Hilbert space H. The terms of this decomposition are estimated with an iterative Fisher scoring procedure that uses the Quasi-Likelihood function and an approximation of the non parametric function by normalized B-splines. The good behaviour of our procedure is illustrated from a theoretical and practical point of view. Asymptotic results indicate that the nonparametric function, the single index coefficient and the terms of the additive decomposition can be estimated without suffering from curse of dimensionality , while some applications to real and simulated data show the high predictive performance of our method.
Keywords: Additive decomposition, asymptotic normality, Fisher scoring algorithm, functional data analysis (FDA), polynomial splines, predictive directions, projection pursuit regression, Quasi-likelihood, single-index model
Mathematics Subject Classification:

1 Introduction

Let H be an Hilbert space which is endowed with the scalar product $<\cdot,\cdot>_H$ and the norm $||\cdot||_H$. Let Y be a scalar response variable and $(X,Z)\in R^d\times H$ be the predictor vector where $X=(X_1,\dots,X_d)$ and Z be a functional random variable which is valued in H. For a fixed $(x,z)\in R^d\times H$, we assume that the conditional density function of the response Y given $(X,Z)=(x,z)$ belongs to the following canonical exponential family

$$f_{Y|X=x,Z=z}(y)=\exp\Big(y\,\xi(x,z)-B(\xi(x,z))+C(y)\Big), \tag{1.1}$$

where B and C are two known functions which are defined from R into R, and $\xi\ :\ R^d\times H\ \longrightarrow\ R$ is the parameter in the generalized parametric linear model which is linked to the dependent variable

$$\mu(x,z)=\mathbb{E}\Big[Y|X=x,Z=z\Big]=B'(\xi(x,z)), \tag{1.2}$$

where B' denotes the first derivative of the function B. In what follows we modelize the scalar response Y as a generalized functional projection pursuit regression for partially linear single-index model (GFPPR-PLSIM). by

$$g(\mu(X,Y))=\eta_0\left(\alpha^\top X\right)+R\left(Z\right)+\varepsilon, \tag{1.3}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in R^d$ is the d-dimensional single-index coefficient vector, η is the unknown single-index link function or the systematic non-linear component which will be assumed to be sufficiently smooth, g is the known link function and R is the regression operator to be estimated which is approximated by a finite sum of terms $R(Z) \approx \sum_{j=1}^{m} g_j(\langle \beta_j, Z\rangle)$ where $x^\top$ denotes the transpose vector of x. β_j is called the $j-$th most predictive direction and g_j is the $j-$th most predictive additive component.

2 Estimation methodology

Let $(X_i, Y_i, Z_i)_{i=1,\ldots,n}$ be a sequence of independent and identically distributed (i.i.d.) vectors as (X, Y, Z) and, for each $i = 1, \ldots, n$,

$$g(\mu(X_i, Y_i)) = \eta\left(\alpha^\top X_i\right) + R(Z_i) + \varepsilon_i. \tag{2.1}$$

We assume that the function η is supported within the interval $[a, b]$ where $a = \inf(\alpha^\top X)$ and $b = \sup(\alpha^\top X)$ and the regression operator R which we will model as $R(Z) = \sum_{j=1}^{J} g_j(\langle \beta_j, Z\rangle)$ where $\beta_j \in H$ with $\|\beta_j\|^2 = 1$, assuming that $E(\varepsilon|X, Z) = 0$ and $E(\varepsilon^2|X, Z) < \infty$.

Let $W_j = \langle \beta_j, Z\rangle, j = 1, ..., J$.
Denote $\varepsilon_{1,\beta_1} = Y - g_1(\langle \beta_1, Z\rangle)$, then ε_{1,β_1} and $\langle \beta_1, Z\rangle$ are uncorrelated. So, in an iterative way, we can define $\varepsilon_{j,\beta_j} = Y - \sum_{k=1}^{j} g_k(\langle \beta_k, Z\rangle), j = 1, ..., J-1$, and by plug-in $g(\mu(X, Z)) = \eta(\alpha^\top X) - \sum_{k=1}^{J} g_k(\langle \beta_k, Z\rangle)$ with $E[\varepsilon_{j,\beta_j}|\langle \beta_j, Z\rangle] = 0$ at each stage $j = 1, ..., J-1$ and $E[\varepsilon_{J,\beta_J}|X, \langle \beta_J, Z\rangle] = 0$. For $j = 1, ..., J-1$, the $j-$th direction β_j is obtained by solving the minimum problem

$$\min_{\|\beta_j\|^2=1} E\big[\big(\varepsilon_{j-1,\beta_{j-1}} - E\big[\varepsilon_{j-1,\beta_{j-1}}|\langle \beta_j, Z\rangle\big]\big)^2\big] \tag{2.2}$$

so, the $j-$th component is defined as follow

$$g_j(u) = E\big[\varepsilon_{j-1,\beta_{j-1}}|\langle \beta_j, Z\rangle = u\big] \tag{2.3}$$

Finally, we estimate α, η and g_J by using the quailikelihood function as will be defined later.

For $j = 1, ..., J-1$, given $\beta_1, ..., \beta_j$, we wish to estimate the functions $g_{j,\beta_j}(u) = E\big[\varepsilon_{j-1,\beta_{j-1}}|\langle \beta_j, Z\rangle = u\big]$ with $\varepsilon_{0,\beta_0} = Y$ and $\varepsilon_{j,\beta_j} = Y - \sum_{k=1}^{j} g_{k,\beta_k}(\langle \beta_k, Z\rangle)$ which can be estimated by using the Nadaraya-Watson kernel approach. For all $j = 1, ..., J-1$, the estimates are constructed as

$$\widehat{g}_{j,\beta_j}(u) = \frac{\sum_{i=1}^{n} \hat{\varepsilon}_{j-1,\beta_{j-1}} K_j\big(\frac{u-\langle \beta_j, Z_i\rangle}{h_j}\big)}{\sum_{i=1}^{n} K_j\big(\frac{u-\langle \beta_j, Z_i\rangle}{h_j}\big)}$$

where, for all $i = 1, ..., n$, $\varepsilon_{0,\beta_0,i} = Y_i$, and $\widehat{\varepsilon}_{j,\beta_j,i} = Y_i - \sum_{k=1}^{j} \widehat{g}_{k,\beta_k}(\langle \beta_s, Z_i\rangle), j = 1, ..., J-1$ $h_j, j = 1, ..., J-1$ are smoothing parameters depending on n, and K_j are standard kernel weighting functions.
We assume that the function η is supported within the interval $[a, b]$ where $a = \inf(\alpha^\top X)$ and $b = \sup(\alpha^\top X)$.
We introduce a sequence of knots (k_m) in the interval $[a, b]$, with J interior knots, such that $k_{-r+1} = \cdots = k_{-1} = k_0 = a < k_1 < \cdots < k_J = k_{J+1} = \cdots = k_{J+r}$, where $J := J_n$ is a sequence of integers which increases with the sample size n. Now, let $N_n = J_n + r$ be the number of knots, $(B_j(u))_{j=1,\ldots,N_n}$ be

the B-spline basis functions of order r, and $h = (b-a)/(J_n+1)$ be the distance between the neighbors knots.
Let $\mathcal{S}_n$ be the space of polynomial splines on $[a,b]$ of order $r \geq 1$. By De Boor [7], we can approximate η, assumed in $\mathcal{H}(p)$ (which will be defined in section 3) by a function $\tilde{\eta} \in \mathcal{S}_n$. So, we can write $\tilde{\eta}(u) = \tilde{\gamma}^\top B(u)$ where $B(u)$ is the spline basis and $\tilde{\gamma} \in R^{N_n}$ is the spline coefficient vector.
We introduce a new knots sequence $t_0 < t_1 < \cdots < t_{k+1}$ of the support of R. Then, there exists $l_J = k+r+1$ functions in the B-splines basis which are normalized and of order r, such that

$$g_J(\cdot) \approx \delta^\top B^{(J)}(.) \text{ where } B^{(J)}(.) = \left(B_1^{(J)}(.), B_2^{(J)}(.), \ldots, B_{l_J}^{(J)}(.)\right)^\top \text{ and } \delta \in R^{l_J}.$$

By setting $W_i^{(J)} = \langle \beta_J, Z_i \rangle$, the mean function estimator $\widehat{m}\,(x,z)$ is then given by the evaluation of the parameter $\theta = \left(\alpha^\top, \gamma^\top, \delta^\top\right)^\top$ and by inverting the following equation
$g(\mu(X_i, Z_i)) - \sum_{j=1}^{J-1} \widehat{g}_j(\langle \beta_j, Z_i \rangle) = \widehat{\gamma}^\top B\left(\widehat{\alpha}^\top x\right) + \widehat{\delta}^\top B^J(W_i^{(J)})$. Notice that the parameter $\theta = \left(\alpha^\top, \gamma^\top, \delta^\top\right)^\top$ is determined by maximizing the following quasi-likelihood rule
$\widehat{\theta} = \left(\widehat{\alpha}^\top, \widehat{\gamma}^\top, \widehat{\delta}^\top\right)^\top = \underset{\theta=(\alpha,\gamma,\delta)\in R^d\times R^{N_n}\times R^{l_J}}{\arg\max} l(\theta)$, where $l(\theta) := l(\alpha,\gamma,\delta) = \frac{1}{n}\sum_{i=1}^n Q\left(g^{-1}(m_i), Y_i\right)$, with $m_i := \gamma^\top B\left(\alpha^\top X_i\right) + \delta^\top B^{(J)}(W_i^{(J)})$, where $U_{0i} = \alpha_0^\top X_i$ with α_0, γ_0, δ_0, η_0 denoting the true values, respectively, of α, γ, δ, and η .
To overcome the constraint $\|\alpha\| = 1$ and $\alpha_1 > 0$ of the d-dimensional index α, we proceed by a re-parameterization, which is similar to Yu and Ruppert $\alpha(\tau) = \left(\sqrt{1-\|\tau\|^2}, \tau^\top\right)^\top$ for $\tau \in R^{d-1}$.
The true value τ_0 of τ, must satisfy $\|\tau_0\| \leq 1$. Then, we assume that $\|\tau_0\| < 1$. The jacobian matrix of $\alpha:\ \tau \to \alpha(\tau)$ of dimension $d \times (d-1)$ is $J(\tau)$. Notice that τ is unconstrained and is one dimension lower than α.
Finally, let $R(\tau) = \begin{pmatrix} J(\tau) & 0 \\ 0 & I_{l_J} \times I_{l_J} \end{pmatrix}$ the jacobian matrix of $\left(\alpha(\tau)^\top, \delta^\top\right)^\top$, which is of dimension $(d+l_J) \times (d+l_J-1)$. Let

$$(\widetilde{\alpha}, \widetilde{\delta}) = \underset{(\alpha,\delta)\in R^d\times R^{l_J},\ \tau\in R^{d-1}}{\arg\max} \frac{1}{n}\sum_{i=1}^n Q\left(\widetilde{\eta}\left(\alpha^\top(\tau)X_i\right) + \delta^\top B^{(J)}(W_i^{(J)}), Y_i\right) \text{ and } T_i = \left(X_i^\top, B^{(J)}(W_i^{(J)})^\top\right)^\top,$$

$(\widetilde{\tau}, \widetilde{\delta}) = \underset{\tau,\delta}{\arg\max}\ \widetilde{l}(\tau,\delta)$ where $\widetilde{l}(\tau,\delta) = \frac{1}{n}\sum_{i=1}^n Q\left(\widetilde{\eta}\left(\alpha(\tau)^\top X_i\right) + \delta^\top B^{(J)}(W_i^{(J)}), Y_i\right)$. Note that $\theta_\tau = \left(\tau^\top, \gamma^\top, \delta^\top\right)^\top$ is a $(d-1)\times N_n \times l_J$-dimensional parameter, while θ is a $d \times N_n \times l_J$-dimensional one. Let $\rho_l(m) = \frac{1}{\sigma^2 V m}$ and denote $q_l(m,y) = \frac{\partial^l}{\partial m^l} Q\,(m,y)$, for $l = 1,2$. Then, $q_1(m,y) = (y-m)\,\rho_1(m)$ and $q_2(m,y) = (y-m)\,\rho_1'(m) - \rho_2(m)$.
So, $l\,(\theta_\tau)$ becomes $l\,(\theta_\tau) = \frac{1}{n}\sum_{i=1}^n Q\left(\gamma^\top B\left(\alpha^\top(\tau)X_i\right) + \delta^\top B^{(J)}(W_i^{(J)}), Y_i\right) = \frac{1}{n}\sum_{i=1}^n Q\left(g^{-1}(m_i), Y_i\right)$
The score vector is then $S\,(\theta_\tau) = \frac{\partial l}{\partial \theta_\tau}\,(\theta_\tau) = \frac{1}{n}\sum_{i=1}^n q_1\,(m_i, Y_i)\,\xi_i(\tau,\gamma,\delta)$, where $\xi_i(\tau,\gamma,\delta) = \begin{pmatrix} \gamma^\top B'\left(\alpha^\top(\tau)X_i\right) J^\top(\tau)X_i \\ B\left(\alpha^\top(\tau)X_i\right) \\ B^{(J)}(W_i^{(J)}) \end{pmatrix}$. The expectation of the Hessian matrix is $H\,(\theta_\tau) = \mathbb{E}\left[\frac{\partial^2}{\partial\theta_\tau^\top \partial\theta_\tau} S\,(\theta_\tau)\right] = -\frac{1}{n}\sum_{i=1}^n \rho_2\,(m_i)\,\xi_i(\tau,\gamma,\delta)\xi_i^\top(\tau,\gamma,\delta)$,

The Fisher Scoring update equations $\theta_\tau^{(k+1)} = \theta_\tau^{(k)} - \left[H\left(\theta_\tau^{(k)}\right)\right]^{-1} S\left(\theta_\tau^{(k)}\right)$, becomes

$$\begin{aligned}\theta_\tau^{(k+1)} &= \theta_\tau^{(k)} + \left[\sum_{i=1}^n \rho_2\left(m_i^{(k)}\right)\xi_i\left(\tau^{(k)},\gamma^{(k)},\delta^{(k)}\right)\xi_i^\top\left(\tau^{(k)},\gamma^{(k)},\delta^{(k)}\right)\right]^{-1} \\ &\quad\times\left[\sum_{i=1}^n\left(Y_i-\mu_i^{(k)}\right)\rho_1\left(m_i^{(k)}\right)\xi_i\left(\tau^{(k)},\gamma^{(k)},\delta^{(k)}\right)\right],\end{aligned}$$

where $m_i^{(k)} = \gamma^{(k)\top}B\left(\alpha^{(k)\top}(\tau^{(k)})X_i\right) + \delta^{(k)\top}B^{(J)}(W_i^{(J)})$, for $1 \le i \le n$.

It follows that

$$\begin{aligned}\widehat{g_J^*}(t) &= \widehat{\delta}^\top B^{(J)}(t) = \delta^{(k)\top}B^{(J)}(t), \qquad \widehat{\eta}(t) = \widehat{\gamma}^\top B(t) = \gamma^{(k)\top}B(t), \\ \widehat{m}_i &= \widehat{\gamma}^\top B\left(\alpha^\top(\widehat{\tau})X_i\right) + \widehat{\delta}^\top B^{(J)}(W_i^{(J)}) = \gamma^{(k)\top}B\left(\alpha^\top\left(\tau^k\right)\right)X_i + \delta^{(k)\top}B^{(J)}(W_i^{(J)}),\end{aligned}$$

where $\widehat{\alpha} = \alpha\left(\tau^{(k)}\right)$ is the estimator of the single-index coefficient vector of the GFPPR-PLSIM model.

3 Asymptotic results

Let $v \in \mathbb{N}^*$ and $e \in (0,1]$ such that $p = v + e > 1.5$. We denote by $\mathcal{H}(p)$ the collection of functions g, which are defined on $[a,b]$ whose v-th order derivative, $g^{(v)}$, exists and satisfies the following e-th order Lipschitz condition $\left|g^{(v)}(m') - g^{(v)}(m)\right| \le C\left|m'-m\right|^e$, for all $a \le m, m' \le b$. Let $\varepsilon = Y - m_0(T)$ where $T = \left(X^\top, W^\top\right)^\top$.
(C1) The single-index link function $\eta_0 \in \mathcal{H}(p)$, where $\mathcal{H}(p)$ is defined as above.
(C2) For all $m \in R$ and for all y in the range of the response variable Y, the function $q_2(m,y)$ is strictly negative, and for $k = 1,2$, there exist some positive constants c_q and C_q such that $c_q < \left|q_2^k(m,y)\right| < C_q$.
(C3) The marginal density function of $\alpha^\top X$ is continuous and bounded away from zero and is infinite on its support $[a,b]$. The v-th order partial derivatives of the joint density function of X satisfy the Lipschitz condition of order α ($\alpha \in (0,1]$).
(C4) For any vector τ, there exist positive constants c_τ and C_τ, such that

$$c_\tau I_{t\times t} \le \mathbb{E}\left[\begin{pmatrix}1\\T\end{pmatrix}\begin{pmatrix}1\\T\end{pmatrix}^\top \middle| \alpha^\top(\tau)X = \alpha^\top(\tau)x\right] \le C_\tau I_{t\times t},$$

where $t = 1 + N_n + l_J$ and $T = \left(X^\top, W^\top\right)^\top$.
(C5) The number of knots N_n satisfy $n^{\frac{1}{2(p+1)}} \ll N_n \ll n^{\frac{1}{8}}$, for $p > 3$.
(C6) The fourth order moment of the random variable Z is finite, i.e., $\mathbb{E}\|Z(.)\|^4 \le C$, where C denotes a generic positive constant.
(C7) The covariance function $K(t,s) = \mathrm{Cov}(Z(t), Z(s))$ is positive definite.
(C8) For some finite positive constants C_ρ, C_ρ^* and M_0

$$|\rho_1(m_0)| \le C_\rho \ \text{ and } \ |\rho_1(m) - \rho_1(m_0)| \le C_\rho^*|m-m_0| \ \text{ for all } \ |m-m_0| \le M_0.$$

(C9) It exists a positive constant C_0, such that $\mathbb{E}(\epsilon^2|U_{\tau,0}) \le C_0$, where $\epsilon = Y - g^{-1}(m_0(T))$.
(C10) We assume that all the random variables $\langle\beta, Z\rangle$, for all $\beta \in \mathcal{H}$, have values on a set $\mathcal{C}$ where $\mathcal{C}$ is a compact subset of R.
(C11) We assume that for some $\delta > 0$ and some $C = C(\beta_1, \beta_2, ..., \beta_J) \in R^+$, we have, for all $j = 1, ..., J$,

$$\forall (u,v) \in \mathcal{C}^2, \ |g_{j,\beta_j}(u) - g_{j,\beta_j}(v)| \le C|u-v|^\delta.$$

(C12) The usual conditional moments requirement, $\forall r \geq 1,\ \exists C < \infty,\ E\big[|Y|^r = \sigma_r(Z)\big] < Cr!$. where $\sigma_r(.)$ is continuous and bounded, and we assume that for all $\beta \in \mathcal{H}$, the distribution of the real random variable $\langle \beta, Z\rangle$ is absolutely continuous with respect to the Lebesgue measur, with density f_β satisfying $0 < \inf_{u\in\mathcal{C}} f_\beta(u) \leq \sup_{u\in\mathcal{C}} f_\beta(u) < \infty$.

(C13) The specific conditions required for the nonparametric kernel smoother are the following: for any $j = 1, ..., J$, K_j is integrable and bounded with support $(-1, 1)$, and

$$\exists \tau_j > 0,\ \exists \alpha_j \geq 0,\ h_j \sim C^{st}\left(\frac{(\log n)^{\alpha_j}}{n}\right)^{\frac{1}{2\tau_j+1}}.$$

(C14) We assumed an extra smoothness of the target g_j: $\forall k = 1, ..., k_j - 1$, $\int u^k K_j(u)du = 0$ and $\int u^{k_j} K_j(u)du > 0$ with $k_j \geq q_j$ and $k_j < j_{j-1}$.

3.0.1 Convergence of the estimated univariate components

The convergence of the estimated univariate components is given by the following theorem.

Theorem 3.1.
(i) Under the conditions (C10-C13), with $\tau_j = \delta$ and $\alpha_j = 1$ in (C13) for $j = 1, ..., J - 1$, we have

$$\sup_{u\in\mathcal{C}} |\widehat{g}_{j,\beta_j}(u) - g_{j,\beta_j}(u)| = O\left(\left(\frac{\log n}{n}\right)^{\frac{\beta}{2\beta+1}}\right) \ a.s.$$

(ii) Under the conditions (C10-C14), with $\tau_j = k_j$ and $\alpha_j = 0$ in (C13) for , $j = 1, ..., J - 1$, we have

$$E\left[\int_{\mathcal{C}} \left(\widehat{g}_{j,\beta_j}(u) - g_{j,\beta_j}(u)\right)^2 du\right] \sim C\left(\frac{1}{n}\right)^{\dfrac{2k_j}{2k_j+1}}.$$

So, for the $J-$th component , we have the following theorem

Theorem 3.2. *Under assumptions $(C_1) - (C_8)$, and $k \sim n^{1/(2r+1)}$, we have*

$$\|\widehat{g}_{J,\beta_J} - g_{J,\beta_J}(\cdot)\|^2 = \mathcal{O}_{Pp}\left(N_n^2\left(h^p + \frac{1}{\sqrt{nh}}\right)^2\right) + \mathcal{O}_{Pp}(n^{-2r/(2r+1)}).$$

3.0.2 Estimation of the non-parametric function

Theorem 3.3. *Under assumptions (C1)-(C7), we have $\|\hat{\eta} - \eta_0\|_2 = O_{\mathbb{P}}\left\{\sqrt{N_n}\left(\frac{1}{\sqrt{nh}} + h^p\right)\right\}$ and $\|\hat{\eta} - \eta_0\|_n = O_{\mathbb{P}}\left\{\sqrt{N_n}\left(\frac{1}{\sqrt{nh}} + h^p\right)\right\}$*

3.0.3 Estimation of the parametric components

Theorem 3.4. *Under assumptions (C1)-(C10), the quasi-likelihood estimator $\hat{\alpha}$ with the constraint $\|\hat{\alpha}\| = 1$ is asymptotically normal i.e., $\sqrt{n}\,(\hat{\alpha} - \alpha_0) \xrightarrow{D} \mathcal{N}\left(0, J(\tau_0)\, D^{-1} J^\top(\tau_0)\right)$, where $D = \mathbb{E}\left[\rho_2\,(m_0(T))\left(\eta_0'\,(U_{\tau,0})\, J^\top(\tau_0)\,\Phi(X)\right)\left(\eta_0'\,(U_{\tau,0})\, J^\top(\tau_0)\,\Phi(X)\right)^\top\right]$.*

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On the conditions starlikeness and close-to convexitv for certain analytic functions

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Abstract: In this study, $f(z) \in A$, $f(z) = z + \sum_{n\geq 2}^{\infty} a_n.z^n$ will be an analytic function in the open unit disc $U = \{z: \ |z| < 1, \ z \in C\}$ normalized by $f(0) = 0, f'(0) = 1$. In this work, starlike functions and cloce-to-convex functions with degree $\frac{1}{4}$ have been studied according to the exact analytic requirements.

Keywords: Analytic function, univalent function, starlike function,. close – to-convex function
Mathematics Subject Classification: 30C45, 23584

1 Introduction

Let class A be the class of analytic function in the open unit disc $U = \{z: \ |z| < 1, \ z \in C\}$ normalized by

$$f(z) = z + \sum_{n\geq 2}^{\infty} a_n.z^n \ \text{ for } \ f(0) = 0, f'(0) = 1 \ \text{ where } \ z \in U = \{z: \ |z| < 1, \ z \in C\}$$

S denotes the class of $f(z)$ functions in A which $f(z)$ is a univalent function. These $f(z) \in A$ functions lie in U as starlike of order $\alpha\,(0 \leq \alpha < 1)$, such that

$$Re(\frac{zf'(z)}{f(z)}) > \alpha, f(z) \in A \ \text{ for all } \ z \in U = \{z: \ |z| < 1, \ z \in C\}$$

In other words$f(z) \in S^{\otimes}(\alpha)$That is $f(z) \in S^{\otimes}(\alpha)$ if and only if $z.f'(z) \in S^{\otimes}(\alpha)$. If there is a convex function $g(z)$ that provides the following uniequal function , then the $f(z)$ is called close-to-convex. Let $K^{\otimes}$ be the class of close-to-convex.

$$Re(\frac{z.f'(z)}{g'(z)}) > \alpha, \ \ z \in U = \{z : |z| < 1\ , \ z \in C\}\,,$$

According to the definitions fort he clasess starlike functions $S^{\otimes}(\alpha)$ and complex functions $K(\alpha)$ which these functions are of α degree. ,we know that $f(z) \in K(\alpha)$ if and only if $z.f'(z) \in S^{\otimes}(\alpha)$. [1, 5, 6, 11]. For the starlike function $f(z)$ with degree $\alpha(0 \leq \alpha < 1)$, we can give the following function as an example

$$f(z) = \frac{z}{1 - z^2} == z + \sum_{n\geq 2}^{\infty} n.z^{(2n-1)} \in S^*$$

Where $f(U)$ is starlike region by origin.
And
For the convex function $f(z)$ with $\alpha(0 \leq \alpha < 1)$, we can give the following function as an example

$$f(z) = \frac{1}{2}, Ln\left(\frac{1+z}{1-z}\right) = z + \sum_{n\geq 2}^{\infty} \frac{1}{2n-1} z^{2n-1} \in K,$$

which K is set of convex functions
Where $f(U)$ is convex region in complekx plane.

Lemma 1.1. *Let $h(z) = 1 + \sum_{n\geq 1}^{\infty} c_n.z^n$ be analytic in the unit disc U and suppose that there exists a point $z_0 \in U$ such that*

$$Reh(z) > 0 \;\; and \;\; Reh(z_0) = 0.$$

Then we have

$$z_0.h\prime(z_0) \leq -\frac{1}{2}(1 + |h(z_0)|^2) \; for \quad |z| < |z_0| \,.[1].$$

Theorem 1.1. ***(Main theorem1)*** *Let $f(z) \in A$ and suppose that there exists a starlike function $g(z)$ such that*

$$Re\left[\frac{z.f'(z)}{g(z)}(1 + \frac{z.f''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right] > -\frac{1}{8}(1 + \left|\frac{zf'(z)}{f(z)}\right|^2),$$

Then $f(z)$ is a close-to-convex function of degree $\frac{1}{4}$ i.e $f(z) \in K^{\otimes}(\frac{1}{4})$.

Proof Let us put $h(z) = 4(\frac{z.f'(z)}{g(z)} - \frac{3}{4})$ for $h(0) = 1$. Then $h(z)$ is analytic in $|z| < 1$ which satisfies the condition. Now using $h(z) = 4(\frac{z.f'(z)}{g(z)} - \frac{3}{4})$.

$$h'(z) = 4(\frac{(f'(z) + zf''(z)).g(z) - g'(z)zf'(z)}{(g(z))^2})$$

$$zh'(z) = 4(\frac{zf'(z)}{g(z)} + \frac{zf''(z)}{f'(z)}.\frac{zf'(z)}{g(z)} - \frac{zf'(z)}{g(z)}.\frac{zg'(z)}{g(z)}) = 4(\frac{zf'(z)}{g(z)}(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)})$$

$$\frac{1}{4}zh'(z) = (\frac{zf'(z)}{g(z)}(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)})$$

From $h(z)$ is analytic in U and $h(0) = 1$ suppose that there exists a complex number $z_0 \in U$ which satisfies the conditions of lemma. And from here

$$(\frac{zf'(z)}{g(z)}(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}) = \frac{1}{4}zh'(z).$$

On the other hand, since the function $h(z)$ and the point $z_0 \in |z| < 1$ satisfy all conditions lemma:1, then we obtain

$$Re(\frac{z_0f'(z_0)}{g(z_0)}(1 + \frac{z_0f''(z_0)}{f'(z_0)} - \frac{z_0g'(z_0)}{g(z_0)}) \leq -\frac{1}{8}(1 + |h(z_0|^2) = -\frac{1}{8}(1 + \left|\frac{z_0f'(z_0)}{g(z_0)}\right|^2).$$

Therefore proof of theorem 1 is completed. ■

Theorem 1.2. *Let $f(z) \in A$, and suppose that there a starlike function $g(z)$ such that $Re(\frac{zf'(z)}{g(z)}(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}) > -\frac{1}{2}(1 + \left|\frac{zf'(z)}{g(z)}\right|^2)$ for $z_0 \in |z| < 1$, Then $f(z)$ is the close-to- convex, so $f(z) \in K^{\otimes}$.*

Proof If $h(z) = \frac{zf'(z)}{g(z)}$ then, and $h(z)$ is analytic in U.
By using $h(z) = \frac{zf'(z)}{g(z)}$, we have

$$\frac{z_0.f'(z_0)}{g(z_0)}(1 + \frac{z_0.f''(z_0)}{f'(z_0)} - \frac{zg'(z_0)}{g(z_0)}) = z_0h'(z_0).$$

Therefore, we obtain

$$Re\left[\frac{z_0.f'(z_0)}{g(z_0)}(1 + \frac{z_0.f''(z_0)}{f'(z_0)} - \frac{zg'(z_0)}{g(z_0)})\right] = z_0h'(z_0) \leq -\frac{1}{2}(1 + \left|h(z_0)^2\right| = -\frac{1}{2}(1 + \left|\frac{z_0f'(z_0)}{g(z_0)}\right|^2).[9]$$

■

Lemma 1.2. *Let $h(z) = 1 + \sum_{n\geq 1}^{\infty} c_n.z^n$ be analytic in $|z| < 1$ and (α which is $0 < \alpha \leq \frac{1}{2}$) be a positive real number. Then suppose that there exists a point $z_0 \in |z| < 1$ such that*

$$Reh(z) > \alpha \quad and \quad Reh(z_0) = \alpha \quad and \quad h(z_0) \neq \alpha \quad for \quad |z| < |z_0| \quad .$$

$$\frac{z_0 h'(z_0)}{h(z_0)} \leq -\frac{\alpha}{2(1-\alpha)} [10]$$

Lemma 1.3. *t(z) being a non- constant analytic function in $|z| < 1$ with $t(0) = 0,$. If $|t(z)|$ attainins its maximum value on the $|z| = r < 1$ at z_0. Then*

$$z_0.w'(z_0) = kw(z) \quad where \quad k \geq 1 \;\; is \quad a\, real\, number[1].$$

Theorem 1.3. *If $f(z) \in A$ satisfies the following inequality*
$Re\left[\frac{z.f'(z)}{f(z)}(1 + \alpha\frac{z.f''(z)}{f'(z)}\right] > -\frac{\alpha^2}{4}(1-\alpha)$, $0 \leq \alpha < 2$ *, then $f(z) \in S^{\otimes}(\frac{1}{2})$ [2, 4].*

Theorem 1.4. *If $f(z) \in A$ satisfies the following inequality*
$Re\left[\frac{z.f'(z)}{f(z)}(1 + \frac{z.f''(z)}{f'(z)}\right] > 0$, *then $f(z) \in S^{\otimes}(\frac{1}{2})$ [2, 3].*

Theorem 1.5. *Let α ($0 < \alpha \leq \frac{1}{4}$) is a pozitive real number and $f(z) \in A$. If*
$Re(1 + \frac{zf''(z)}{f'(z)}) > Re(\frac{zf'(z)}{f(z)}) - \frac{1}{6}$. *Then, we have $f(z) \in S^{\otimes}(\frac{1}{4})$.*

Proof If $h(z) = \frac{zf'(z)}{f(z)}$. Then $h(z)$ is analytic in $|z| < 1$ and $h(0) = 1$. Suppose that there exists a complex number $z_0 \in |z| < 1$ which satisfies the conditions

$$Reh(z) > \frac{1}{4} \quad and \quad Reh(z_0) = \frac{1}{4} \quad and \quad h(z_0) \neq \frac{1}{4} \quad for \quad |z| < |z_0| \quad .$$

Really, now using $h(z) = \frac{zf'(z)}{f(z)}$, it follows that

$$h'(z) = \frac{(f'(z) + zf'(z))f(z) - z(f'(z))^2}{(f(z))^2}$$

$$zh'(z) = \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}.\frac{zf'(z)}{f(z)} - \frac{zf'(z)}{f(z)}.\frac{zf'(z)}{f(z)}$$

$$\frac{zh'(z)}{h(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \text{ for } h(0) = \frac{zf'(0)}{f(0)} = 1. \tag{1.1}$$

Since the function $h(z)$ and $z_0 \in |z| < 1$ satisfy all conditions lemma 2, therefore in view of

$$\frac{zh'(z)}{h(z)} \text{ and 1.1 gives } Re(1 + \frac{z_0 f''(z_0)}{f'(z_0)}) = Re(\frac{z_0 h'(z_0)}{h(z_0)} + h(z_0)$$

This is a contradiction and therefore proof of the theorem 8 is completed. ■

Theorem 1.6. *Let $h(z) = 1 + \sum_{n\geq 1}^{\infty} b_n.z^n$ be analytic in $U = \{z : |z| < 1\}$ and suppose that there exists $z_0 \in U$ such that $Re(h(z)) > 0$ for $|z| < |z_0|$, $Re(h(z_0)) = 0$. Then $z_0 h'(z_0) \leq -\frac{1}{4}(1 + |h(z_0)|^2)$.*

Proof Let's define $p(z) = 2.\frac{1-h(z)}{1+h(z)}$ function which satisfies the following conditions in its $|z| < |z_0|$ region $p(0) = 0$, $|p(z)| < 1$ *and* $|p(z_0)| = 1$.

$$p'(z) = 2\frac{-h'(z)(1 + h(z) - h'(z)(1 - h(z)}{(1 + h(z))^2} = \frac{-4h'(z)}{(1 + h(z))^2}$$

From $Reh(z) > 0 \quad and \quad Reh(z_0) = 0 \quad for \quad |z| < |z_0|$
$\frac{z.p'(z)}{p(z)} = \frac{-4zh'(z)}{(1-h(z))(1+h(z))}$ or $\frac{z_0.p'(z_0)}{p(z_0)} = \frac{-4z_0 h'(z_0)}{(1-h(z_0))(1+h(z_0))} \geq 1$, for $|z| < |z_0|$
Therefore, we have $z_0 h'(z_0) \leq -\frac{1}{4}(1 + |h(z_0)|^2)$. ■

Theorem 1.7. *Let's assume that the function $f(z) \in A$ satisfies the conditions $f(z).f'(z) \neq 0$ and $Re\left[\frac{z.f'(z)}{f(z)}(1+\frac{z.f''(z)}{f'(z)}\right] > -\frac{1}{4}\left|\frac{z.f'(z)}{f(z)}\right|^2$ for $0<|z|<1$. Then $f(z) \in S^{\otimes}(\frac{1}{4})$.*

Proof Let's define the function $h(z) = \frac{2z.f'(z)}{f(z)} - 1$ which holds $h(0) = 1$ Using this value of $h(z)$, we can consider the following equality $Re\left[\frac{z_0.f'(z_0)}{f(z_0)}(1+\frac{z_0.f''(z_0)}{f'(z_0)}\right] = Re\left[\frac{1}{4}z_0h'(z_0) + \frac{1}{8}(1+h(z_0)^2\right]$ where $z_0 \in U$ is a complex number which satisfies $Reh(z) > 0 \quad for \quad |z| < |z_0| \quad and \quad Reh(z_0) = 0$. By using relations $Reh(z_0) = 0$, $z_o.h'(z_o) \leq -\frac{1}{4}(1+|h(z_0)|^2)$, the following inequality can be written. $Re\left[\frac{z_0.f'(z_0)}{f(z_0)}(1+\frac{z_0.f''(z_0)}{f'(z_0)}\right] \leq -\frac{1}{8}(1+|h(z_0)|^2) - \frac{1}{8}|h(z_0)|^2 + \frac{1}{8}$

$$\leq -\frac{1}{4}|h(z_0)|^2 \leq -\frac{1}{4}\left|\frac{z_0.f'(z_0)}{f(z_0)}\right|.$$

Therefore, we have $Reh(z) > 0$ or $Re(\frac{z.f'(z)}{f(z)}) > \frac{1}{4}$,so $f(z) \in S^{\otimes}(\frac{1}{4})$ [2]. ■

Theorem 1.8. ***(Main Theorem 2)*** *If $f(z)$ is a function which satisfies the following conditions $(z).f'(z) \neq 0$ and $Re\left[\frac{z.f'(z)}{f(z)}(1+\alpha\frac{z.f''(z)}{f'(z)}\right] > -\frac{\alpha}{8}(3-\alpha)(2-\alpha)(1-\alpha)$ in $0<|z|<1$ for $0<\alpha<3$, then $f(z)$ is $\frac{1}{4}$order starlike function. That is $f(z) \in S^{\otimes}(\frac{1}{4})$.*

Proof Let's define $\frac{z.f'(z)}{f(z)} = (1-h(z))\frac{\alpha}{4} + h(z)$ for $h(0) = 1$. Then the following equality can be written such that $Reh(z) > 0$ and $Reh(z_0) = 0$ for $|z| < |z_0|$Then from here,

$$\frac{d}{dz}\frac{z.f'(z)}{f(z)} = \frac{f'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}.\frac{f'(z)}{f(z)} - z(\frac{f'(z)}{f(z)})^2 = (1-\frac{\alpha}{4})h'(z)$$

$$\frac{zf'(z)}{f(z)} + \frac{f''(z)}{f'(z)}.\frac{zf'(z)}{f(z)} - (\frac{zf'(z)}{f(z)})^2 = (1-\frac{\alpha}{4})z.h'(z)$$

$$\frac{zf'(z)}{f(z)}(1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}) = (1-\frac{\alpha}{4})z.h'(z)$$

Suppose that there exists a complex point $z_0 \in |z| < 1$ such that it satisfies the conditions $Reh(z) > 0$ and $Reh(z_0) = 0$ and lemma 1, lemma 2 and lemma 3. Then we have

$$Re\left[\frac{z_0.f'(z_0)}{f(z_0)}(1+\alpha\frac{z_0.f''(z_0)}{f'(z_9)}\right] =$$

$$Re\left[\alpha(1-\frac{\alpha}{4})z_0h'(z_0) + \alpha(1-\frac{\alpha}{4})^2(h(z))^2 + (1-\frac{\alpha}{4})(\alpha^2+1-\alpha)h(z_0) + \frac{\alpha^3}{8} + (1-\alpha)\frac{\alpha}{4})\right]$$

If the relations $Reh'(z_o) = 0$ and $z_o.Reh'(z_o) \leq -\frac{1}{4}(1+|h(z_0)|^2)$ are used in the above equation, then

$$Re\left[\frac{z_0.f'(z)}{f(z_0)}(1+\alpha\frac{z_0.f''(z_0)}{f'(z_0)}\right] = Re(\frac{z_0.f'(z)}{f(z_0)}).Re((1+\alpha\frac{z_0.f''(z_0)}{f'(z_0)})$$
$$\leq -\frac{\alpha}{4}(1-\frac{\alpha}{4})(1+|h(z_0)|^2) - \alpha(1-\frac{\alpha}{4})^2|h(z_0)|^2 + \frac{\alpha^3}{8} + \frac{\alpha}{4}(1-\alpha)$$

$$\leq -\frac{\alpha}{4}(1-\frac{\alpha}{4}) + \frac{\alpha^3}{8} + \frac{\alpha}{4}(1-\alpha) \leq -\frac{\alpha^3}{8}(3-\alpha)(2-\alpha)(1-\alpha)$$

$Re(\frac{z.f'(z)}{f(z)}) > \frac{\alpha}{4}$ is also obtained, which is $f(z) \in S^{\otimes}(\frac{1}{4})$. ■

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has been held regularly since 2011. It was interrupted only in 2020 due to the Covid-19 Pandemic. It was decided to be held online in 2022 due to the continuation of the pandemic. At the conference, people from different parts of the world and from different countries had the opportunity to work together. This conference is a prime example of how people can contribute to science together.