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A New Insight into Meromorphic Functions: An Exploration of 2^{-r}order Convex and Starlike Dynamics

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Abstract: Analytical and univalent functions play a critical role in the study of complex analysis. Specifically, functions that fall under the $S$and $\sum$classes exhibit unique characteristics, making them important subjects of study. In this research, we focus on these specific classes and utilize certain inherent properties associated with them. A fundamental area of interest is the open $D = \{z : 0 < |z| < 1, z \in C\}$unit disk in the complex plane. In this domain, our primary focus is on understanding the behavior of these functions under specific conditions, namely, logarithmic derivative conditions. Logarithmic derivatives are an essential tool in determining the nature and behavior of a function within its domain. In this context, we have been successful in deriving 2^{-r} (r =1,2,3,..) order special starlike and convex functions. Starlike functions are a subset of univalent functions that exhibit a specific shape-preserving property, while convex functions are those for which the line segment between any two points on the graph of the function lies above or on the graph itself. By leveraging certain properties of the $S$and $\sum$classes and applying the logarithmic derivative conditions within the open D unit disk, this research provides new insights and results into the study of 2^{-r}-order special starlike and convex functions.

Keywords: Analytic functions, starlike functions, convex functions.

1 Introduction

Univalent Functions Theory, which started to be studied with coefficient estimates at the beginning of the 20th century, is one of the most important branches of the Geometric Function Theory and deals with analytical and univalent functions in a certain complex region. The first important work in this field was made by P. Koebe in 1907. This work was followed by J.W.Alexander and L. Bieberbach in 1915 and 1916, respectively.

It is generally used as the range of order for starlike and convex functions defined on functions belonging to class A, and very few special cases of this order have been studied. We used specialvalue $\alpha = 2^{-r}$ $(r = 1,2,3,..)$ in this study.

Let Adenote the class of functions of the form [1], [3], [4], [5]:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{n\geq 2} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $D = \{z : |z| < 1, z \in \mathrm{C}\}$ then a function $f(z)$ belonging to A is said to be starlike of order 2^{-r} $(0<2^{-r}<1)$ if its satisfies the following inequality,

$$Re\left[\frac{zDf(z)}{f(z)}\right] \geq \frac{1}{2^r}, \quad z \in D\,, \quad r = 1,2,3,.. \tag{2}$$

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Additionally, we denote by $S^{*}$the subclass of A consisting of functions which are starlike of order 2^{-r} in D. Moreover, a function $f(z)$ belonging to A in said to be convex of order αif it satisfies the following inequality;

$$Re\left[1+\frac{zD^{2}f(z)}{Df(z)}\right]\geq\frac{1}{2^{r}},\quad z\in D. \tag{3}$$

Now, let us define the differential operators as below:

$$\begin{aligned}&Df(z)=\tfrac{df(z)}{dz}\quad, for\quad \tfrac{d}{dz}=D,\\&D^{0}f(z)=f(z),\\&D^{n}f(z)=D(D^{n-1}f(z)),\ n=1,2,....\end{aligned}$$

A function$f(z)$ which is univalent and regular in $D=\{z:0<|z|<1,z\in C\}$ of the form [6], [7], [8], [10], [11] :

$$f(z)=\frac{1}{z}+\sum_{n\geq1}a_{n}.z^{n}, \tag{4}$$

is said to be a meromorphic univalent belonging to the class $\sum$. Now let's define the following function:

$$h(z)=zf(z)=1+\sum_{n\geq1}a_{n}.z^{n}=\frac{1}{1-g(z)},\ \ g(z)\neq1,z\in D. \tag{5}$$

Definition 1: [1] A function $f(z)$ is said to be member of the class$B(\alpha)$ if and only if the following criteria satisfied: [9]

$$\left|\frac{z^{2}f'(z)}{f^{2}(z)}-1\right|<1-\alpha. \tag{6}$$

For some $0\leq\alpha<1$ and all $z\in D$. This condition implies $Re(\frac{z^{2}f'(z)}{f^{2}(z)})>\alpha$.

Lemma 1. [3]. Let w(z) be analytic in D and such that w(0)=0. If $|w(z)|$has its maximum value on the circle $|z|<1$, a point $z_{0}\in D$ then, we have:

$$z_{0}w'(z_{0})=kw(z_{0})\quad and\quad w(z_{0})=e^{i\theta},$$

where $k\geq1$ is a real number.

Lemma 2: [1]. Let$f(z)\in A$ satisfy the following condition:

$$\left|\frac{z^{2}f'(z)}{f^{2}(z)}-1\right|<1,z\in D, \tag{7}$$

then $f(z)\in A$ is said to be univalent in $z\in D$.

Theorem 1. $Re\left[1+\frac{zD^{2}f(z)}{Df(z)}\right]\leq\frac{1}{2^{r}},\quad(r=1,2,3,...)$ is for a function $f(z)\in A$then the function $f(z)$is a starlike function of order 2^{-1}.

Proof:

$$f(z)=z+\sum_{n\geq2}a_{n}z^{n}=z+a_{2}z^{2}+a_{3}z^{3}+...,$$

$$zf'(z)=z+2a_{2}z^{2}+3a_{3}z^{3}+...,$$

$$\frac{zDf(z)}{f(z)}=1+a_{2}z+...,$$

$$\frac{zDf(z)}{f(z)}=\frac{1}{1-g(z)},\quad g(z)\neq1. \tag{8}$$

By applying the logarithmic derivative of both sides of the above equation

$$\frac{1}{z}+\frac{D^2 f(z)}{Df(z)}-\frac{Df(z)}{f(z)}=\frac{Dg(z)}{1-g(z)},$$

$$1+\frac{zD^2 f(z)}{Df(z)}=\frac{Df(z)}{f(z)}+\frac{1}{1-g(z)}$$

$$=\frac{1}{1-g(z)}+\frac{zDg(z)}{g(z)}.\frac{g(z)}{1-g(z)},\left(\frac{zDf(z)}{f(z)}=\frac{1}{1-g(z)}\right) \tag{9}$$

$\max\limits_{|z|\le|z_0|}|g(z_0)|=|g(z_0)|=1$ and $g(z_0)=e^{i\theta}, z_0Dg(z_0)=kg(z_0)$for $k\ge 1$.

Then we can write the following inequality:

$$Re\left[1+\frac{zD^2 f(z)}{Df(z)}\right]=Re\left[\frac{1}{e^{i\theta}}+k\frac{e^{i\theta}}{1-e^{i\theta}}\right]$$

$$=\frac{1}{2}+k\frac{\cos\theta-1}{2(1-\cos\theta)}=\frac{1}{2}(1-k)\le 0, \quad k\ge 1. \tag{10}$$

So,

$\frac{1}{2^r}>\frac{1}{2}(1-k)$ and $Re\left(1+\frac{D^2 f(z)}{Df(z)}\right)<\frac{1}{2^r}$.

On the other hand, according to the data obtained above, use reach the following result.

$$1+\frac{zD^2 f(z)}{Df(z)}=\frac{zDf(z)}{f(z)}+\frac{zDg(z)}{1-g(z)}=\frac{zDf(z)}{f(z)}+\frac{zDf(z)}{g(z)}.\frac{g(z)}{1-g(z)}$$

$$Re\left[1+\frac{z_0D^2 f(z_0)}{Df(z_0)}\right]=Re\left[\frac{z_0Df(z_0)}{f(z_0)}+k\frac{e^{i\theta}}{1-e^{i\theta}}\right]$$

$$\frac{1}{2}(1-k)=Re\left[\frac{zDf(z)}{f(z)}\right]+k\frac{\cos\theta-1}{2(1-\cos\theta)}$$

$$=Re\left[\frac{zDf(z)}{f(z)}\right]-\frac{k}{2} \tag{11}$$

$$Re\left[\frac{zDf(z)}{f(z)}\right]=\frac{1}{2}(1-k)+\frac{k}{2}=\frac{1}{2} \tag{12}$$

Thus, the proof is completed. **Theorem 2:** For the analytical and univalent function $f(z)=z+\sum_{n\ge 2}a_nz^n \quad (f(z)\in S)$ defined on the ring region $D=\{z:0<|z|<1, z\in \mathrm{C}\}$, if there is a $z_0\in D$with $0<z^{-r}<1,(r=1,2,3,...)$

$z_0=2^{-r}e^{i\theta_0}$ and $\min\limits_{|z|<2^{-r}}|f(z_0)|<|f(z_0)|$,then

$$\frac{z_0Df(z_0)}{f(z_0)}=1-k\le 0, Re\left[1+\frac{z_0D^2 f(z_0)}{Df(z_0)}\right]\ge 1-k<2^{-r}$$

and

$$k\ge\frac{|z_0-f(z_0)|^2}{2^{-r}-|f(z_0)|^2}\ge\frac{2^{-r}-|f(z_0)|}{2^{-r}+|f(z_0)|}, \quad (k=1,2,3,...). \tag{13}$$

Proof: Let's define the function $h(z)=\frac{z}{f(z)}$. By taking logarithmic derivatives from both sides of this equation:

$$\ln z-\ln f(z)=\ln h(z)$$

$$1-\frac{zDf(z)}{f(z)}=\frac{zDh(z)}{h(z)}$$

$$\frac{z_0Df(z_0)}{f(z_0)}=1-\frac{zDh(z)}{h(z)}$$

$$\frac{z_0Df(z_0)}{f(z_0)}=1-\frac{ke^{i\theta}}{e^{i\theta}}=1-k\leq 0<2^{-r}, \tag{14}$$

which $h(z_0)=e^{i\theta}$, $z_0h(z_0)=ke^{i\theta}$. On the other hand, $f(z)=z+\sum_{n\geq 2}a_nz^n$ and $Df(z)=1+\sum_{n\geq 2}na_nz^{n-1}$ $\Rightarrow$ $zDf(z)=z+\sum_{n\geq 2}na_nz^n$,

the equation can be written.

$\frac{zDf(z)}{f(z)}=\frac{1}{1-g(z)}$ for $z_0\in D$. (**??**)

By taking logarithmic derivatives from both sides of this last equation and doing the necessary operations and if Lemma 1 is used;

$$\frac{1}{z}+\frac{D^2f(z)}{Df(z)}-\frac{Df(z)}{f(z)}=\frac{Dg(z)}{1-g(z)}$$

$$1+\frac{zD^2f(z)}{Df(z)}-\frac{zDf(z)}{f(z)}=\frac{zDg(z)}{1-g(z)}$$

$$1+\frac{zD^2f(z)}{Df(z)}=\frac{1}{1-g(z)}+\frac{zDg(z)}{g(z)}.\frac{g(z)}{1-g(z)}$$

$$1+\frac{zD^2f(z)}{Df(z)}=\frac{1}{1-e^{i\theta}}+\frac{ke^{i\theta}}{e^{i\theta}}\frac{e^{i\theta}}{1-e^{i\theta}}$$

$$=\frac{1}{1-e^{i\theta}}+\frac{ke^{i\theta}}{1-e^{i\theta}}$$

$$Re\left[1+\frac{z_0D^2f(z_0)}{Df(z_0)}\right]=Re\left[\frac{1}{1-e^{i\theta}}+\frac{ke^{i\theta}}{1-e^{i\theta}}\right]$$

$$=\frac{1}{2}-\frac{k}{2}=\frac{1}{2}(1-k)\leq 0<2^{-r}$$

$$Re\left[1+\frac{z_0D^2f(z_0)}{Df(z_0)}\right]\geq 1-k<2^{-r}. \tag{15}$$

Moreover, for $h(z_0)=\frac{z_0}{f(z_0)}$, we can write:

$$\frac{\left|\frac{z_0}{f(z_0)}-1\right|^2}{\left|\frac{z_0}{f(z_0)}\right|^2-1}=\frac{|z_0-f(z_0)|^2}{|z_0|^2-|f(z_0)|^2}\geq\frac{|z_0-f(z_0)|\,|z_0+f(z_0)|}{|z_0|^2-|f(z_0)|^2}$$

$$\geq\frac{|z_0-f(z_0)|\,|z_0+f(z_0)|}{(|z_0|-|f(z_0)|)(|z_0|+|f(z_0)|)}$$

$$k\geq 1,\quad k\geq\frac{1-2^{-r}|f(z_0)|}{1+2^{-r}|f(z_0)|}. \tag{16}$$

Theore3: If the function $f(z)\in A$ is a convex function in region $D=\{z:|z|<1,z\in C\}$, that is;

$$0<Re\left[1+\frac{z.D^2f(z)}{Df(z)}\right]\leq\frac{1}{2^r},\ r=1,2,3,..., \tag{17}$$

then the function $f(z)\in A$ is a starlike function of order $\frac{1}{2}$ on the D.

Proof:

$$f(z)=z+\sum_{n\geq 2}a_n.z^n \Rightarrow zDf(z)=z+\sum_{n\geq 2}n.a_n.z^n$$

$$\frac{z.Df(z)}{f(z)}=\frac{1}{1-h(z)}, h(z)\neq 1 \tag{18}$$

Equality can always be written. The equation can always be written by taking the logarithmic derivative of both sides of the above equation we get:

$$1+\frac{z.D^2f(z)}{Df(z)}=\frac{z.Dh(z)}{f(z)}+\frac{zDf(z)}{1-h(z)}$$

$$Re(1+\frac{z_0.D^2f(z_0)}{Df(z_0)})=Re(\frac{z_0.Dh(z_0)}{1-h(z_0)}+\frac{z_0Df(z_0)}{f(z_0)})$$

$$Re(1+\frac{z_0.D^2f(z_0)}{Df(z_0)})=Re(\frac{1}{1-h(z_0)}.\frac{z_0.Dh(z_0)}{h(z_0)}+\frac{h(z_0)}{1-h(z_0)})$$

$$Re(1+\frac{z_0.D^2f(z_0)}{Df(z_0)})=Re(\frac{1}{1-e^{i\theta}}.\frac{k.e^{i\theta}}{e^{i\theta}}+\frac{e^{i\theta}}{1-e^{i\theta}}), \tag{19}$$

which is $h(z_0)=e^{i\theta} \quad and \quad z_0Dh(z_0)=k.e^{i\theta}$ for lemma 1. From here too

$$Re(1+\frac{z_0.D^2f(z_0)}{Df(z_0)})=\frac{1}{2}-\frac{k}{2}=\frac{1}{2}(1-k)\leq 0. \tag{20}$$

This situation is contrary to the given condition of the theorem 3 no such $z_0\in D$ in D.

In that case,

$Re(1+\frac{z.D^2f(z)}{Df(z)})=\frac{1}{2}$ which is $|h(z)|<1$ and $z\in D$. For example,

the koebe function $f(z)=\frac{1}{(1-z)^2}=z+\sum_{n\geq 2}n.z^n$ which is analytically univalent of class A, satisfies this theorem. Really,

$$f(z)=\frac{1}{(1-z)^2}=z+\sum_{n\geq 2}n.z^n \Rightarrow Df(z)=\frac{1+z}{(1-z)^3}\Rightarrow D^2f(z)=\frac{2z+4}{(1-z)^4}.$$

From here we can write the following equation

$$1+\frac{z.D^2f(z)}{Df(z)}=\frac{1+2z}{1-z}+\frac{z}{1+z}. \tag{21}$$

On the other hand, for $o<\frac{1}{2^r}<1$ and $z=\frac{1}{2^r}.e^{i\theta}$

$$1+\frac{z.D^2f(z)}{Df(z)}=\frac{1+2\frac{1}{2^r}e^{i\theta}}{1-\frac{1}{2^r}e^{i\theta}}+\frac{\frac{1}{2^r}e^{i\theta}}{1+\frac{1}{2^r}e^{i\theta}}=\frac{(1+2\frac{1}{2^r}e^{i\theta})(1+\frac{1}{2^r}e^{i\theta})+\frac{1}{2^r}e^{i\theta}(1-\frac{1}{2^r}e^{i\theta})}{(1+\frac{1}{2^r}e^{i\theta})(1-\frac{1}{2^r}e^{i\theta})}$$

$$=\frac{1+4\frac{1}{2^r}Cos\theta+(\frac{1}{2^r})^2Cos^2\theta}{1-(\frac{1}{2^r})^2Cos^2\theta}>\frac{1-4\frac{1}{2^r}+(\frac{1}{2^r})^2}{1-(\frac{1}{2})^2}>0. \tag{22}$$

In the final we get:

$$1+\frac{z.D^2f(z)}{Df(z)}<\frac{1}{2^2}, \tag{23}$$

which completes the proof of the theorem.

Theorem: 4. If the function $f(z)\in A$ satisfies the following inequality

$\left|\frac{z.D^2f(z)}{Df(z)}-\frac{2z.Df(z)}{f(z)}\right|<\frac{1-\frac{1}{2^r}}{2-\frac{1}{2^r}}$, $z\in D$ and $r=1,2,3,...$ (**??**)

then $f(z)$ is of order $\frac{1}{2^r}$ and univalent in D.
Proof: For the proof of this theorem. Let's define the function$g(z)$ with the following equation:

$$f(z)=z+\sum_{n\geq 2}a_nz^n \Rightarrow g(z)=Df(z)=1+\sum_{n\geq 2}na_nz^{n-1}$$

$$z^2g(z)=z^2Df(z)=z^2+\sum_{n\geq 2}na_nz^{n+1}.$$

From here, let's define the function $h(z)$ as follows,

$$\frac{z^2Df(z)}{f^2(z)}=1+(1-2^{-r})h(z), \tag{24}$$

we can write the following equation by taking the logarithmic derivative from both sides of the above:

$$\frac{D^2(zf(z))}{Df(z)}-\frac{2zDf(z)}{f(z)}=\frac{(1-2^{-r})zDh(z)}{1+(1-2^{-r})h(z)}. \tag{25}$$

If there is a $z_0\in D$ to be $\max\limits_{|z|<|z_0|}|h(z)|=|h(z_0)|=1$ then we get the following inequality by writing $h(z_0)=e^{i\theta}$ and $z_0Dh(z_0)=ke^{i\theta},(k\geq 1)$ in the above equation from the lemma

$$\left|\frac{D^2(z_0f(z_0))}{Df(z_0)}-\frac{2z_0Df(z_0)}{f(z_0)}\right|=\left|\frac{(1-2^{-r})ke^{i\theta}}{1+(1-2^{-r})e^{i\theta}}\right|\geq\frac{1-2^{-r}}{2-2^{-r}}. \tag{26}$$

As a result, using lemma1 and lemma2 and $|h(z)|<1\ for\ all\ z\in D$ we get the following result which means $f(z)$ is of order $\frac{1}{2^r}$ and univalent in D.

2 Conclusion:

In this study, starlike and convex functions are obtained by defining a new order and using univalent and sigma classes created on class A.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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