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Matrix Krylov subspace methods for image restoration

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Abstract: In the present paper, we consider some matrix Krylov subspace methods for solving ill-posed linear matrix equations and in those problems coming from the restoration of blurred and noisy images. Applying the well known Tikhonov regularization procedure leads to a Sylvester matrix equation depending the Tikhonov regularized parameter. We apply the matrix versions of the well known Krylov subspace methods, namely the Least Squared (LSQR) and the conjugate gradient (CG) methods to get approximate solutions representing the restored images. Some numerical tests are presented to show the effectiveness of the proposed methods.

Keywords: Global Lanczos, LSQR, Ill-posed, Image restoration.

1 Introduction

Image restoration is the process of removing blur and noise from degraded images to recover an approximation of the original image. This field of imaging technology is becoming increasingly significant in many scientific applications [1, 24, 25]. Since blurring is the degradation from the process of image formation, it is a deterministic process which has a sufficiently accurate mathematical model for its description. The goal of the image restoration is to recover a good approximation of the original image X which is $m \times n$, for a given the degraded image B of size $m \times n$, the blur matrix H of size $mn \times mn$, and the statistics of the noise matrix E. The mathematical model [18, 21, 23, 22] that relates the given blurred and noisy image to the unknown true image is given as follows

$$Hx = b + e. \tag{1}$$

The key for obtaining this general linear model is to rearrange the elements of the images X, B and the noise matrix E into column vectors by stacking the columns of these images into three long vectors $x = vec(X)$, $b = vec(B)$ and $e = vec(E)$, respectively, of length $N = mn$. See [22, 3, 8] for more details concerning image representation and modeling. By solving the inverse problem for x, an approximation to the true image can be computed; however, this is not so simple due to the severe ill-conditioning and large dimensions of the matrix H [**?**].
Since the blurring matrix H is ill-conditioned, the image restoration problem will be extremely sensitive to perturbations in the right hand side vector. In order to diminish the effects of the noise in the data, we replace the original operator by a better conditioned one. One of the most popular regularization methods is due to Tikhonov [15, 17, 20]. The method replaces the problem (1) by the new one

$$\min_x \left(\| Hx - g \|_2^2 + \mu^2 \| Lx \|_2^2 \right), \tag{2}$$

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where $g = b + e$, L is a regularization operator chosen to obtain a solution with desirable properties. The matrix L could be the identity matrix or a discrete form of first or second derivative. In the first case, the parameter μ acts on the size of the solution, while in the second case μ acts on the smoothness of the solution.

2 Backgrounds

In Tikhonov regularization, the solution of the problem (1) is computed as the unique solution of the following linear least squares problem

$$\hat{x} = \underset{x}{\operatorname{argmin}} \left\| \begin{bmatrix} H \\ \mu L \end{bmatrix} x - \begin{bmatrix} g \\ 0 \end{bmatrix} \right\|_2^2 . \tag{3}$$

See [11,12,17,18] for more details about theory of Tikhonov regularization. The minimizer of the problem (3) is computed as the solution of the following normal equations

$$(H^T H + \mu^2 L^T L)x = H^T g. \tag{4}$$

The Kronecker product of two matrices $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times n}$ is the matrix $H \in \mathbb{R}^{mn\times mn}$ of the following bloc form

$$H = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{21}B & \cdots & a_{2m}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix},$$

where a_{ij} are the coefficients of the matrix A.
Here we list some of the most important properties of Kronecker products [30]:

$$\begin{gathered} (A\otimes B)\mathrm{vec}(X) = \mathrm{vec}(BXA^T), \\ (A\otimes B)^T = A^T \otimes B^T, \qquad (A\otimes B)^{-1} = A^{-1}\otimes B^{-1}, \\ (AB)\otimes(CD) = (A\otimes C)(B\otimes D), \end{gathered}$$

where the *vec* operator transforms the matrix A of size $m\times n$ to a vector a of size $mn\times 1$ by stacking the columns of A. For A and B in $\mathbb{R}^{m\times n}$, we define the inner product $< A, B >_F = tr(A^T B)$, where $tr(Z)$ denotes the trace of the square matrix Z. The Frobenius norm is defined by $\| A \|_F = (< A, A >_F)^{1/2}$.
We assume that the matrices H and L are decomposed as a Kronecker product of squares matrices H_2, L_2 of size $m\times m$, and H_1, L_1 of size $n\times n$ such that

$$H = H_2 \otimes H_1, \text{ and } L = L_2 \otimes L_1.$$

Therefore, by using the connection between the properties of the Kronecker product and linear operator *vec*, the problem (4) can be written as

$$(H_1^T H_1)X(H_2^T H_2) + \mu^2 (L_1^T L_1)X(L_2^T L_2) = H_1^T G H_2, \tag{5}$$

where X and G are matrices such that $vec(X) = x$ and $vec(G) = g$. The linear matrix (5) is referred to as the generalized Sylvester matrix equation and is written in the following form

$$AXB + CXD = E, \tag{6}$$

where

$$A = H_1^T H_1, \quad B = H_2^T H_2, \quad C = \mu^2 L_1^T L_1, \quad D = L_2^T L_2 \quad \text{and} \quad E = H_1^T G H_2.$$

For solving the generalized Sylvester matrix equation (6), we define the following linear operator

$$\mathscr{L} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$$
$$X \to \mathscr{L}(X) = AXB + CXD.$$

Its transpose is defined by $\mathscr{L}^T(X) = A^T X B^T + C^T X D^T$. Hence, the generalized Sylvester matrix equation (6) is rewritten as

$$\mathscr{L}(X) = E. \tag{7}$$

The linear operator $\mathscr{L}$ is symmetric positive definite, whenever $\mathscr{L}$ is symmetric, i.e. $\mathscr{L} = \mathscr{L}^T$, and the following condition is satisfied

$$< \mathscr{L}(X), X >_F > 0, \quad 0 \neq X \in \mathbb{R}^{m\times n},$$

and then the matrix equation could be solved by a matrix version of the conjugate gradient method. If $\mathscr{L}$ is not symmetric, then one can consider the following symmetric problem

$$\hat{\mathscr{L}}(X) = (\mathscr{L}^T \circ \mathscr{L})(X). \tag{8}$$

3 The global LSQR for solving generalized Sylvester matrix equations

The global least squares (Gl-LSQR) method is a generalization of the classical least squares (LSQR) method [33] for solving the problem

$$\min_X \| \mathscr{L}(X) - E \|_F \ . \tag{9}$$

Let V_1 and U_1 be two matrices of size $m\times n$, let us denote by $\mathscr{K}_k(\mathscr{L}, V_1)$ and $\mathscr{K}_k(\mathscr{L}^T, U_1)$ the matrix Krylov subspaces generated by $\{V_1, \mathscr{L}(V_1), ..., \mathscr{L}^{k-1}(V_1)\}$ and $\{U_1, \mathscr{L}^T(U_1), ..., \mathscr{L}^{Tk-1}(U_1)\}$, respectively. We note that $\mathscr{L}^i(V)$ is defined recursively as $\mathscr{L}(\mathscr{L}^{i-1}(V))$. The global bidiagonalization process construct two F-orthonormal basis $\{V_1, V_2, ..., V_k\}$ and $\{U_1, U_2, ..., U_k\}$ of the matrix Krylov subspaces $\mathscr{K}_k(\mathscr{L}, V_1)$ and $\mathscr{K}_k(\mathscr{L}^T, U_1)$, respectively.

The global bidiagonalization steps are summarized in the following algorithm.

Algorithme 1: The global bidiagonalization process

1. Set $\beta_1 = ||E||_F$, $U_1 = E/\beta_1$, $\alpha_1 = ||\mathscr{L}^T(U_1)||_F$, $V_1 = \mathscr{L}^T(U_1)/\alpha_1$.
2. For $j = 1, 2, ..., k$
   (a) $\tilde{U}_j = \mathscr{L}(V_j) - \alpha_j U_j$
   (b) $\beta_{j+1} = ||\tilde{U}_j||_F$
   (c) $U_{j+1} = \tilde{U}_j/\beta_{j+1}$
   (d) $\tilde{V}_j = \mathscr{L}^T(U_{j+1}) - \beta_{j+1} V_j$
   (e) $\alpha_{j+1} = ||\tilde{V}_j||_F$
   (f) $V_{j+1} = \tilde{V}_j/\alpha_{j+1}$
   EndFor

We now define the matrices $\mathscr{V}_k \equiv [V_1, V_2, ..., V_k]$, $\mathscr{U}_k \equiv [U_1, U_2, ..., U_k]$ and the lower bidiagonal matrix

$$T_k \equiv \begin{bmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \\ & & \beta_k & \alpha_k \\ & & & \beta_{k+1} \end{bmatrix}. \quad (10)$$

Let $*$ denotes the following product

$$\mathscr{V}_k * t = \sum_{j=1}^{k} V_j t_j, \quad t \in \mathbb{R}^k,$$

where $t \in \mathbb{R}^k$. It is easy to see that the following relations are satisfied:

$$\mathscr{V}_k * (t+s) = (\mathscr{V}_k * t) + (\mathscr{V}_k * s), \quad s \in \mathbb{R}^k \quad (11)$$

$$(\mathscr{V}_k * T_k) * t = \mathscr{V}_k * (T_k t), \quad (12)$$

$$||\mathscr{V}_k * t||_F = ||t||_2, \quad (13)$$

where

$$\mathscr{V}_k * T_k = [\mathscr{V}_k * T_{.,1}, \mathscr{V}_k * T_{.,2}, ..., \mathscr{V}_k * T_{.,k}],$$

where $T_{.,j}$ is jth column of the matrix T_k.
Now, according to the notation $*$, the recurrence steps in Algorithme 1 may be rewritten as

$$\mathscr{U}_{k+1} * (\beta_1 e_1) = E, \quad (14)$$

$$\left[\mathscr{L}(V_1), \mathscr{L}(V_2), ..., \mathscr{L}(V_k)\right] = \mathscr{U}_{k+1} * T_k, \quad (15)$$

$$\left[\mathscr{L}^T(U_1), \mathscr{L}^T(U_2), ..., \mathscr{L}^T(U_{k+1})\right] = \mathscr{V}_k * T_k^T + \alpha_{k+1} V_{k+1} * e_{k+1}^T, \quad (16)$$

where $e_{k+1} = (0, ..., 0, 1)^T \in \mathbb{R}^{k+1}$.
Now, we seek an approximate solution X_k to (9) such that $X_k \in span(\mathscr{V}_k)$ and write

$$X_k = \mathscr{V}_k * y_k, \quad y_k \in \mathbb{R}^k, \quad (17)$$

Then the corresponding residual matrix of the equation (9) is

$$\begin{aligned} R_k &= E - \mathscr{L}(X_k) \\ &= E - \left[\mathscr{L}(V_1), \mathscr{L}(V_2), ..., \mathscr{L}(V_k)\right] * y_k \\ &= \beta_1 U_1 - (\mathscr{U}_{k+1} * T_k) * y_k \\ &= \beta_1 U_1 - \mathscr{U}_{k+1} * (T_k y_k) \\ &= \mathscr{U}_{k+1} * (\beta_1 e_1 - T_k y_k) \end{aligned}$$

The global LSQR algorithm chooses the vector y_k which minimizes $||R_k||_F$. Thus from the last relatio, we get

$$\min||R_k||_F = \min_{y_k\in\mathbb{R}^k}||\beta_1 e_1 - T_k y_k||_2. \tag{18}$$

This minimization problem is accomplished by using the QR decomposition

$$Q_k\begin{bmatrix}T_k & \beta_1 e_1\end{bmatrix} = \begin{bmatrix}R_k & f_k\\ 0 & \bar{\phi}_{k+1}\end{bmatrix},$$

where the matrix Q_k is a product $G_{k,k+1}G_{k-1,k}...G_{1,2}$ chosen to eliminate the subdiagonal element $\beta_2,...\beta_{k+1}$ of T_k and

$$R_k = \begin{bmatrix}\rho_1 & \theta_2 & & & \\ & \rho_2 & \theta_3 & & \\ & & \ddots & \ddots & \\ & & & \rho_{k-1} & \theta_k\\ & & & & \rho_k\end{bmatrix}, \quad \text{and} \quad f_k = \begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_{k-1}\\ \phi_k\end{bmatrix},$$

The minimizer y_k of (18) can then be obtained from $R_k y_k = f_k$. Therefore an approximate solution is formed as

$$X_k = \mathscr{V}_k * y_k = \mathscr{V}_k * (R_k^{-1} f_k) = (\mathscr{V}_k * R_k^{-1}) * f_k.$$

Letting $\mathscr{P}_k \equiv \mathscr{V}_k * R_k^{-1} \equiv \begin{bmatrix}P_1, P_2, ..., P_k\end{bmatrix}$, the approximate solution is given by $X_k = \mathscr{P}_k * f_k$. With the initial guess $P_0 = X_0 = 0, X_k$ can be obtained by the relation

$$X_k = X_{k-1} + P_k\phi_k.$$

The last block column P_k of $\mathscr{P}_k$ can be updated by the previous P_{k-1} and V_k,

$$P_k = (V_k - P_{k-1}\theta_k)\rho_k^{-1}, \tag{19}$$

and

$$f_k = \begin{bmatrix}f_{k-1}\\ \phi_k\end{bmatrix},$$

where $\phi_k = c_k\bar{\phi}_k$.
The matrix residual norm $||R_k||_F$ is computed directly from the quantity

$$||R_k||_F = |\bar{\phi}_{k+1}|$$

The Gl-LSQR algorithm steps for resolving the linear operator equation (9) are summarized in Algorithme 2.

Algorithme 2: The Gl-LSQR algorithm for solving the generalized Sylvester equation (6)

1. Set $X_0 = 0_{m\times n}$
2. $\beta_1 = ||E||_F$, $U_1 = E/\beta_1$, $\alpha_1 = ||\mathscr{L}^T(U_1)||_F$, $V_1 = \mathscr{L}^T(U_1)/\alpha_1$.
3. Set $W_1 = V_1$, $\bar{\phi}_1 = \beta_1$, $\bar{\rho}_1 = \alpha_1$
4. For $j = 1, 2, ..., k$
   (a) $\tilde{U}_j = \mathscr{L}(V_j) - \alpha_j U_j$

(b) $\beta_{j+1} = ||\tilde{U}_j||_F$
(c) $U_{j+1} = \tilde{U}_j/\beta_{j+1}$
(d) $\tilde{V}_j = \mathscr{L}^T(U_{j+1}) - \beta_{j+1}V_j$
(e) $\alpha_{j+1} = ||\tilde{V}_j||_F$
(f) $V_{j+1} = \tilde{V}_j/\alpha_{j+1}$
(g) $\rho_j = (\bar{\rho}^2 + \beta_{j+1}^2)^{1/2}$, $c_j = \bar{\rho}_j/\rho_j$, $s_j = \beta_{j+1}/\rho_j$
(h) $\theta_{j+1} = s_j\alpha_{j+1}$, $\bar{\rho}_{j+1} = c_j\alpha_{j+1}$
(i) $\phi_j = c_j\bar{\phi}_j$, $\bar{\phi}_{j+1} = -s_j\bar{\phi}_j$
(j) $X_j = X_{j-1} + (\phi_i/\rho_i)W_i$
(k) $W_{j+1} = V_{j-1} + (\theta_{i+1}/\rho_i)$
(l) If $|\bar{\phi}_{j+1}|$ is small enough, then stop.

4 The global CG method for solving large general Sylvester matrix equations

The global conjugate gradient (Gl-CG) method is a generalization of CG method [9] for solving sparse symmetric positive definite (SPD) linear system of equations

$$\mathscr{M}(X) = E, \tag{20}$$

where E and X are $m \times n$ matrices and the linear operator $\mathscr{M}$ is assumed to be symmetric and postive definite. We demonstrate how to employ the Gl-CG method to get an approximate solution of the linear matrix operator equation (6). The global Lanczos process is a method for transforming the linear operator $\mathscr{M}$ to a tridiagonal matrix. The algorithm is given as follows

Algorithme 3: The global Lanczos process

1. Set V_1 of size $m \times n$ such that $||V_1||_F = 1$.
2. Set $\beta_1 = 0$ and $V_0 = 0$.
3. For $j = 1, 2, .., k$
   (a) $W = \mathscr{M}(V_j) - \beta_j V_{j-1}$,
   (b) $\alpha_j = < V_j, W >_F$,
   (c) $W = W - \alpha_j V_j$,
   (d) $\beta_{j+1} = ||W||_F$,
   (e) $V_{j+1} = W/\beta_{j+1}$,

4.EndFor

This algorithm constructs an F-orthogonal basis $\mathscr{V}_k \equiv [V_1, V_2, ..., V_k]$ of the matrix Krylov subspace $\mathscr{K}_k(\mathscr{M}, V_1)$. Let T_k be the tridiagonal matrix constructed by this algorithm and given by

$$T_k \equiv \begin{bmatrix} \alpha_1 & \beta_2 & & \\ \beta_2 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_k \\ & & \beta_k & \alpha_k \end{bmatrix}. \tag{21}$$

Now, according to the notation $*$, the recurrence steps in Algorithme 3 may be rewritten as

$$\left[\mathscr{M}(V_1), \mathscr{M}(V_2), ..., \mathscr{M}(V_k)\right] = \mathscr{V}_k * T_k + \beta_{k+1}\left[0, ..., 0, V_{k+1}\right] \tag{22}$$

$$= \mathscr{V}_{k+1} * \tilde{T}_k \tag{23}$$

where

$$\tilde{T}_k = \begin{bmatrix} T_k \\ \beta_{k+1} e_k^T \end{bmatrix},$$

where $e_k = (0,...,0,1)^T \in \mathbb{R}^k$.
Starting from an initial guess $X_0 \in \mathbb{R}^{m\times n}$ and the corresponding residual $R_0 = E - \mathscr{M}(X_0)$, the approximate solution X_k of (6) is defined as follows:

$$X_k = X_0 + Z_k, \quad \text{with} \quad Z_k \in \mathscr{K}_k(\mathscr{M}, V_1), \tag{24}$$

where $V_1 = \frac{R_0}{||R_0||_F}$.
From relation (24), the approximation X_k can be written as

$$X_k = X_0 + \mathscr{V}_k * y_k, \tag{25}$$

$$= X_0 + \mathscr{V}_k(y_k \otimes I_n), \tag{26}$$

where $y_k \in \mathbb{R}^k$ is the solution of the following linear system

$$T_k y_k = \| R_0 \|_F \; e_1, \tag{27}$$

where e_1 is the first unit vector of $\mathbb{R}^k$. From the LU factorization of T_k

$$\begin{aligned} T_k &= L_k U_k, \\ &= \texttt{tridiag}(\lambda_i, 1, 0) \times \texttt{tridiag}(0, \eta_i, \beta_{i+1}), \end{aligned}$$

the approximate solution X_k is expressed as

$$\begin{aligned} X_k &= X_0 + \mathscr{V}_k * (U_k^{-1} L_k^{-1}) ||R_0||_F e_1, \\ &= X_0 + (\mathscr{V}_k * U_k^{-1}) * (L_k^{-1} ||R_0||_F e_1), \\ &= X_0 + \mathscr{P}_k * z_k, \\ &= X_{k-1} + \zeta_k \mathscr{P}_k, \end{aligned}$$

where $\mathscr{P}_k = \mathscr{V}_k * U_k^{-1} = [P_1, P_2, ..., P_k]$ and $z_k = L_k^{-1} ||R_0||_F e_1 = \begin{bmatrix} z_{k-1} \\ \zeta_k \end{bmatrix}$.
The last block column P_k of $\mathscr{P}_k$ can be updated by the previous P_{k-1} and V_k as

$$P_k = \frac{1}{\eta_k}[V_k - \beta_k P_{k-1}], \tag{28}$$

and

$$\lambda_k = \frac{\beta_k}{\eta_{k-1}}, \quad \eta_k = \alpha_k - \lambda_k \beta_k.$$

From the relation

$$X_{j+1} = X_j + \alpha_j P_j,$$

the residual matrix of normal matrix must satisfy the recurrence

$$R_{j+1} = R_j - \alpha_j \mathscr{M}(P_j),$$

and the next search direction P_{j+1} is a linear combination of R_{j+1} and P_j,

$$P_{j+1} = R_{j+1} + \beta_j P_j.$$

Thus the F-orthogonality of the R_j brings

$$\alpha_j = \frac{< R_j, R_j >_F}{< \mathscr{M}(P_j), P_j >_F},$$

and

$$\beta_j = \frac{< R_{j+1}, R_{j+1} >_F}{< R_j, R_j >_F}.$$

If we apply the global CG algorithm for solving the problem (6) given as

$$\mathscr{M}(X) = (H_1^T H_1)X(H_2^T H_2) + \mu^2 (L_1^T L_1)X(L_2^T L_2)$$

and $E = H_1^T G H_2$, we get the following algorithm.

Algorithme 4: Gl-CG for sloving the generalized Sylvester equation (6)

1. Choose $X_0 \in \mathbb{R}^{m\times n}$
2. Compute $R_0 = F - \mathscr{M}(X_0)$, $P_0 = R_0$
3. For $j = 1, 2, .., k$ until convergence Do
   (a) $\alpha_j = \frac{<R_j, R_j>_F}{<\mathscr{M}(P_j), P_j>_F}$,
   (b) $X_{j+1} = X_j + \alpha_j P_j$
   (c) $R_{j+1} = R_j - \alpha_j \mathscr{M}(P_j)$,
   (d) $\beta_j = \frac{<R_{j+1}, R_{j+1}>_F}{<R_j, R_j>_F}$
   (e) $P_{j+1} = R_{j+1} + \beta_j P_j$,

4.EnDo

5 A parameter selection method for Tikhonov regularization

We now consider a parameter choice method. An appropriate selection of the regularization parameter μ is important in Tikhonov regularization. The well-known method for choosing suitable regularization parameters is due to generalized cross validation (GCV) method [14]. GCV is a widely used and very successful predictive method for choosing the smoothing parameter. The basic idea is that, if one datum point is dropped, then a good value of the regularization parameter should predict the missing datum value fairly well. For this method, the regularization parameter is chosen to minimize the GCV function

$$GCV(\mu) = \frac{||Hx_\mu - g||_2^2}{[tr(I - HH_\mu^{-1}H^T)]^2} = \frac{||(I - HH_\mu^{-1}H^T)g||_2^2}{[tr(I - HH_\mu^{-1}H^T)]^2},$$

where $H_\mu = H^T H + \mu^2 L^T L$ and x_μ is the solution of $H_\mu x = H^T g$. Let $H = H_2 \otimes H_1$ and $L = L_2 \otimes L_1$ where H_2, L_2 and H_1, L_1 are of size $m \times m$ and $n \times n$, respectively. The GCV function can be simplified for Tikhonov regularization method using the generalized singular value decompositions (GSVD) [13] of the pairs (H_1, L_1) and (H_2, L_2). Thus, there exist orthonormal matrices U_1, U_2, V_1, V_2 and invertible matrices X_1, X_2 such that

$$U_1^T H_1 X_1 = C_1 = \mathrm{diag}(c_{1,1},...,c_{m,1}), \quad c_{i,1} \geq 0$$
$$U_2^T H_2 X_2 = C_2 = \mathrm{diag}(c_{1,2},...,c_{n,2}), \quad c_{i,2} \geq 0$$

and

$$V_1^T L_1 X_1 = S_1 = \mathrm{diag}(s_{1,1},...,s_{m,1}), \quad s_{i,1} \geq 0$$
$$V_2^T L_2 X_2 = S_2 = \mathrm{diag}(s_{1,2},...,s_{n,2}), \quad s_{i,2} \geq 0$$

Then the GSVD of the pair (H,L) is given by

$$U^T HX = C = \mathrm{diag}(c_1,...,c_N), \quad c_i \geq 0$$
$$V^T LX = S = \mathrm{diag}(s_1,...,s_N), \quad s_i \geq 0$$

where $U = U_2 \otimes U_1$, $V = V_2 \otimes V_1$, $C = C_2 \otimes C_1$, $S = S_2 \otimes S_1$ and $N = mn$. Therefore, the GCV function when used with Tikhonov regularization can be simplified to

$$GCV(\mu) = \frac{\sum_{i=1}^{N}(s_i^2 \hat{g}_i/(c_i^2+\mu^2 s_i^2))^2}{(\sum_{i=1}^{N} s_i^2/(c_i^2+\mu^2 s_i^2))^2}, \tag{29}$$

with $\hat{g} = U^T g$. For the particular case where the matrix L reduces to the identity I, the GSVD of the pair (H,I) reduces to the SVD of the matrix H and the expression of GCV is given by the following formula

$$GCV(\mu) = \frac{\sum_{i=1}^{N}(\hat{g}_i/(\sigma_i^2+\mu^2))^2}{(\sum_{i=1}^{N} 1/(\sigma_i^2+\mu^2))^2}, \tag{30}$$

where σ_i is the ith singular value of the matrix H.
Because $GCV(\mu)$ in this case is a continuous function, we use the Matlab function `fminbnd`, which is based on a combination of golden section search and quadratic interpolation search, to find the value of μ at which $GCV(\mu)$ is minimized.

6 Numerical results

This section presents a culmination of all the numerical tests and experiments we performed. We provide numerical experiments with application of Gl-LSQR and Gl-CG for solving generalized Sylvester equation appearing in the image restoration problem. In order to understand these numerical experiments, we frst recall the problem (1)

$$Hx = b + e$$

where b is the blurred image, H is a blurring matrix, x is the true image and e is the noise. The noise, which is generated by Matlab's random function, is composed of normally distributed random numbers. Various noise levels can be utilized by taking a percentage of the generated noise; usually we pick from 0,01 or 0,001. The blurring matrix H is determined from two ingredients: the PSF [22], which defines how each pixel is blurred, and the boundary conditions, which specify our assumptions on the scene just outside our image. In order to obtain a high-quality deblurred image we must choose the boundary conditions appropriately. Each boundary condition makes the PSF matrix H having a different special structure. In this work we consider Neumann boundary conditions (the pixels outside image have mirror image values of the scene inside the image borders), the coefficient matrix H is a sum of block Toeplitz with Toeplitz blocks (BTTB), block Toeplitz with Hankel blocks (BTHB), block Hankel with Toeplitz blocks (BHTB), and block Hankel with Hankel blocks (BHHB). Let $\hat{x} = vec(\hat{X})$ be the vector whose entries are the pixel values of the original image $\hat{X}$. We would like to determine an approximate solution of the unavailable system $H\hat{x} = b$ by computing an approximate solution of the available linear system of equations (1) which is equivalent to find an approximate solution of the linear operator equation (6) by using GCV method for the optimal regularization parameter. To compare the effectiveness of our approach, it is hard to determine whether one method is better than the others just by looking at the images; therefore, it is necessary to compute the relative error of each solution

$$\frac{||\hat{X} - X_k||_F}{||\hat{X}||_F},$$

where X_k is the approximate solution of the linear operator equation (6).

In some cases the horizontal and vertical components of the blur PSF can be separated. If this is the case, the blurring matrix H given in (1) can be decomposed as a Kronecker product $H = H_2 \otimes H_1$ and the blurred image is then given by $H_1^T \hat{X} H_2$. In the nonseparable case, one can approximate the matrix H by solving the Kronecker product approximation (KPA) problem [37] $(\hat{H}_1, \hat{H}_2) = \arg\min_{H_1, H2} \| H - H_2 \otimes H_1 \|_F$,

Example 1. For a test we use a PSF array for out-of-focus blur [22], where the entris are given by

$$(\text{PSF})_{ij} = \begin{cases} \frac{1}{(\pi r^2)}, & \text{if} \quad (i-k)^2 + (j-l)^2 \leq r^2, \\ 0 & \qquad\qquad\qquad \text{else} \end{cases}$$

where (k, l) is the center of PSF, and r is the radius of the blur. For a test we set $r = 3$. In this example the original image is the cameraman image of dimension 256×256 from Matlab and it is shown in the left of Figure 1. The blurred and noisy image of Figure 1 has been built by the product $H_1^T \hat{X} H_2$ and adding a 0.001 noise level. In this example, the matrix L in the linear discrete ill-posed problem is chosen to be the identity of size $256^2 \times 256^2$. Using the GCV method, the

Fig. 1: Original image (left), degraded image (center) and restored image (right).

regularization parameter is given by: $\mu = 8.7 \times 10^{-3}$. For this optimization parameter value, the restored image determined by solving the linear operator equation with Gl-CG is shown in the right of Figure 1. The relative error of restored image was 5.13×10^{-2}.

Example 2. In this example the original image is the vendredi image of dimension 256×256 from Matlab and it is shown in the left of Figure 2. The blurring matrix H is given by $H = H_2 \otimes H_1 \in \mathbb{R}^{256^2 \times 256^2}$, where $H_1 = H_2 = [h_{ij}]$ and $[h_{ij}]$ is the Toeplitz matrix of dimension 256×256 given by

$$h_{ij} = \begin{cases} \frac{1}{2r-1}, |i-j| \leq r, \\ 0 \qquad \text{otherwise} \end{cases}$$

The blurring matrix H models a uniform blur. In our example we set $r = 5$. The blurred and noisy image of Figure 2 has been built by the product $H_1^T \hat{X} H_2$ and adding a 0.01 noise level. In this example, the matrix L in the linear discrete ill-posed problem is chosen to be the identity of size $256^2 \times 256^2$. Using the GCV method, the regularization parameter

Fig. 2: Original image (left), degraded image (center) and restored image (right).

is given by: $\mu = 9.7907 \times 10^{-4}$. For this optimization parameter value, the restored image determined by solving the linear operator equation (6) with Gl-CG is shown in the right of Figure 2. The relative error of restored image was 1.392×10^{-1}.

Example 3. In this example the original image is the iograyBorder image of dimension 256×256 from Matlab and it is shown in the left of Figure 3. The blurring matrix H is given by $H = H_2 \otimes H_1 \in \mathbb{R}^{256^2 \times 256^2}$, where $H_1 = H_2 = [h_{ij}]$ and $[h_{ij}]$ is the Toeplitz matrix of dimension 256×256 given by

$$h_{ij} = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(i-j)^2}{2\sigma^2}\right), |i-j| \leq r, \\ 0 \qquad\qquad\qquad\qquad \text{sinon} \end{cases}$$

The blurring matrix H models a blur arising in connection with the degradation of digital images by atmospheric turbulence blur. We let $\sigma = 5$ and $r = 35$. The blurred and noisy image of Figure 3 has been built by the product $H_1^T \hat{X} H_2$ and adding a 0.001 noise level. In this example, the matrix L in the linear discrete ill-posed problem is chosen to be the identity of size $256^2 \times 256^2$. Using the GCV method, the regularization parameter is given by: $\mu = 9.2149 \times 10^{-4}$. For this optimization parameter value, the restored image determined by solving the linear operator equation (6) with Gl-LSQR is shown in the right of Figure 3. The relative error of restored image was 8.04×10^{-2}.

Fig. 3: Original image (left), degraded image (center) and restored image (right).

7 Conclusion

We have considered two filtering methods for image restoration and a method for choosing the regularization parameter. We have demonstrated how to apply the Gl-LSQR and Gl-CG algorithm to solve the generalized Sylvester matrix equation. The implementation of these methods illustrate that our techniques are efficient.

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