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Taylor polynomial solution of difference equation with constant coefficients via time scales calculus

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Abstract: In this study, we present a practical matrix method to find an approximate solution of higher order linear difference equation with constant coefficients under the initial-boundary conditions in terms of Taylor polynomials. To obtain this goal, we first present time scale extension of previous polynomial approach, then restrict the formula to the Integers with h step. This method converts the difference equation to a matrix equation, which may be considered as a system of linear algebraic equations.

Keywords: Time Scales Calculus, Difference Equations, Matrix Method.

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [1] in order to unify continuous and discrete analysis. This theory is appealing because it provides a useful tool for modeling dynamical processes. Time scale calculus has been created in order to unify the study of differential and difference equations, so called dynamic equations [2,3]. The concept of dynamic equations has motivated a huge size of research work in recent years [4,5,6,7]. Since time scale calculus has main purposes as unification, extension and discretization [8], it allows us to use the theory to solve a difference equation by the methods are used to solve differential equations. Several numerical methods were used such as the successive approximations, Adomian decomposition, Chebyshev and Taylor collocation, Haar Wavelet, Tau and Walsh series etc [9]–[17]. Since the beginning of the 1994, Taylor and Chebyshev matrix methods have also been used by Sezer et. al. to solve linear differential, Fredholm integral and Fredholm integro differential-difference equations.

Since time scale calculus has unification purpose, it allows us to consider so called dynamic equations as the unification of differential and difference equations. Also, the Taylor matrix method has been used to find the approximate solutions of differential, integral and integro-differential equations in recent years. In this paper we present the time scale analogue of this matrix method in terms of delta derivatives and restrict this method to Integers with h step to solve higher order difference equations with constant coefficients.

This approach is based on the matrix relations between Taylor polynomials and their derivatives which are modified to

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solve the n-th order linear ODE with constant coefficients

$$\sum_{k=1}^{n} P_k y^{(k)}(x) = g(x) \tag{1}$$

under the initial-boundary conditions

$$\sum_{k=1}^{n-1} a_{ik} y^k(a) + b_{ik} y^k(b) = \lambda_i \ , \ \ i = 1,2,\ldots,n-1$$

where a_{ik} , b_{ik}, and λ_i are constants. In the method of Taylor polynomial solution to differential equations; it is assumed that the solution can be constructed by the Taylor polynomial form

$$y(x) = \sum_{n=0}^{N} y_n (x-c)^n, \ \ y_n = \frac{y^{(n)}(c)}{n!}, \ \ , a \le c \le b$$

such that the Taylor coefficients to be determined are y_n for $n = 1,2,\ldots,N$.

In this paper, we study a matrix method to find an approximate solution of higher order linear difference equation with constant coefficients under the initial-boundary conditions in terms of Taylor polynomials via the time scales calculus extension of the method. In Section 2, we give the extension of the matrix method for general time scales. The solution to a generalized dynamic equation of higher order with constant coefficients. We also study the accuracy of the error. In Section 3, we consider the dynamic equation as a difference equation with constant coefficient by choosing the generalized time scale as a particular one, that is $h\mathbb{Z}$. Then, we obtain the matrix relations for the present method. In section 4, we give our conclusions about the method.

2 Time scales calculus extension of taylor polynomial approach

Let us consider the time scales analogue of Eq. 1; i.e. n-th order dynamic equation with constant coefficients:

$$\sum_{k=1}^{n} P_k y^{\Delta^k}(t) = g(t) \tag{2}$$

where Δ^k denotes k-th order delta derivative of $y : \mathbb{T} \to \mathbb{R}$, $\mathbb{T}$ is in an arbitrary time scale, and $\forall t \in \mathbb{T}^\kappa$.

In [2], Agarwal et. al. presented Taylor polynomial for arbitrary time scales as follows:

$$y(t) \approx \sum_{k=1}^{n} h_k(t,c) y^{\Delta^k}(c), \tag{3}$$

where $t \in \mathbb{T}^\kappa$, $c \in \mathbb{T}$ for an arbitrary time scale; and $h_k(t,c)$ can be defined recursively as

$$h_0(t,c) = 1, \ \ h_{k+1}(t,c) = \int_t^c h_k(\tau,c)\Delta\tau.$$

If we rewrite the equation (3) in the matrix form, we may obtain the matrix relation

$$y(t) = \mathrm{H}(t)\mathrm{Y}, \tag{4}$$

where

$$\mathrm{H}(t) = \left[h_0(t,c)\ h_1(t,c)\ \cdots\ h_N(t,c) \right] \tag{5}$$

and

$$\mathrm{Y} = \left[y(c)\ y^{\Delta}(c)\ \cdots\ y^{\Delta^k}(c) \right]^{\mathrm{T}}. \tag{6}$$

On the other hand, it is straightforward that the relation between the matrix $\mathrm{H}(t)$ and its delta derivative $\mathrm{H}^{\Delta}(t)$ is

$$\mathrm{H}^{\Delta}(t) = \mathrm{H}(t)\mathrm{B}, \tag{7}$$

where

$$\mathrm{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the means of the matrix equation (7), it follows that

$$\begin{aligned} \mathrm{H}^{\Delta}(t) &= \mathrm{H}(t)\mathrm{B} \\ \mathrm{H}^{\Delta^2}(t) &= \mathrm{H}^{\Delta}(t)\mathrm{B} = \mathrm{H}(t)\mathrm{B}^2 \\ &\vdots \\ \mathrm{H}^{\Delta^k}(t) &= \mathrm{H}(t)\mathrm{B}^k. \end{aligned}$$

By using the equations (4), (7), and delta derivative relations of $\mathrm{H}(t)$, we obtain the recurrence relations

$$y^{\Delta^k}(t) = \mathrm{H}^{\Delta^k}(t)\mathrm{Y} = \mathrm{H}(t)\mathrm{B}^k\mathrm{Y}, \ \ k = 1,2,\ldots,n. \tag{8}$$

Substituting the equation (8) to (2) leads us to the matrix relation

$$\sum_{k=1}^{n} \mathrm{P}_k \mathrm{H}(t)\mathrm{B}^k\mathrm{Y} = g(t). \tag{9}$$

Now let us consider the Taylor expansion of $g(t)$:

$$g(t) \approx \sum_{k=1}^{n} h_k(t,c) g^{\Delta^k}(c).$$

It is also possible to consider this expansion as the matrix form

$$g(t) = \mathrm{H}(t)\mathrm{G}, \tag{10}$$

where $\mathrm{G}=\left[g(c)\ g^{\Delta}(c)\ \cdots\ g^{\Delta^k}(c)\right]^{\mathrm{T}}$.

We are now able to construct the fundamental matrix equation corresponding to the equation (2). Substituting the matrix relation (10) into the equation (2), then simplifying lead us to fundamental matrix equation:

$$\left\{\sum_{k=0}^{n}\mathrm{P}_k\mathrm{B}^k\right\}\mathrm{Y}=\mathrm{G}. \tag{11}$$

Briefly, equation (11) can be written in the form of a augmented matrix as

$$[\mathrm{W}:\mathrm{G}], \tag{12}$$

where

$$\mathrm{W}=[\omega_{pq}]=\sum_{k=0}^{n}\mathrm{P}_k\mathrm{B}^k,\ \ p,q=1,2,\ldots,N.$$

Now let us consider matrix representation of the initial boundary conditions. We can obtain the corresponding matrix form for the conditions

$$\sum_{k=0}^{m-1}a_k y^{\Delta^k}(a)+b_k y^{\Delta^k}(b)=\lambda_i,\ \ i=1,2,\ldots,m-1.$$

This forms leads us to following matrix equations:

$$\sum_{k=0}^{m-1}\left(a_{ik}\mathrm{H}(a)+b_{ik}\mathrm{H}(a)\right)\mathrm{B}^k\mathrm{Y}=[\lambda_i],$$

where $i=1,2,\ldots,m-1$. We can also consider this equation in the form of

$$\mathrm{U}_i\mathrm{Y}=\lambda_i,\ \ i=1,2,\ldots,m-1\ , \tag{13}$$

where

$$\mathrm{U}_i=\sum_{k=0}^{m-1}\left(a_{ik}\mathrm{H}(a)+b_{ik}\mathrm{H}(a)\right)\mathrm{B}^k,\ \ i=1,2,\ldots,m-1.$$

To obtain the solution of equation (2) under the initial-boundary conditions, if we replace the row matrices (13) by the last m rows of the matrix (12), then we have the new augmented matrix

$$[\overline{\mathrm{W}}:\overline{\mathrm{G}}]=\begin{bmatrix}\omega_{00} & \omega_{01} & \cdots & \omega_{0N} & : & g_0\\ \omega_{10} & \omega_{11} & \cdots & \omega_{1N} & : & g_1\\ \vdots & \vdots & \ddots & \vdots & & \vdots\\ \omega_{N-m,0} & \omega_{N-m,1} & \cdots & \omega_{N-m,N} & : & g_{N-m}\\ U_{00} & U_{01} & \cdots & U_{0N} & : & \lambda_0\\ U_{10} & U_{11} & \cdots & U_{1N} & : & \lambda_1\\ \vdots & \vdots & \ddots & \vdots & & \vdots\\ U_{m-1,0} & U_{m-1,1} & \cdots & U_{m-1,N} & : & \lambda_N\end{bmatrix}.$$

If $rank\overline{\mathrm{W}}=rank[\overline{\mathrm{W}}:\overline{\mathrm{G}}]=N+1$, then

$$\mathrm{Y}=\left(\overline{\mathrm{W}}\right)^{-1}\overline{\mathrm{G}}, \tag{14}$$

and the matrix Y is uniquely determined. This solution is given by the Taylor polynomial solution.

2.1 Accuracy of the solution

We can easily check the accuracy of this solution method. Since the truncated Taylor series is an approximated solution of the equation (2), when the function $y_N(t)$ and its delta derivatives are substituted in equation (2), the resulting equation must be satisfied approximately, that is, for $\xi_q \in [a,b]$ and $q = 1,2,\ldots$

$$Error(\xi_q) \leq 10^{-\xi_q}.$$

If $\max\left\{10^{-\xi_q}\right\} = 10^{-\xi}$ for a positive integer ξ is prescribed, then the truncation limit N is increased, until the $\Delta Error(\xi_q)$ at each of the points becomes smaller than the prescribed $10^{-\xi}$. On the other hand, the error can be estimated by the function

$$E_N(t) = \sum_{k=0}^{n} P_k y_N^{\Delta^k}(t) - g(t).$$

For the sufficiently large enough N, if E_N is approaching to zero then the error decreases.

3 Taylor polynomial solution of higher order difference equation

In this section, we present approximate solution of higher order difference equation. For this purpose, we assume the time scale is integers with h step. Then, delta derivative of $y(t)$ becomes $\Delta_h y(t)$; i.e. h-difference of $y(t)$. We can also consider discrete analogue of Taylor expansion as follows:

$$y(t) \approx \sum_{k=0}^{N} \binom{t-c}{k} \Delta_h^k y(c).$$

For the sake of simplicity, we assume $h = 1$, and then corresponding difference equation with constant coefficient is

$$\sum_{k=1}^{n} P_k \Delta^k y(t) = g(t). \tag{15}$$

It is possible to construct the integer analogues of the previous method. The matrix relation of the $y(t)$ stills same with

$$\mathrm{H}(t) = \left[\, 1 \;\; t-c \;\; \binom{t-c}{2} \;\; \cdots \;\; \binom{t-c}{N} \,\right] \tag{16}$$

and

$$\mathrm{Y} = \left[\, y(c) \;\; \Delta y(c) \;\; \Delta^2 y(c) \;\; \cdots \;\; \Delta^N y(c) \,\right]^{\mathrm{T}}. \tag{17}$$

Recurrence relations of delta derivative of $y(t)$ which is presented in previous section also becomes

$$\Delta^k y(t) = \Delta^k \mathrm{H}(t) \mathrm{B}^k \mathrm{Y}, \;\; k = 1,2,\ldots,n. \tag{18}$$

By the equations (15–18) and the discrete Taylor expansion of the function $g(t)$ which is

$$g(t) \approx \sum_{k=0}^{N} \binom{t-c}{k} \Delta^k g(c),$$

we obtain that the fundamental matrix equation is equal to the equation (11).

4 Conclusions

In this study, we present a new technique to solve a difference equation with constant coefficients by using the extension property of the time scales calculus. The most important advantage of this method is that the result is obtained in terms of arbitrary time scales. Therefore, to obtain approximate solutions of various types of difference equations with constant coefficients one may just need to change the matrix $\mathrm{H}(t)$ rather than to change whole computation procedure. Also, it is possible to apply this method to other equations with constant coefficients since the method is fully dynamic.

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