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On conformable fractional versions of trapezoid-type inequalities according to twice-differentiable functions

Fatih Hezenci and Hüseyin Budak

Department of Mathematics, Faculty of Science and Arts, Duzce University, Turkiye

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Abstract: This paper proves an equality for the case of twice-differentiable convex mappings with respect to the conformable fractional integrals. With the help of this equality, several trapezoid-type inequalities are established by convex functions involving conformable fractional integrals. Sundry significant inequalities are acquired with taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Furthermore, we present several new results connected with trapezoid-type inequalities by using the special choices of obtained results.

Keywords: Trapezoid type inequality, fractional conformable integrals, fractional conformable derivatives, convex function.

1 Introduction

Inequalities theory represents a long-standing topic in many mathematical fields and remains an attractive research with many applications. The theory of convexity stages a central and stunning role in several fields of research. This theory provides us with a powerful tool in order to solve many problems. Furthermore, the concept of convexity has been extended and improved in many directions.

Another significant result connected with convex function is the *Hermite–Hadamard-type inequality*, which has been first introduced by C. Hermite and J. Hadamard for the case of convex functions. Let $f : I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then, the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is valid. If f is concave, then both inequalities in (1) hold in the reverse direction. Trapezoidal inequality, which is the right side of this inequality, and Midpoint inequality, which is the left side, have pioneered many scientific studies.

Fractional calculus has been the center of attraction for researchers in mathematical sciences on account of its fundamental definitions, properties, and applications in tackling real-life problems. Moreover, fractional calculus is the answer to the question of whether fractional integrals and fractional derivatives can be taken. Hence, it has been offered large number of solutions to many problems. The most famous of the fractional integrals that are growing day by day are the Riemann-Liouville, Conformable fractional, and Caputo integrals.

Several papers have been focused on acquiring trapezoid-type and midpoint-type inequalities that demonstrate the bounds by the right-hand side and left-hand side of the Hermite–Hadamard inequality, respectively. For instance, Dragomir first acquired trapezoidal inequalities in [6], while Kırmacı first derived midpoint-type inequalities in [17].

* Corresponding author e-mail: fatihezenci@gmail.com

Sarikaya et al. and Iqbal et al. established several fractional trapezoid and midpoint-type inequalities for convex functions in papers [21] and [9], respectively.

With the help of the only derivative's fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative is provided in paper [15]. The conformable derivative fulfills several important requirements that cannot be fulfilled by the Riemann-Liouville fractional operator and Caputo fractional operator definitions. In addition to these, in paper [1] the researchers investigated that the conformable approach in [15] cannot yield good results when compared to the Caputo definition via specific functions. This defect in the conformable description is avoided by several refinements of the conformable approach [23,8].

Twice-differentiable functions have been studied by many mathematicians. Hermite–Hadamard-type inequalities are proved by twice-differentiable convex functions in papers [3] and [4]. Several generalized fractional integral inequalities of midpoint and trapezoid-type with respect to twice-differentiable convex mappings are established in paper [18]. Moreover, the authors [19] presented several new inequalities of the Simpson and the Hermite–Hadamard-type for functions whose modulus of derivatives are convex. By using generalized fractional integrals in [5], presented some midpoint and trapezoid-type inequalities via mappings whose second derivatives in modulus are convex. For further information about fractional integral inequalities, see [13,7] and the references cited therein.

The purpose of this investigation is to derive several new trapezoid-type inequalities with the help of the twice-differentiable functions involving conformable fractional integrals. We also acquire that the newly established outcomes are the generalization of the existing trapezoid-type inequalities. This paper contains four chapters along with the introduction. In the second part, some fundamental information that we will use in our outcomes is mentioned. More precisely, we will give the definitions of Riemann-Liouville integral and conformable fractional integral. In the third part, an equality will be established in the case of twice-differentiable convex function including the conformable fractional integrals. By using this equality, sundry trapezoid-type inequalities will be proved by convex functions with respect to conformable fractional integrals. To be more precise, several trapezoid-type inequalities are acquired with taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Furthermore, we will be given some new results about trapezoid-type inequalities by using the special choices of obtained results. In the last section, the conclusions obtained from the research will be presented. In addition, ideas for future research will be given.

2 Preliminaries

This section considers the basics for building our outcomes. More precisely, definitions of Riemann-Liouville integrals and conformable integrals, which are well known in the literature, are given. From the fact of fractional calculus theory, mathematical preliminaries will be present as follows:
The well-known *gamma function*, *beta function* and *incomplete beta function* are defined

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt,$$

$$\mathfrak{B}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

and

$$\mathscr{B}(x,y,r) := \int_0^r t^{x-1}(1-t)^{y-1} dt,$$

respectively for $0 < x, y < \infty$ and $x, y \in \mathbb{R}$.

Kilbas et al. [16] defined fractional integrals, also called *Riemann-Liouville integrals* as follows:

Definition 1.[16] *For $f \in L_1[a,b]$, the Riemann-Liouville integrals $J^{\beta}_{a+}f(x)$ and $J^{\beta}_{b-}f(x)$ of order $\beta > 0$ are respectively given by*

$$J^{\beta}_{a+}f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)dt, \quad x > a \tag{2}$$

and

$$J^{\beta}_{b-}f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t)dt, \quad x < b. \tag{3}$$

Here, Γ denotes the Gamma function. Riemann-Liouville integrals reduces to the classical integrals for the case of $\beta = 1$.

In paper [12], Jarad et al. introduced the following fractional conformable integral operators. They also provided certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined by as follows:

Definition 2.[12] *For $f \in L_1[a,b]$, the fractional conformable integral operator ${}^{\beta}\mathscr{J}^{\alpha}_{a+}f(x)$ and ${}^{\beta}\mathscr{J}^{\alpha}_{b-}f(x)$ of order $\beta \in \mathbb{C}$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are respectively presented by*

$${}^{\beta}\mathscr{J}^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt, \quad t > a, \tag{4}$$

and

$${}^{\beta}\mathscr{J}^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt, \quad t < b. \tag{5}$$

Note that the fractional integral in (4) reduces to the Riemann-Liouville fractional integral in (2) if we assign $\alpha = 1$. Moreover, the fractional integral in (5) coincides with the Riemann-Liouville fractional integral in (3) if we select $\alpha = 1$. Sundry recent results with respect to fractional integral inequalities see [11,14,2,10] and the references cited therein.

3 Principal Outcomes

Lemma 1. *If $f : [a,b] \to \mathbb{R}$ is a twice-differentiable function on (a,b) such that $f'' \in L_1[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}} \left[{}^{\beta}\mathscr{J}^{\alpha}_{a+}f(b) + {}^{\beta}\mathscr{J}^{\alpha}_{b-}f(a) \right]$$

$$= \frac{(b-a)^2 \alpha^{\beta}}{2} \int\limits_0^1 \left(\int\limits_t^1 \left[\left(\frac{1-(1-s)^{\alpha}}{\alpha} \right)^{\beta} - \left(\frac{1-s^{\alpha}}{\alpha} \right)^{\beta} \right] ds \right) f''(ta+(1-t)b)\, dt. \tag{6}$$

Proof. Let us consider

$$\frac{(b-a)^2 \alpha^{\beta}}{2} \left\{ \int\limits_0^1 \left(\int\limits_t^1 \left(\frac{1-(1-s)^{\alpha}}{\alpha} \right)^{\beta} ds \right) f''(ta+(1-t)b)\, dt \right.$$

$$\left. - \int\limits_0^1 \left(\int\limits_t^1 \left(\frac{1-s^{\alpha}}{\alpha} \right)^{\beta} ds \right) f''(ta+(1-t)b)\, dt \right\} = \frac{(b-a)\alpha^{\beta}}{2} [I_1 - I_2]. \tag{7}$$

Integrating by parts, we have

$$I_1=\int_0^1\left(\int_t^1\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)f''(ta+(1-t)b)\,dt$$

$$=-\frac{1}{b-a}\left(\int_t^1\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)f'(ta+(1-t)b)\Big|_0^1$$

$$-\frac{1}{b-a}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta f'(ta+(1-t)b)\,dt$$

$$=\frac{1}{b-a}\left(\int_0^1\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)f'(b)-\frac{1}{b-a}\left\{-\frac{1}{b-a}\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta f(ta+(1-t)b)\Big|_0^1\right.$$

$$\left.+\frac{\beta}{b-a}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f(ta+(1-t)b)\,dt\right\}.$$

Considering $x=ta+(1-t)b$, we get

$$I_1=\frac{1}{b-a}\left(\int_0^1\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)f'(b)+\frac{1}{(b-a)^2\alpha^\beta}f(a)$$

$$-\frac{\Gamma(\beta+1)}{(b-a)^{\alpha\beta+2}}\frac{1}{\Gamma(\beta)}\int_a^b\left(\frac{(b-a)^\alpha-(x-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(x)}{(x-a)^{1-\alpha}}dx$$

$$=\frac{1}{b-a}\left(\int_0^1\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta ds\right)f'(b)+\frac{1}{(b-a)^2\alpha^\beta}f(a)-\frac{\Gamma(\beta+1)}{(b-a)^{\alpha\beta+2}}\,{}^\beta\mathscr{J}_{a+}^\alpha f(b). \tag{8}$$

Using the argument outlined above, we obtain

$$I_2=\int_0^1\left(\int_t^1\left(\frac{1-s^\alpha}{\alpha}\right)^\beta ds\right)f''(ta+(1-t)b)\,dt$$

$$=\frac{1}{b-a}\left(\int_0^1\left(\frac{1-s^\alpha}{\alpha}\right)^\beta ds\right)f'(b)-\frac{1}{(b-a)^2\alpha^\beta}f(b)+\frac{\Gamma(\beta+1)}{(b-a)^{\alpha\beta+2}}\,{}^\beta\mathscr{J}_{b-}^\alpha f(a). \tag{9}$$

If we substitute (8) and (9) in the equality (7), then it can be arrived the following equality

$$\frac{(b-a)\alpha^{\beta}}{2}[I_1-I_2]=\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)+{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)\right],$$

which completes the proof of Lemma 1.

Theorem 1. *Suppose that $f:[a,b]\to\mathbb{R}$ denotes a twice-differentiable mapping on (a,b) such that $f''\in L_1([a,b])$ and $|f''|$ is convex on $[a,b]$. The, the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)+{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^2\alpha^{\beta}}{2}\left[\varphi_1(\alpha,\beta)\left|f''(a)\right|+\varphi_2(\alpha,\beta)\left|f''(b)\right|\right] \tag{10}$$

is valid. Here,

$$\begin{cases}\varphi_1(\alpha,\beta)=\int\limits_0^1\left|\int\limits_t^1\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-s^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|tdt\\[2ex] =\frac{1}{\alpha^{\beta+1}}\int\limits_0^1\left|\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-t)^{\alpha}\right)-\left(\mathfrak{B}\left(\frac{1}{\alpha},\beta\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,t^{\alpha}\right)\right)\right|tdt,\\[2ex] \varphi_2(\alpha,\beta)=\int\limits_0^1\left|\int\limits_t^1\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-s^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|(1-t)dt\\[2ex] =\frac{1}{\alpha^{\beta+1}}\int\limits_0^1\left|\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-t)^{\alpha}\right)-\left(\mathfrak{B}\left(\frac{1}{\alpha},\beta\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,t^{\alpha}\right)\right)\right|(1-t)dt,\end{cases} \tag{11}$$

where $\mathfrak{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof. Let us take the absolute value of both sides of (6). Then, we derive

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)+{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^2\alpha^{\beta}}{2}\int\limits_0^1\left|\int\limits_t^1\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-s^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left|f''(ta+(1-t)b)\right|dt. \tag{12}$$

If we use the convexity of the $|f''|$ on $[a,b]$, then we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)+{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^2\alpha^{\beta}}{2}\int\limits_0^1\left|\int\limits_t^1\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-s^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|\left[t\left|f''(a)\right|+(1-t)\left|f''(b)\right|\right]dt$$

$$=\frac{(b-a)^2\alpha^{\beta}}{2}\left[\varphi_1(\alpha,\beta)\left|f''(a)\right|+\varphi_2(\alpha,\beta)\left|f''(b)\right|\right].$$

*Remark.*if it is chosen $\alpha=1$ in (10), then Theorem 1 reduces to

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{a+}^{\beta}f(b)+J_{b-}^{\beta}f(a)\right]\right|\leq\frac{(b-a)^{2}\beta}{4(\beta+1)(\beta+2)}\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

which is given by Usta et al. in [22, Corollary 5.5].

*Remark.*Let us consider $\alpha=1$ and $\beta=1$ in (10). Then, Theorem 1 is equal to

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|\leq\frac{(b-a)^{2}}{24}\left[\left|f''(a)\right|+\left|f''(b)\right|\right],$$

which is given by Sarikaya and Aktan in [19, Proposition 2].

Theorem 2.*Let $f:[a,b]\to\mathbb{R}$ denote a twice-differentiable function on (a,b) so that $f''\in L_1[a,b]$ with $a<b$ and $|f''|^q$ be convex on $[a,b]$ with $q>1$. Then, the following double inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)+{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2}\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-s^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|^{p}dt\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^{q}+|f''(b)|^{q}}{2}\right)^{\frac{1}{q}}. \tag{13}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.*By using Hölder inequality in (12), we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)+{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2}\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-s^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f''(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}.$$

From the fact that $|f''|^q$ is convex on $[a,b]$. It yields

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)+{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^{2}\alpha^{\beta}}{2}\left(\int_{0}^{1}\left|\int_{t}^{1}\left[\left(\frac{1-(1-s)^{\alpha}}{\alpha}\right)^{\beta}-\left(\frac{1-s^{\alpha}}{\alpha}\right)^{\beta}\right]ds\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_{0}^{1}\left[t\left|f''(a)\right|^{q}+(1-t)\left|f''(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)^2\alpha^\beta}{2}\left(\int_0^1\left|\int_t^1\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\left(\frac{1-s^\alpha}{\alpha}\right)^\beta\right]ds\right|^p dt\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Finally, the proof of Theorem 2 is completed.

*Remark.*If we choose $\alpha=1$ in Theorem 2, then we derive

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{2(b-a)^\beta}\left[J_{a+}^\beta f(b)+J_{b-}^\beta f(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{2(\beta+1)}\left(1-\frac{2}{p(\beta+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}},$$

which is given in [22, Corollary 5.9].

Corollary 1.*If we take $\alpha=1$ and $\beta=1$ in Theorem 2, then we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)^2}{4}\left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 3.*Assume that $f:[a,b]\to\mathbb{R}$ is a twice-differentiable function on (a,b) so that $f''\in L_1[a,b]$. If $|f''|^q$ is convex on $[a,b]$ with $q\geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^\beta\mathscr{J}_{a+}^\alpha f(b)+{}^\beta\mathscr{J}_{b-}^\alpha f(a)\right]\right|$$

$$\leq\frac{(b-a)^2\alpha^\beta}{2}(\varphi_3(\alpha,\beta))^{1-\frac{1}{q}}\left(\varphi_1(\alpha,\beta)\left|f''(a)\right|^q+\varphi_2(\alpha,\beta)\left|f''(b)\right|^q\right)^{\frac{1}{q}}. \tag{14}$$

Here, $\varphi_1(\alpha,\beta)$ and $\varphi_2(\alpha,\beta)$ are defined in (11) and

$$\varphi_3(\alpha,\beta)=\int_0^1\left|\int_t^1\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\left(\frac{1-s^\alpha}{\alpha}\right)^\beta\right]ds\right|dt$$

$$=\frac{1}{\alpha^{\beta+1}}\int_0^1\left|\mathscr{B}\left(\frac{1}{\alpha},\beta+1,(1-t)^\alpha\right)-\left(\mathfrak{B}\left(\frac{1}{\alpha},\beta\right)-\mathscr{B}\left(\frac{1}{\alpha},\beta+1,t^\alpha\right)\right)\right|dt,$$

where $\mathfrak{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

*Proof.*With the help of the power-mean inequality in (12), we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^\beta\mathscr{J}_{a+}^\alpha f(b)+{}^\beta\mathscr{J}_{b-}^\alpha f(a)\right]\right|$$

$$\leq\frac{(b-a)^2\alpha^\beta}{2}\left(\int_0^1\left|\int_t^1\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\left(\frac{1-s^\alpha}{\alpha}\right)^\beta\right]ds\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|\int_t^1\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\left(\frac{1-s^\alpha}{\alpha}\right)^\beta\right]ds\right|\left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

It is known that $|f''|^q$ is convex on $[a,b]$. Then, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\alpha^\beta\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}\mathscr{J}_{a+}^{\alpha}f(b)+{}^{\beta}\mathscr{J}_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^2\alpha^\beta}{2}\left(\int_0^1\left|\int_t^1\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\left(\frac{1-s^\alpha}{\alpha}\right)^\beta\right]ds\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|\int_t^1\left[\left(\frac{1-(1-s)^\alpha}{\alpha}\right)^\beta-\left(\frac{1-s^\alpha}{\alpha}\right)^\beta\right]ds\right|\left[t\left|f''(a)\right|^q+(1-t)\left|f''(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)^2\alpha^\beta}{2}\left(\varphi_3(\alpha,\beta)\right)^{1-\frac{1}{q}}\left(\varphi_1(\alpha,\beta)\left|f''(a)\right|^q+\varphi_2(\alpha,\beta)\left|f''(b)\right|^q\right)^{\frac{1}{q}}.$$

This ends the proof of Theorem 3.

*Remark.*Let us consider $\alpha=1$ in Theorem 3. Then, we derive

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{2(b-a)^\beta}\left[J_{a+}^{\beta}f(b)+J_{b-}^{\beta}f(a)\right]\right|\leq\frac{(b-a)^2\beta}{2(\beta+1)(\beta+2)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}},$$

which is given by Usta et al. in [22, Corollary 5.13].

Corollary 2.*If we take $\alpha=1$ and $\beta=1$ in Theorem 3, then Theorem 3 becomes to*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{12}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

4 Concluding Remarks

In this presented article, an equality is established for the case of twice-differentiable convex mappings involving the conformable fractional integrals. With the help of this equality, sundry trapezoid-type inequalities are given by convex functions with respect to conformable fractional integrals. Several trapezoid-type inequalities are acquired with taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Furthermore, we give some new results about trapezoid-type inequalities by using the special choices of obtained results.

The ideas and strategies via our outcomes concerning trapezoid-type inequalities based on conformable fractional integrals may open new avenues for further research in this area. It will appeal to mathematicians that the inequalities produced in the research include both conformable fractional integrals and twice differentiable functions. To the best of our knowledge, these results are new in the literature. We expect that the ideas and techniques of this paper will inspire ones working in this field. With the techniques used in the obtained inequalities, different types of fractional integrals can be used to obtain new inequalities in the future. In addition to these, new inequalities can be obtained by considering different order derivatives of the functions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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