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The Group of Transformation Which Preserving Distance on Cuboctahedron and Truncated Octahedron Space

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Abstract: Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions. In a Minkowski geometry the unit ball is a symmetric, convex closed set instead of the usual sphere in Euclidean space. In [14], it is shown that there are some geometries which unit spheres are cuboctahedron and truncated octahedron-which are Archimedean solids-, they are also Minkowski geometries. In geometry determining the group of isometries of a space with a metric is a fundamental problem. In this article we show that the group of isometries of the $3-$dimensional spaces covered by $CO-metric$ and $TO-metric$ are the semi-direct product of O_h and $T(3)$, where octahedral group O_h is the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the $3-$dimensional space.

Keywords: Archimedean solids; Isometry group; Octahedral Symmetry; Cuboctahedron; Truncated octahedron.

1 Introduction

Polyhedra have attracted the attention in time and in various fields of art, science and philosophy because of their symmetries since symmetry is the primary matter of aesthetic. Polyhedra consist of faces, edges and vertices; polygons that bound polyhedra are faces, line segments where faces meet are edges and the meeting points of edges are vertices.

If for any two points of a polyhedron the line segment joining the points is in the polyhedron then it is called ”convex”. Many thinkers have worked with convex polyhedra since the ancient Greeks. The Greeks defined two classes of convex equilateral polyhedron; the Platonic and the Archimedean. Johannes Kepler found a third class, the rhombic polyhedra and as a fourth class Eug ène Catalan defined ‘dual’ polyhedra of Archimedean solids and they are named Catalan solids. Unlike Platonic and Archimedean solids the faces of Catalan solids are not regular polygons, but just like Archimedean solids there are thirteen Catalan solids.

In commonly studied Euclidean space, distances are equally determined for all directions, that is Euclidean space is ”isotropic”. Non-isotropic situations can be modelled by the assumption that units for measuring lengths are different in different directions. The taxicab space, with taxicab metric determined for $P_1=(x_1,y_1z_1)$, $P_2=(x_2,y_2,z_2)$ in $\mathbb{R}^3$

$$d_T(P_1,P_2)=|x_1-x_2|+|y_1-y_2|+|z_1-z_2|$$

is an example of such a situation. For this situation unit circles and unit spheres are no longer similar to Euclidean ones, it turned out that they have polygonal or polyhedral shapes. By the mathematicians worked on metric geometry many

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examples of this kind of spaces have been found using different metrics [3,4,15]. For example, in taxicab and maximum space geometry the unit spheres are the octahedron and the hexahedron,respectively, which are the Platonic solids[12,9]. In CC-space geometry the unit sphere is a deltoidal icositetrahedron which is a Catalan solid [13]. In [14], the authors give the spaces which their spheres are truncated octahedron and cuboctahedron by equipping analytical 3-dimensional space with TO (truncated octahedron) and CO (cuboctahedron) metrics. Also, $\mathbb{R}^3_{CO}$ and $\mathbb{R}^3_{TO}$ denotes these spaces. These spaces are finite-dimensional real normed spaces or as they called are Minkowski spaces. Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry (and from the Minkowskian geometry of space-time). In a Minkowski geometry the linear structure is the same as the Euclidean one but distance is not uniform in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a certain symmetric closed convex set [15].

One of the fundamental problem in geometry for S, which is a space with d metric, is to define the G group of isometries. If S is Euclidean 3-dimensional space with usual metric then it is obviously known that G consist of translations, rotations, reflections, glide reflections and screw of the 3- dimensional space. One can see [1,2,3,4,6,8,9,10,11,12,13, 14,15,16,17,18,19,21,22,24] for some paper about isometry group This problem enforce us to find group of isometries of spaces which unit spheres are cuboctahedron and truncated octahedron. Thus we show that the group of isometries of $\mathbb{R}^3_{CO}$ and $\mathbb{R}^3_{TO}$ are the semi-direct product of O_h and $T(3)$, where octahedral group O_h is the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the $3-$dimensional space.

2 Preliminaries

It has been stated in [27], an Archimedean solid is a symmetric, semiregular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. A polyhedron is called semiregular if its faces are all regular polygons and its corners are alike. And, identical vertices are usually means that for two taken vertices there must be an isometry of the entire solid that transforms one vertex to the other. The Archimedean solids are the only 13 polyhedra that are convex, have identical vertices, and their faces are regular polygons (although not equal as in the Platonic solids). Five Archimedean solids are derived from the Platonic solids by truncating (cutting off the corners) a percentage less than 1/2. Two special Archimedean solids can be obtained by full truncating (percentage 1/2) either of two dual Platonic solids: the Cuboctahedron, which comes from truncating either a Cube, or its dual an Octahedron. And the icosidodecahedron, which comes from truncating either an Icosahedron, or its dual a Dodecahedron. Hence their "double name". Cuboctahedron is an Archimedean solid with 14 faces. 8 of these faces are triangular and 6 of them are square. Truncated octahedron is another 14 faces Archimedean solid and 8 of its faces are hexagonal and 6 of them are square (See Figure 1(a)-(b)).

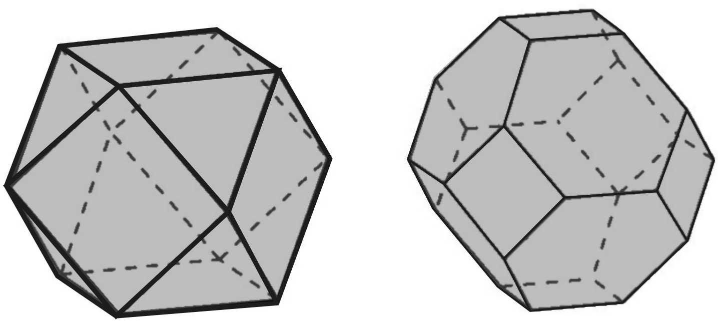

Figure 1(a): Cuboctahedron Figure 1(b): Truncated octahedron

The taxicab (Manhattan) and the maximum (Chebyshev) norms are defined as $\|X\|_1 = |x| + |y| + |z|$ and $\|X\|_\infty = \max\{|x|, |y|, |z|\}$, respectively and they are special cases of l_p-norm; $\|X\|_p = (|x|^p + |y|^p + |z|^p)^{1/p}$, where

$X = (x,y,z) \in \mathbb{R}^3$. Among l_p-metrics only crystalline metrics, i.e., metrics having polygonal unit balls are l_1- and $l_\infty-$ metrics [7].

In [14] CO and $TO-$metrics for $3-$dimensional analytical space were introduced. Now, the CO and $TO-$metrics and their some properties are given briefly from [14]. First we give some notions that will be used in the descriptions of distance functions we define. For $P_1 = (x_1,y_1,z_1)$, $P_2 = (x_2,y_2,z_2) \in \mathbb{R}^3$, M denotes $\|P_1 - P_2\|_\infty$ and S denotes $\|P_1 - P_2\|_1$. Moreover $X-Y-Z-X$ and $Z-Y-X-Z$ orientations are called positive (+) direction and negative (-) direction, respectively. M^+ and M^- expresses the next term in the respective direction according to M. For example, if $M = |x_1 - x_2|$, then $M^+ = |y_1 - y_2|$ and $M^- = |z_1 - z_2|$. Let $\mathbb{R}^3_{CO}$ and $\mathbb{R}^3_{TO}$ which are called cuboctahedron and truncated octahedron space denote $3-$dimensional analytical space furnished by cuboctahedron and truncated octahedron metric, respectively. CO and TO metrics between points $P_1 = (x_1,y_1 z_1)$ and $P_2 = (x_2,y_2,z_2)$ are defined as follows

$$d_{CO}(P_1,P_2)=\max\left\{M \ , \ \frac{1}{2}S\right\}$$

and

$$d_{TO}(P_1,P_2)=\max\left\{M \ , \ \frac{2}{3}S\right\}$$

According to definition of $d_{CO}-$distance and $d_{TO}-$distance there are two possible ways for the shortest paths between the points P_1 and P_2 as shown in Figure 2(a)-(b). These paths follows:

i) A line segment which is parallel to one of the coordinate axis,

ii) Union of three line segments each of which is parallel to one of the coordinate axis .

Thus, the shortest $d_{CO}-$distance between P_1 and P_2 is the Euclidean length of such a line segment in case (i) or $\frac{1}{2}$ times sum of the Euclidean lengths of three line segments in case (ii). Similarly, the shortest $d_{TO}-$distance between P_1 and P_2 is the Euclidean length of such a line segment in case (i) or $\frac{2}{3}$ times sum of the Euclidean lengths of three line segments in case (ii). Figure 2(a)-(b) shows the cuboctahedron and truncated octahedron ways from P_1 to P_2.

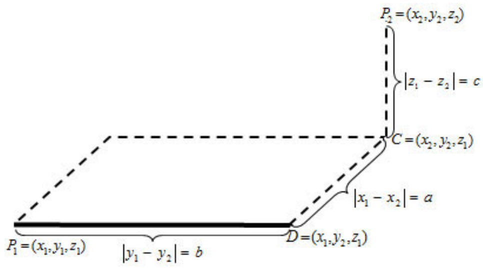

Figure 2(a)

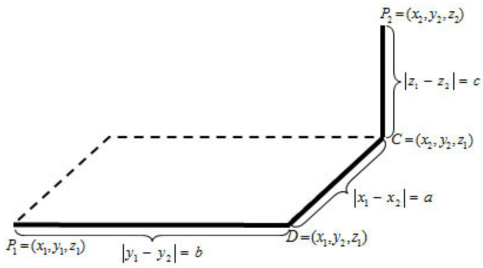

Figure 2(b)

Let $M_0 = \|X - X_0\|_\infty$ for $X = (x,y,z)$ and $X_0 = (x_0,y_0,z_0)$. A cuboctahedron sphere with center (x_0,y_0,z_0) and radius r in $\mathbb{R}^3_{CO}$ is the set of points (x,y,z) in the $3-$dimensional space satisfying the equation

$$\max\left\{M_0 \ , \ \frac{1}{2}\left(M_0 + M_0^+ + M_0^-\right)\right\} = r$$

Coordinates of the vertices are translations to (x_0,y_0,z_0) such that all permutations of the three axis components and all possible $+/-$ sign changes of each axis component of $(r,r,0)$. Figure 3(a) shows the $CO-$sphere which center is $O = (0,0,0)$.

Similarly, a truncated octahedron sphere with center (x_0,y_0,z_0) and radius r in $\mathbb{R}^3_{TO}$ is the set of points (x,y,z) in the

3−dimensional space satisfying the equation

$$\max\left\{M_0 \ ,\ \frac{2}{3}\left(M_0+M_0^{+}+M_0^{-}\right)\right\}=r$$

Coordinates of the vertices of a truncated octahedron sphere with radius r in $\mathbb{R}^3_{TO}$ are translations to (x_0,y_0,z_0) such that all permutations of the three axis components and all possible $+/-$ sign changes of each axis component of $\left(\frac{r}{2},r,0\right)$. Figure 3(b) shows the $TO-$ sphere which center is $O=(0,0,0)$.

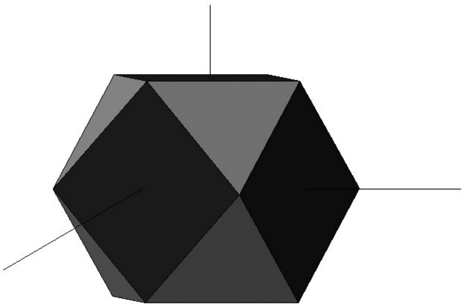

Figure 3(a): Cuboctahedron

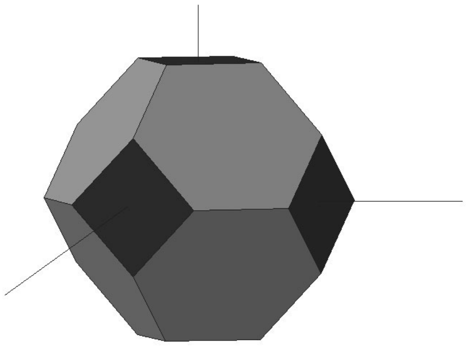

Figure 3(b): Truncated octahedron

The following lemma gives the relations between Euclidean metric and cuboctahedron metric or truncated octahedron metric.

Lemma 1. *Let $M_d=\|P\|_\infty$ for $P=(p,q,r)$. Let l be the line through the points $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ in $\mathbb{R}^3$. If l has direction vector (p,q,r) then*

$$d_{CO}(P_1,P_2)=\mu_{CO}(P_1P_2)\,d_E(P_1,P_2)$$

and

$$d_{TO}(P_1,P_2)=\mu_{TO}(P_1P_2)\,d_E(P_1,P_2)$$

where

$$\mu_{CO}(P_1P_2)=\frac{\max\left\{M_d \ ,\ \frac{1}{2}\left(M_d+M_d^{+}+M_d^{-}\right)\right\}}{\sqrt{p^2+q^2+r^2}}$$

and

$$\mu_{TO}(P_1P_2)=\frac{\max\left\{M_d \ ,\ \frac{2}{3}\left(M_d+M_d^{+}+M_d^{-}\right)\right\}}{\sqrt{p^2+q^2+r^2}}.$$

The above lemma states that d_{CO} and $d_{TO}-$distances along any line are some positive constant multiple of Euclidean distance along the same line. Consequently one can reach the following corollaries:

Corollary 1. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then*

$$d_E(P_1,X)=d_E(P_2,X) \iff d_{CO}(P_1,X)=d_{CO}(P_2,X)$$

and

$$d_E(P_1,X)=d_E(P_2,X) \iff d_{TO}(P_1,X)=d_{TO}(P_2,X).$$

Corollary 2. *If P_1, P_2 and X are any three distinct collinear points in $\mathbb{R}^3$, then*

$$\frac{d_E(X,P_1)}{d_E(X,P_2)}=\frac{d_{CO}(X,P_1)}{d_{CO}(X,P_2)}=\frac{d_{TO}(X,P_1)}{d_{TO}(X,P_2)}.$$

In the following part of this work, we will study the isometries of $\mathbb{R}^3_{CO}$ and $\mathbb{R}^3_{TO}$ and determine their groups of isometries.

3 Isometries of CO and TO Spaces

S be a space with d metric, that is S be a geometry. One of the fundamental problem for S is to define the G group of isometries. If S is Euclidean geometry, which is 3-dimensional space with usual metric d_E, then it is certainly known that G consists of translations, rotations, reflections, glide reflections and screw of the $3-$dimensional space. We used definitions of transformations from [25].

∘ A *transformation* is one to one equivalence from the set of points in space onto itself. If $d(X,Y) = d(\alpha(X), \alpha(Y))$ for every point X and Y, then α transformation is named an *isometry.*

∘ For all points X, if $\iota(X) = X$, then ι is called *identity.*

∘ If α fixes which set of points then α isometry is called a symmetry.

∘ For Δ plane, If $\sigma_\Delta(X) = X$ for point X on Δ and if $\sigma_\Delta(X) = Y$ for point X off Δ and Δ is perpendicular bisector of XY line segment, then σ_Δ, which is mapping on the points in $\mathbb{R}_{TH}$, is called reflection.

∘ $\sigma_\Delta \sigma_\Gamma$ is defined *a rotation* about axis *l,* if Γ and Δ are two intersecting planes at line *l.*

∘ $\sigma_\Pi \sigma_\Delta \sigma_\Gamma$ is defined *a rotary reflection* about the common point to Γ, Δ and Π if Γ, Δ ,which each one perpendicular to Π, are intersecting planes.

∘ If $\sigma_N(X) = Y$ for every X points and N is midpoint of X and Y, then σ_N inversion about N is called *a transformation.* At the same time σ_N is defined *a point reflection.*

In [20] the author gives the following theorem:

Theorem 1.*If the unit ball C of* $(V, \|\|)$ *does not intersect a two-plane in an ellipse, then the group* $I(3)$*of isometries of* $(V, \|\|)$ *is isomorphic to the semi-direct product of the translation group* $T(3)$ *of* $\mathbb{R}^3$ *with a finite subgroup of the group of linear transformations with determinant* ± 1.

After this theorem remains a single question. This question is that what is the relevant subgroup?

Now we will show that all isometries of the $\mathbb{R}^3_{CO}$ and $\mathbb{R}^3_{TO}$ are in $T(3).G(CO)$ and $T(3).G(TO)$, respectively. In the rest of article, we take $\triangle = CO$ or $\triangle = TO$. That is, $\triangle \in \{CO, TO\}$.

Definition 1.*Let A and B be two points in* $\mathbb{R}^3_\triangle$. *The minimum distance set of* A, B *is defined by*

$$\left\{X \mid d_\triangle(A,X) + d_\triangle(B,X) = d_\triangle(A,B)\right\}$$

and denoted by $[AB]$.

In $\mathbb{R}^3_{CO}$, $[AB]$ is a parallelepiped and in $\mathbb{R}^3_{TO}$, $[AB]$ is a hexagonal dipyramid like as seen in Figure 4(a)-(b), respectively.

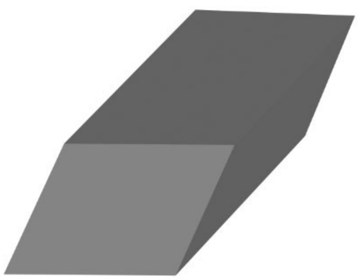

Figure 4(a): CO Min. Dis. Set

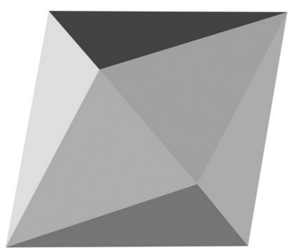

Figure 4(b): TO Min. Dis. Set

Proposition 1.*Let $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ be an isometry and let $[AB]$ be the minimum distance set of A, B. Then*

$$\phi([AB]) = [\phi(A)\phi(B)]$$

*Proof.*Let $Y \in \phi([AB])$.Then,

$$\begin{aligned} Y \in \phi([AB]) &\Longleftrightarrow \exists X \in [AB] \ni Y = \phi(X) \\ &\Longleftrightarrow d_\triangle(A,X) + d_\triangle(B,X) = d_\triangle(A,B) \\ &\Longleftrightarrow d_\triangle(\phi(A),\phi(X)) + d_\triangle(\phi(B),\phi(X)) = d_\triangle(\phi(A),\phi(B)) \\ &\Longleftrightarrow Y = \phi(X) \in [\phi(A)\phi(B)]\,. \end{aligned}$$

Corollary 3.*Let $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ be an isometry and $[AB]$ be the minimum distance set of A, B. Then ϕ maps vertices to vertices and preserves the lengths of the edges of $[AB]$.*

Proposition 2.*Let $f : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ be an isometry such that $f(O) = O$. Then $f \in G(\triangle)$.*

*Proof.*Since $\triangle \in \{CO, TO\}$, there are two possibility for $\triangle$. Let $\triangle = CO$, and let $V_1 = (1,0,1)$, $V_5 = (1,1,0)$, $V_9 = (0,1,1)$ and $D = (2,2,2)$ be four points in $\mathbb{R}^3_{CO}$. Consider $[OD]$ that is the parallelepiped with diagonal OD. (See Figure 5(a)).

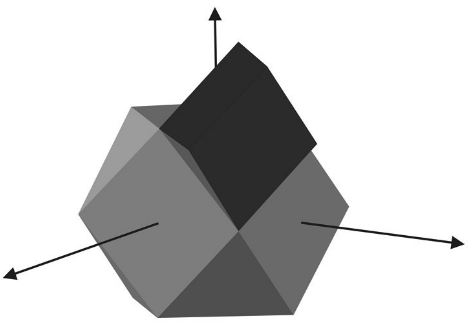

Figure 5(a)

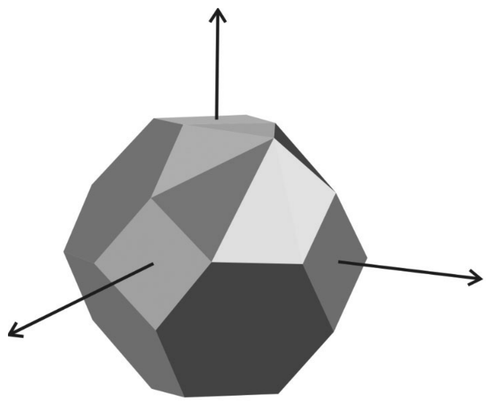

Figure 5(b)

Also points V_1, V_5, V_9 are on minimum distance set $[OD]$ and unit sphere with center at origin. Moreover these three points are the corner points of a cuboctahedron's face which is an equilateral triangle. f maps points V_1, V_5, V_9 to the vertices of a cuboctahedron by Corollary 3. Since f preserve the lengths of the edges, $f(V_1) = V_i$, $f(V_5) = V_j$ and $f(V_9) = V_k$ such that $i, j, k \in \{1,2,...,12\}$. Since cuboctahedron have 8 equilateral triangle faces, there are 8 possibility to points which they can map, and also there are six possibility to points which they can map on the equilateral triangle face of cuboctahedron. Therefore total number of possibility are 48. Some of these cases can be seen as follows:

If $f(V_1) = V_1$, $f(V_5) = V_5$ and $f(V_9) = V_9$, then f is identity.

If $f(V_1) = V_1$, $f(V_5) = V_6$ and $f(V_9) = V_{11}$ then $f = \sigma_\Delta$ is the reflection about the plane $\Delta : y = 0$.

If $f(V_1) = V_1$, $f(V_5) = V_{11}$ and $f(V_9) = V_6$ then $f = r_\pi$ is the rotation with rotation axis $\parallel \left(1/\sqrt{2}, 0, 1/\sqrt{2}\right)$

The remaining cases can be similarly given.

Let $\triangle = TO$, and let $V_1 = \left(\frac{1}{2},0,1\right)$, $V_{17} = \left(1,0,\frac{1}{2}\right)$, $V_5 = \left(1,\frac{1}{2},0\right)$, $V_{21} = \left(\frac{1}{2},1,0\right)$,$V_9 = \left(0,1,\frac{1}{2}\right)$, $V_{13} = \left(0,\frac{1}{2},1\right)$ and $D = (1,1,1)$ be seven points in $\mathbb{R}^3_{TO}$. Consider $[OD]$ that is the hexagonal dipyramid (See Figure 5(b)). Also points V_1, V_{17}, V_5, V_{21} ,V_9, V_{13} are on minimum distance set $[OD]$ and unit sphere with center at origin. Moreover these six points are the corner points of a cuboctahedron's face which is a regular hexagon. f maps points V_1, V_{17}, V_5, V_{21} ,V_9, V_{13} to the vertices of a cuboctahedron by Corollary 3. Since f preserve the lengths of the edges, $f(V_1)=V_i$, $f(V_{17})=V_j$, $f(V_5)=V_k$, $f(V_{21})=V_l$, $f(V_9)=V_m$, $f(V_{13})=V_n$ such that $i,j,k,l,m,n\in\{1,2,\ldots,24\}$ and ,V_iV_j, V_jV_k, V_kV_l, V_lV_m, V_mV_n and V_nV_i are six edges of the truncated octahedron. Since truncated octahedron have 8 regular hexagon faces, there are 8 possibility to points which they can map, and also there are 6 possibility to points which they can map on the face of truncated octahedron. Therefore total number of possibility are 48. Some of these cases can be seen as follows:

$f(V_1)=V_1$, $f(V_{17})=V_{17}$, $f(V_5)=V_6$,$f(V_{21})=V_{22}$, $f(V_9)=V_{11}$ and $f(V_{13})=V_{15}$ then $f=\sigma_\Delta$ the reflection about the plane $\Delta: y=0$

$f(V_1)=V_1$, $f(V_{17})=V_{17}$, $f(V_5)=V_5$,$f(V_{21})=V_{21}$, $f(V_9)=V_9$ and $f(V_{13})=V_{13}$ then f is identity

The remaining cases can be similarly given.

Theorem 2.*Let* $f:\mathbb{R}^3_\triangle\to\mathbb{R}^3_\triangle$ *be an isometry. Then there exists a unique* $T_A\in T(3)$ *and* $g\in G(\triangle)$ *where* $f=T_A\circ g$

*Proof.*Let $f(O)=A$ such that $A=(a_1,a_2,a_3)$. Define $g=T_{-A}\circ f$. We know that $g(O)=O$ and g is an isometry. Thereby, $g\in G(\triangle)$ and $f=T_A\circ g$ by Proposition 2. The proof of uniqueness is trivial.

Therefore, all isometries of the $\mathbb{R}^3_{CO}$ and $\mathbb{R}^3_{TO}$ are in $T(3).G(CO)$ and $T(3).G(TO)$, respectively. That is, the isometry groups of $\mathbb{R}^3_{CO}$ and $\mathbb{R}^3_{TO}$ are the semi-direct product of O_h and $T(3)$, where octahedral group O_h is the (Euclidean) symmetry of the octahedron and $T(3)$ is the group of all translations of the $3-$dimensional space.

4 Conclusion

In this paper, the spaces of which their sphere are the cuboctahedron and the truncated octahedron are introduced and some properties of metrics which are used setting up these space are given. Also, isometry groups of these spaces are given. So each of the groups of isometries of the spaces covering with the cuboctahedron and the truncated octahedron is the semi-direct product of the icosahedral group O_h and $T(3)$, where O_h is the (Euclidean) symmetry group of the octahedron and $T(3)$ is the group of all translations of the 3-dimensional space. In the future works, handled solids in this paper are Archimedian solids, the new metric space by considering different solids from these solids can be constructed and invastigate their some properties which are related to metrics.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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