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Some Solution Methods for Lane-Emden Differential Equation

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Abstract: The Lane-Emden equations have been used to model some phenomena in astrophysics, such as mathematical physics and stellar structure theory. In addition, the Lane-Emden equation is a central equation in the theory of stellar structures. In this paper, we introduce the Lane-Emden differential equations. A special second order Lane-Emden differential equation is solved by He's variational iteration, adomian decomposition method, homotopy analysis method, homotopy perturbation method, and finite difference method respectively. The results obtained are compared with each other and it is analyzed which method gives more reliable results and is more useful than the others.

Keywords: Homotopy analysis, homotopy perturbation, He's variational iteration, Adomian decomposition method, Lane-Emden differential equations.

1 Introduction

In recent years, many scientists have worked on the solution of second-order nonlinear differential equations problems in applied sciences. One of such equation is the Lane-Emden type equations. The Lane-Emden type equations, first published by Jonathan Homer Lane in 1870 [1] and studied by Emden [2]. The general form of the Lane-Emden equations with initial conditions is as follows:

$$y''(x)+\frac{k}{x}y'(x)+f(x,y)=g(x), \qquad kx\geq 0, k\in\mathbb{R} \tag{1}$$

$$y(0)=a, \quad y'(0)=0 \tag{2}$$

where $g(x,y)$ is a nonlinear function depending on the variables x and y. The analytical solution of the problem (1)-(2) is known to be in the neighborhood of the singular point $x=0$.
In the case of $k=2, \quad f(x,y)=y^n, \quad g(x)=0, \quad a=1$ the problem (1)-(2) becomes the following form

$$y''(x)+\frac{2}{x}y'(x)+y^n=0, \tag{3}$$

$$y(0)=1, y'(0)=0 \tag{4}$$

Equation (3) can be rewritten as

$$\frac{1}{x^2}\frac{d}{dx}(x^2\frac{dy}{dx})+y^n=0 \tag{5}$$

Equation (5) is also known as the Lane-Emden equation. In astrophysics, this equation is also called Poisson's equation [3]. A number of techniques and methods have been used to solve the Lane-Emden equations. Some of them are He's variational iteration method [4-5], adomian decomposition method [6], homotopy analysis method [7-8], homotopy perturbation method [9-10], finite difference methods [11-12], and other studies.

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In this paper, a special type of Lane-Emden equation is solved by different methods and comparisons of the methods are presented. The following special type of Lane-Emden equations

$$y''(x)+\frac{2}{x}y'(x)+y^5=0, \tag{6}$$

with boundary conditions are given by

$$y(0)=1, y'(0)=0 \tag{7}$$

is solved by different methods and comparisons are made.
The exact solution of the equations (6)-(7) is [13]

$$y(x)=\sqrt{\frac{3}{3+x^2}} \tag{8}$$

Some numerical values from the exact solution are in the following table:

Table 1 Numerical values of (8)

x	Exact Solution
0.1	0.99833
0.2	0.99339
0.3	0.98532
0.4	0.97435
0.5	0.96077

Now the same differential equation will be solved by different methods and then results of these methods are compared.

1.1 The Solution with He's Variational Method

The correction function for equation (6) is defined as

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(x,s)[y_s''(s)+\frac{2}{s}y'_s(s)+y^5{}_s(s)]ds \tag{9}$$

where Lagrange multiplier is

$$\lambda(x,s)=\frac{s(s-x)}{x} \tag{10}$$

Since the first approximation function must satisfy the initial conditions of the problem, we can take $y_0(x)=1$.
Then we have

$$y_1(x)=1-\frac{x^2}{6}, \tag{11}$$

$$y_2(x)=1-\frac{x^2}{6}+\frac{x^4}{24}+O(x^6), \tag{12}$$

$$y_3(x)=1-\frac{x^2}{6}+\frac{x^4}{24}-\frac{5x^6}{432}+O(x^8), \tag{13}$$

$$y_4(x)=1-\frac{x^2}{6}+\frac{x^4}{24}-\frac{5x^6}{432}+\frac{35x^8}{10368}+O(x^{10}), \tag{14}$$

$$\vdots$$

By continuing in this way, the function $y_n(x)$ is obtained as

$$y_n(x)=1-\frac{x^2}{6}+\frac{x^4}{24}-\frac{5x^6}{432}+\frac{35x^8}{10368}-\frac{7x^{10}}{6912}+\dots \quad . \tag{15}$$

It can be seen that this series corresponds to the Taylor expansion of the exact solution of function (8) around zero. Approximate values for different numbers of terms taken from the series are shown in the following table

Table 2 Numerical values of equation (3) for He's VIM

x	y_1	y_2	y_3	y_4
0.1	0.9983	0.9983	0.99833	0.99833
0.2	0.9983	0.9934	0.99339	0.99339
0.3	0.985	0.98533	0.98532	0.98532
0.4	0.9733	0.9744	0.97435	0.97435
0.5	0.9583	0.96093	0.96075	0.96077

1.2 The Solution with Adomian Decomposition Method

L is a linear operator and L^{-1} is a inverse operator defined by

$$L(.) = x^{-2}\frac{d}{dx}(x^2\frac{d}{dx})(.) \tag{16}$$

$$L^{-1}(.) = \int_0^x x^{-2}\int_0^x x^2(.)\,dxdx. \tag{17}$$

We have

$$Ly = -y^5, \tag{18}$$

$$y(x) = y(0) - L^{-1}y^5 \tag{19}$$

Adomian formulas are as follows

$$\sum_{n=0}^{\infty} y_n = 1 - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{20}$$

$$y_{n+1} = L^{-1}A_n, \tag{21}$$

$$y_{n+1} = -\int_0^x x^{-2}(\int_0^x x^2 A_i\,dx)dx, \qquad i = 0,1,2,.... \tag{22}$$

The Adomian polynomials for $f(x,y) = y^5$ are thus given by

$$A_0 = 1, \tag{23}$$

$$A_1 = -\frac{x^2}{6}, \tag{24}$$

$$A_2 = \frac{35x^4}{72}, \tag{25}$$

$$A_3 = -\frac{5x^6}{432}, \tag{26}$$

$$A_4 = -\frac{35x^8}{144}, \tag{27}$$

$$\vdots$$

There fore the first five solutions are obtained by

$$y_0 = 1, \tag{28}$$

$$y_1 = -\frac{x^2}{6}, \tag{29}$$

$$y_2 = \frac{x^4}{24}, \tag{30}$$

$$y_3 = -\frac{5x^6}{432}, \tag{31}$$

$$y_4 = -\frac{35x^8}{10368}, \tag{32}$$

$$\vdots$$

Hence, the approximate solution of $y(x)$ obtained using the $y_i \quad (i = 0,1,2,...)$ is given by

$$y = y_0 + y_1 + y_2 + y_3 + y_4 + ...$$

$$y = 1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432} + \frac{35x^8}{10368} + ... \quad . \tag{33}$$

It can be seen that this series corresponds to the Taylor expansion of the exact solution of function (8) around zero. Approximate values for different numbers of terms taken from the series are shown in the table that follows.

Table 3 Numerical values of equation (3) for ADM

x	y_1	y_2	y_3	y_4
0.1	0.9983	0.9983	0.99833	0.99833
0.2	0.9933	0.9934	0.99339	0.99339
0.3	0.985	0.98533	0.98532	0.98532
0.4	0.9733	0.9744	0.97435	0.97435
0.5	0.9583	0.96093	0.96075	0.96077

1.3 The Solution with Homotopy Perturbation Method

Substituting the nonlinear differential equation in He's formula gives

$$H(v,p) = (1-p)[L(v) - L(y_0)] \; + p[A(y) - f(r)] = 0 \tag{34}$$

$$H(v,p) = (1-p)[v'' + \frac{2}{x}v' - y_0'' - \frac{2}{x}y_0'] + p[v'' + \frac{2}{x}v' - v^5] = 0. \tag{35}$$

Taking $p = 0,1$ in equation (35) we obtain

$$H(v,0) = [v'' + \frac{2}{x}v' - y_0'' - \frac{2}{x}y_0'] \tag{36}$$

$$H(v,1) = v'' + \frac{2}{x}v' - v^5 = 0. \tag{37}$$

Assuming the solution is in the form of power series, we have

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + O(p^4), \tag{38}$$

$$v' = v_0' + pv_1' + p^2v_2' + p^3v_3' + O(p^4), \tag{39}$$

$$v'' = v_0'' + pv_1'' + p^2v_2'' + p^3v_3'' + O(p^4). \tag{40}$$

Substitute Equation (38), (39), (40) into equation (34)

$$(v_0'' + pv_1'' + p^2v_2'' + p^3v_3'' + ...) + \frac{2}{x}(v_0' + pv_1' + p^2v_2' + p^3v_3') - y_0'' - \frac{2}{x}y_0' + p[y_0'' + \frac{2}{x}y_0'] + p(v_0 + pv_1 \; + p^2v_2 + p^3v_3)^5] = 0. \tag{41}$$

For p_0,

$$v_0'' + \frac{2}{x}v_0' - y_0'' - \frac{2}{x}y_0' = 0 \tag{42}$$

$$v_0(0) = 1,\ v_0'(0) = 0.$$

Then we have

$$v_0(x) = 1[10]. \tag{43}$$

For p_1,

$$v_1'' + \frac{2}{x}v_1' + y_0'' + \frac{2}{x}y_0' + v_0{}^5 = 0 \tag{44}$$

$$v_1(0) = 0,\ v_1'(0) = 0.$$

We apply Taylor series to obtain the solution for $v_1(x)$,

$$v_1'' + \frac{2}{x}v_1' + 1^5 = 0, \tag{45}$$

$$v_1' = \frac{-1}{2}x - \frac{1}{2}xv_1'', \tag{46}$$

$$v_1'' = \frac{-1}{3}(x)' - \frac{1}{3}xv_1''', \tag{47}$$

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + ... \tag{48}$$

Then, we have

$$v_1(x) = -\frac{1}{6}x^2[10]. \tag{49}$$

For p_2,

$$v_2'' + \frac{2}{x}v_2' + 5v_1v_0{}^4 = 0 \tag{50}$$

$$v_2(0) = 0,\ v_2'(0) = 0.$$

We apply Taylor series to obtain the solution for $v_2(x)$,

$$v_2'' + \frac{2}{x}v_2' + v_1'' + 5v_1v_0{}^4 = 0, \tag{51}$$

$$v_2'' = \frac{1}{3}(\frac{5}{6}x^3)' - \frac{1}{3}xv_2''', \tag{52}$$

$$v_2''' = \frac{1}{4}(\frac{5}{6}x^3)'' - \frac{1}{4}xv_2{}^{(4)}, \tag{53}$$

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + ... \tag{54}$$

Then we have

$$v_2(x) = \frac{x^4}{24}[10]. \tag{55}$$

For p_3,

$$v_3'' + \frac{2}{x}v_3' + 5v_2v_0{}^4 + \frac{5\cdot 4}{2}v_1{}^2v_0{}^3 = 0 \tag{56}$$

$$v_3(0) = 0,\ v_3'(0) = 0.$$

We apply Taylor series to obtain the solution for $v_3(x)$,

$$v_3'' + \frac{2}{x}v_3' + 5v_2v_0{}^4 + \frac{5\cdot 4}{2}v_1{}^2v_0{}^3 = 0, \tag{57}$$

$$v_3' = -\frac{1}{2}(\frac{5^2}{120} + \frac{5\cdot 4}{72})x^4 - \frac{1}{2}xv_3'', \tag{58}$$

$$v_3'' = -\frac{1}{3}(\frac{5^2}{120} + \frac{5\cdot 4}{72})x^4 - \frac{1}{3}xv_3''', \tag{59}$$

$$v_3''' = -\frac{1}{4}(\frac{5^2}{120} + \frac{5\cdot 4}{72})x^4 - \frac{1}{4}xv_3{}^{(4)}, \tag{60}$$

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \ldots. \tag{61}$$

Then, we obtain

$$v_3(x) = \frac{-5x^6}{432}[10]. \tag{62}$$

By continuing in this way, the function y is obtained as

$$y \approx v = v_0 + pv_1 + p^2v_2 + p^3v_3 + O(p^4). \tag{63}$$

For

$$y = \lim_{p\to 1} v, \tag{64}$$

we get

$$y = v_0 + pv_1 + p^2v_2 + p^3v_3 + O(p^4) \tag{65}$$

$$y = 1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432} + \ldots . \tag{66}$$

It can be seen that this series corresponds to the Taylor expansion of the (8) exact solution function around zero. Approximate values for different numbers of terms taken from the series are shown in the table that follows.

Table 4 Numerical values of equation (3) for HPM

x	p	p^2	p^3
0.1	0.9983	0.9983	0.99833
0.2	0.9933	0.9934	0.99339
0.3	0.985	0.98533	0.98532
0.4	0.9733	0.9744	0.97435
0.5	0.9583	0.96093	0.96075

1.4 The Solution with Homotopy Analysis Method

Integration constants

$$L[\phi(x;q)] = \phi''(x;q) + \frac{2}{x}\phi'(x) \tag{67}$$

and

$$L[\frac{-c_1}{x} + c_2] = 0, \quad c_i(i = 1,2) \tag{68}$$

where L is a linear operator. We define the nonlinear operator as

$$N[\phi(x;q)] = \phi''(x;q) + \frac{2}{x}\phi'(x) + \phi^5. \tag{69}$$

Then, we construct the zeroth-order deformation equation

$$(1-q)L[\phi(x;q)-y_0(x)] = q\tilde{h}H(x)N[\phi(x;q)] \tag{70}$$

and

$$\phi(x;q)=\phi(x;0)=y_0(x), \quad \phi(x;1)=y(x). \tag{71}$$

The exact solution of the (8) is

$$y(x)=y_0(x)+\sum_{m=1}^{\infty} y_m(x) \tag{72}$$

Then we have m th-order deformation equation

$$y_m(x)=\chi_m y_{m-1}(x)+\tilde{h}L^{-1}R_m(\vec{y}_{m-1}(x)), \tag{73}$$

where

$$R_1(\vec{y}_0(x)) = {y_0}'' + \frac{2}{x}{y_0}' + {y_0}^5 \tag{74}$$

$$R_2(\vec{y}_1(x)) = {y_1}'' + \frac{2}{x}{y_1}' + 5y_1{y_0}^4 \tag{75}$$

$$R_3(\vec{y}_2(x)) = {y_2}'' + \frac{2}{x}{y_2}' + 5y_2{y_0}^4 + 20\frac{{y_1}^2}{2!}{y_0}^3, \tag{76}$$

$$R_4(\vec{y}_3(x)) = {y_3}'' + \frac{2}{x}{y_3}' + 5y_3{y_0}^4 + 20y_2y_1{y_0}^3 + 60\frac{{y_1}^3}{3!}{y_0}^3, \tag{77}$$

$$\vdots$$

are obtained when $m=1,2,3,....$

If we substitute these expressions in (73), we get as follows

$$y_1 = \frac{\tilde{h}x^2}{6} \tag{78}$$

$$y_2 = \frac{\tilde{h}x^2}{6} + \tilde{h}\left(\frac{\tilde{h}x^2}{6} + \frac{\tilde{h}x^4}{24}\right) \tag{79}$$

$$y_3 = \frac{\tilde{h}x^2}{6} + \tilde{h}\left(\frac{\tilde{h}x^2}{6} + \frac{\tilde{h}x^4}{24}\right) + \tilde{h}\left(\frac{\tilde{h}x^2}{6} + \frac{\tilde{h}^2x^2}{6} + \frac{\tilde{h}x^4}{24} + \frac{\tilde{h}^2x^4}{12} + \frac{5\tilde{h}^2x^6}{432}\right), \tag{80}$$

$$\vdots$$

For $\tilde{h}=-1$, we get,

$$y = y_0+y_1+y_2+y_3+y_4+...$$

$$y = 1-\frac{x^2}{6}+\frac{x^4}{24}-\frac{5x^6}{432}+\frac{35x^8}{10368}+... \tag{81}$$

It can be seen that this series corresponds to the Taylor expansion of the (8) exact solution function around zero. Approximate values for different numbers of terms taken from the series are shown in the table that follows.

Table 5 Numerical values of equation (3) for HAM

x	$\tilde{h}=-1$	$\tilde{h}=1$	$\tilde{h}=-0.5$
0.1	0.9983	1.02507	0.99844
0.2	0.9933	1.10114	0.993796
0.3	0.9853	1.2308	0.986167
0.4	0.9743	1.41847	0.975719
0.5	0.9607	1.67055	0.962672

1.5 The Solution with Finite Difference Method

If centered finite difference approximations are applied to the derivatives of y' and y'' , and the errors are neglected we have

$$\frac{y(x_i-h)-2y(x_i)+y(x_i+h)}{h^2}+\frac{2}{x_i}\frac{y(x_{i+1})-y(x_{i-1})}{2h}+y(x_i)^5=0 \tag{82}$$

where

$$h=\frac{(1-0)}{(n+1)},\quad x_i=0+ih \quad i=1,\dots,n. \tag{83}$$

Substituting equation (83) into equation (82), we have

$$-2w_1+2w_2h^2{w_1}^5=0 \tag{84}$$

$$(1-\frac{h}{x_2})w_1-2w_2+(1+\frac{h}{x_2})w_3+h^2{w_2}^5=0 \tag{85}$$

$$(1-\frac{h}{x_3})w_2-2w_3+(1+\frac{h}{x_3})w_4+h^2{w_3}^5=0 \tag{86}$$

$$\vdots$$

$$(1-\frac{h}{x_n})w_{n-1}-2w_n+(1+\frac{h}{x_n})\sqrt{\frac{3}{4}}+h^2{w_n}^5=0. \tag{87}$$

Thus n systems of nonlinear equations are obtained. If this system is solved by Newton's method, the following numerical solution are obtained.

Table 6 Numerical values of equation (3) for FDM

x	$n=20$	$n=50$	$n=100$
0.1	1.000	0.9983	0.9986
0.2	0.9966	0.9947	0.9940
0.3	0.9900	0.987	0.9862
0.4	0.9804	0.9767	0.9755
0.5	0.9679	0.9635	0.9621

2 Conclusion

In this paper, when we compare the solutions of are one of the special type of Lane-Emden (Poisson) equation by He-Variational iteration, homotopy perturbation, homotopy analysis, adomian decomposition method and finite difference methods, we can derive the following conclusions. It is seen that closer results can be obtained by calculating higher-order approximations for all methods used.

The Adomian decomposition method can be easily applied to differential equations. The approximate solutions are almost the same as the analytical solutions of nonlinear equations. For different types of Lane-Emden equations, Adomian polynomials can be found separately and approximate solutions can be obtained.

The homotopy analysis method can be easily applied to nonlinear differential equations. The approximate solutions are almost identical to the analytical solutions of Lane-Emden (Poisson) equation. Close results can be obtained by writing in the approximations by the homotopy analysis method.

Convergence in the homotopy perturbation method depends on the initial approximation being chosen well enough. In addition, convergence also depends on the homotopy path. It is seen that when , close results are obtained. Also, the computational burden is higher in this method compared to others.

The cost of the finite difference method is also higher than the others. It can achieve close results when we choose a large value. Since the computations are on high value, computer assistance may be required.

He's variational iteration can converge to an approximate solution after only a few iterations. The convergence of the solutions is related to the good determination of the initial approximation and the Lagrange multiplier. Compared to the other given methods, the He variational iteration method seems to give close results to the exact one with fewer computations than the other solutions. The computational cost may vary depending on the chosen function. The initial approximation for different types of Lane-Emden equations is found by changing the Lagrange multiplier.

Calculations were performed with the help of Mathematica and MATLAB programs. In the light of the results obtained, it is seen that the methods can be applied to nonlinear Lane-Emden problems.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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