New Trends in Mathematical Sciences

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Obtaining soliton solutions of the perturbed Radhakrishnan-Kundu-Lakshmanan model with M-truncated derivative

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Abstract: This study explores the soliton solutions for the perturbed Radhakrishnan-Kundu-Lakshmanan (RKL) equation which includes M-truncated derivative. Initially, a wave transformation technique is employed for the RKL equation, yielding a nonlinear ordinary differential equation (NODE). Then, the NODE is solved using the new Kudryashov method. A candidate solution and its related derivatives are incorporated into the NODE, resulting in a polynomial expression. A system of algebraic equations emerges by equating coefficients of terms with similar degrees to zero. Solving this system provides the identification of unknown variables in the candidate solution, thereby yielding the solutions to the RKL equation. Diverse illustrations of the obtained solutions are presented via contour, two-dimensional, and three-dimensional plots. The findings of this research may present potential implications for future studies in nonlinear optics.

Keywords: New Kudryashov, partial differential equation, nonlinearity, optics.

1 Introduction

In a variety of scientific disciplines—including but not limited to physics, engineering, biology, and economics—nonlinear partial differential equations (NLPDEs) are essential for the mathematical representation of complex systems [1,2,3]. These equations make it possible to describe occurrences that do not follow linear or straightforward patterns, giving researchers a more precise grasp of the systems they study.

Among the diverse solutions that NLPDEs admit, solitons have emerged as a particularly intriguing and relevant class of solutions. Characterized by their stable and localized waveforms, solitons maintain their shape during propagation and interaction, properties that have garnered substantial academic attention. Its discovery in real life is traced back to the 19th century when a Scottish engineer John Scott Russell observed a steady, lone wave moving without dispersion in a Scottish canal [4].

Various analytical techniques have been developed in the literature to solve NLPDEs and have been successfully applied to NLPDEs such as Calogero-Bogoyavlenskii-Schiff [5], Sasa-Satsuma [6,7,8], Biswas–Arshed [9], Ginzburg-Landau [10], and Fokas–Lenells [11] and Schrödinger-Hirota [12].

The dimensionless RKL equation with the M-truncated derivative and perturbation terms is defined as follows [13,14]:

$$i\mathscr{D}_{M,t}^{\gamma,\beta}q+\alpha\frac{\partial^2 q}{\partial x^2}+\beta|q|^2q=i\sigma\frac{\partial q}{\partial x}+i\lambda\frac{\partial}{\partial x}\left(|q|^2q\right)+i\mu\frac{\partial}{\partial x}\left(|q|^2\right)q-i\delta\frac{\partial^3 q}{\partial x^3},\quad 0<\gamma\leq 1, \tag{1}$$

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Here, $i=\sqrt{-1}$ and $q=q(x,t)$ represents a complex-valued function describing the wave's propagation. $\mathscr{D}_{M,t}^{\gamma,\beta}$ stands for the M-truncated derivative with respect to t. α denotes the group-velocity dispersion, β accounts for the nonlinearity, σ represents the inter-modal dispersion, λ shows self-steepening effects, λ stands for the higher-order dispersion, and δ stands for the coefficient third-order dispersion term.

The RKL equation has been used to model the propagation of light in optical fibers [15]. When the literature is examined, there are not many studies on the perturbed genealized RKL equation, which includes different types of derivative operators such as conformable, beta, and M-truncated. In [14], the researchers obtained optical solitons for the conformable fractional perturbed RKL equation model by utilizing the extended sinh-Gordon equation expansion method. In [16], soliton solutions of conformable time-fractional perturbed Radhakrishnan-Kundu-Lakshmanan equation were studied via the extended (G'/G)-expansion, polynomial, and first integral methods. The main motivation of this article is to obtain optical soliton solutions of the perturbed M-truncated RKL equation with M-truncated derivative using the new Kudryashov method.

2 Preliminary

Definition 1. *The truncated Mittag-Leffler function (TMLF) is given by [17, 18]:*

$$ {}_{i}\mathbb{E}_{\varepsilon}(f)=\sum_{j=0}^{i}\frac{f^{j}}{\Gamma(\varepsilon j+1)}, \tag{2}$$

in which $\varepsilon>0$ and $f\in C$.

Definition 2. *Suppose that $v:[0,\infty)\to\mathbb{R}$ is a function, the local M-truncated derivative of v of order $\gamma\in(0,1)$ with respect to (w.r.t.) y is given [17, 18]:*

$$\mathscr{D}_{M,y}^{\gamma,\varepsilon}v(y)=\lim_{h\to 0}\frac{v\left(y+{}_{i}\mathbb{E}_{\varepsilon}\left(hy^{-\gamma}\right)\right)-v(y)}{h}, \tag{3}$$

in which $\varepsilon,y>0$ and ${}_{i}\mathbb{E}_{\varepsilon}(\cdot)$ is a TMLF.

Theorem 1. *Let us assume that $v(y)$ is γ order differentiable function at $y_0>0$ for $\gamma\in(0,1]$ and $\varepsilon>0$. Then, $v(y)$ is continuous at y_0 [17, 18].*

Theorem 2. *[17, 18] Let $0<\gamma\leq 1,\varepsilon>0,r,s\in\mathbb{R}$ and ϕ,ψ be $\gamma-$ differentiable at a point $y>0$. Then,*

(1) $\mathscr{D}_{M,y}^{\gamma,\varepsilon}(r\phi+s\psi)(y)=r\,\mathscr{D}_{M,y}^{\gamma,\varepsilon}\phi(y)+s\,\mathscr{D}_{M,y}^{\gamma,\varepsilon}\psi(y)$ *in which r,s are reals,*
(2) $\mathscr{D}_{M,y}^{\gamma,\varepsilon}(\phi\psi)(y)=\phi(y)\,\mathscr{D}_{M,y}^{\gamma,\varepsilon}\psi(y)+\psi(y)\,\mathscr{D}_{M,y}^{\gamma,\varepsilon}\phi(y)$,
(3) $\mathscr{D}_{M,y}^{\gamma,\varepsilon}\left(\frac{\phi}{\psi}\right)(y)=\frac{\phi(y)\,\mathscr{D}_{M,y}^{\gamma,\varepsilon}\psi(y)-\psi(y)\,\mathscr{D}_{M,y}^{\gamma,\varepsilon}\phi(y)}{\psi(y)^2}$,
(4) $\mathscr{D}_{M,y}^{\gamma,\varepsilon}\left(\frac{\phi}{\psi}\right)(y)=\frac{\phi(y)\,\mathscr{D}_{M,y}^{\gamma,\varepsilon}\psi(y)-\psi(y)\,\mathscr{D}_{M,y}^{\gamma,\varepsilon}\phi(y)}{\psi(y)^2}$,
(5) *If ϕ is differentiable, then* $\mathscr{D}_{M,y}^{\gamma,\varepsilon}(\phi)(y)=\frac{y^{1-\gamma}}{\Gamma(\varepsilon+1)}\frac{d\phi(y)}{dy}$.

3 Wave Transformation

Inserting the following wave transformation into 1:

$$q(x,t)=\mathscr{Q}(\xi)e^{i\Phi},\ \xi=x-\frac{vt^{\gamma}\Gamma(\varepsilon+1)}{\gamma},\ \Phi=-\kappa x+\frac{\omega t^{\gamma}\Gamma(\varepsilon+1)}{\gamma}+\phi_0. \tag{4}$$

Here, κ,ω,ϕ_0, and v are arbitrary real numbers that stand for the wave number, frequency, phase component, and phase constant, respectively.

After substituting the wave transformation equation 4 into the main partial differential equation 1 and decomposing the real and imaginary parts, we get the following two equations:

$$(\alpha+3\delta\kappa)\mathscr{Q}''-\left(\alpha\kappa^2+\delta\kappa^3+\kappa\sigma+\omega\right)\mathscr{Q}+(\beta-\kappa\lambda)\mathscr{Q}^3=0, \tag{5}$$

and

$$\delta\mathscr{Q}^{(3)}-\left(2\alpha\kappa+3\delta\kappa^2+\sigma+(3\lambda+2\mu)\mathscr{Q}^2+v\right)\mathscr{Q}'=0, \tag{6}$$

where $\mathscr{Q}=\mathscr{Q}(\xi)$. Let us integrate it, so we have:

$$\delta\mathscr{Q}''-\frac{1}{3}(3\lambda+2\mu)\mathscr{Q}^3-\left(2\alpha\kappa+3\delta\kappa^2+\sigma+v\right)\mathscr{Q}=0. \tag{7}$$

From **??**, we get the following constraints conditions:

$$v=\frac{-2\alpha^2\kappa-\alpha\left(8\delta\kappa^2+\sigma\right)+\delta\left(-8\delta\kappa^3-2\kappa\sigma+\omega\right)}{\alpha+3\delta\kappa},\ \beta=-\frac{\alpha(3\lambda+2\mu)}{3\delta}-2\kappa(\lambda+\mu). \tag{8}$$

Substituting the constraint conditions into 5, we get:

$$(\alpha+3\delta\kappa)\mathscr{Q}''-\left(\alpha\kappa^2+\delta\kappa^3+\kappa\sigma+\omega\right)\mathscr{Q}+\left(-\frac{\alpha(3\lambda+2\mu)}{3\delta}-2\kappa(\lambda+\mu)-\kappa\lambda\right)\mathscr{Q}^3=0. \tag{9}$$

4 New Kudryashov Method

Assume that the NLODE in 9 has solutions [19] in the form:

$$\mathscr{Q}(\xi)=\sum_{i=0}^{\mu}A_i\varphi^i(\xi),\quad A_\mu\neq 0. \tag{10}$$

Balancing the terms $\mathscr{Q}''$ and $\mathscr{Q}^3$ in 9, one gets $\mu=1$, therefore, 10 turns into

$$\mathscr{Q}(\xi)=A_0+A_1\varphi(\xi),\quad A_1\neq 0. \tag{11}$$

Here, $\varphi(\xi)$ admits 12.

$$\left(\varphi'(\xi)\right)^2=\delta^2\varphi(\xi)^2\left(1-\eta\varphi(\xi)^2\right), \tag{12}$$

where δ,γ are non-zero real constants and 12 has the following solution:

$$\varphi(\xi)=\frac{4\psi}{\eta\exp(-\delta\xi)+4\psi^2\exp(\delta\xi)}, \tag{13}$$

in which η is non-zero real constant. Let us substitute 13 and its related derivatives to 1. So, the following system of equations is obtained by grouping all terms of the same power of $\varphi(\xi)$ and then equatinq all coefficients equal to zero:

$$\begin{aligned}
&\varphi^0(\xi):\ A_0^3(3\lambda+2\mu)(\alpha+3\delta\kappa)+3\delta A_0(\kappa(\kappa(\alpha+\delta\kappa)+\sigma)+\omega)=0,\\
&\varphi^1(\xi):\ A_1\left(\delta\left(\alpha(\delta-\kappa)(\delta+\kappa)-\kappa\left(-3\delta^3+\delta\kappa^2+\sigma\right)-\omega\right)-A_0^2(3\lambda+2\mu)(\alpha+3\delta\kappa)\right)=0,\\
&\varphi^2(\xi):\ A_0A_1^2(3\lambda+2\mu)(\alpha+3\delta\kappa)=0,\\
&\varphi^3(\xi):\ A_1(\alpha+3\delta\kappa)\left(A_1^2(3\lambda+2\mu)+6\delta^3\eta\right)=0.
\end{aligned}$$

The above algebraic system may be solved using Maple to get the set below:

Set 1$^{\mp}$.

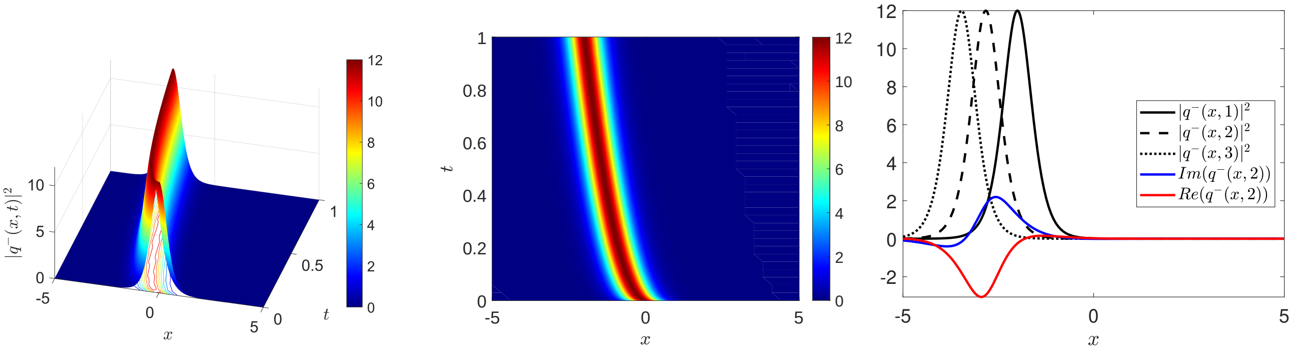
$$\omega=\alpha\left(\delta^2-\kappa^2\right)-\kappa\left(-3\delta^3+\delta\kappa^2+\sigma\right), A_0=0, A_1=\mp\frac{\sqrt{6}\delta^{3/2}\sqrt{\eta}}{\sqrt{-3\lambda-2\mu}}. \tag{14}$$

Using the Set 1 above and substituting φ and wave transformations in 4, one gets the analytical solutions of 1:

$$q^{\mp}(x,t)=\frac{4\sqrt{6}\delta^{3/2}\sqrt{\eta}\psi\exp\left(i\left(\frac{t^{\gamma}\Gamma(\varepsilon+1)\left(\alpha\delta^2-\alpha\kappa^2+3\delta^3\kappa-\delta\kappa^3-\kappa\sigma\right)}{\gamma}-\kappa x+\phi_0\right)\right)}{\sqrt{-3\lambda-2\mu}\left(\eta\exp\left(-\delta\xi\right)+4\psi^2\exp\left(\delta\xi\right)\right)}. \tag{15}$$

where

$$\xi=x-\frac{t^{\gamma}\Gamma(\varepsilon+1)\left(-2\alpha^2\kappa+\delta\left(\alpha\delta^2-\alpha\kappa^2+3\delta^3\kappa-9\delta\kappa^3-3\kappa\sigma\right)-\alpha\left(8\delta\kappa^2+\sigma\right)\right)}{\gamma(\alpha+3\delta\kappa)}, \tag{16}$$



(a) 3*D* surface. **(b)** Contour plot. **(c)** 2D sketchs.

Fig. 1: The plot of $q^{-}(x,t)$ where $\alpha=1,\gamma=1,\delta=2,\varepsilon=1,\eta=2,\kappa=1,\lambda=-2,\mu=1,\sigma=1,\psi=1$, and $\phi_0=1$.

In 1, the bright soliton representation of $q^{-}(x,t)$ is demonstrated for $\alpha=1,\gamma=1,\delta=2,\varepsilon=1,\eta=2,\kappa=1,\lambda=-2,\mu=1,\sigma=1,\psi=1$, and $\phi_0=1$.

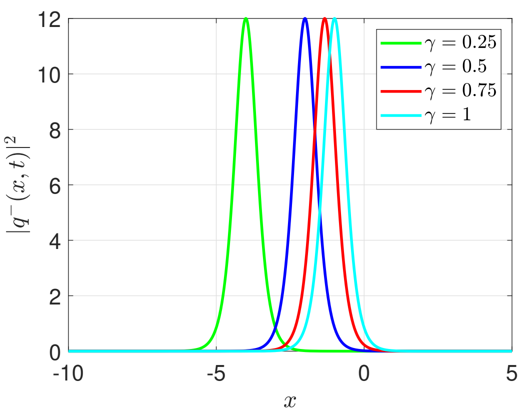


Fig. 2: A comparison for different derivative order γ where $t=1$.

In 2, the impact of the derivative order γ is compared for $t=1, \alpha=1, \delta=2, \varepsilon=1, \eta=2, \kappa=1, \lambda=-2, \mu=1, \sigma=1, \psi=1$, and $\phi_0=1$.

5 Conclusion

This study aimed to explore the soliton solutions of the perturbed Radhakrishnan-Kundu-Lakshmanan (RKL) equation with M-truncated derivative with the help of new Kudryashov method. The research successfully derived solutions and visually represented their properties. Graphical representations provided further insight into the properties of these solutions. The limited attention given to the RKL equation with M-truncated derivatives in existing literature accentuates the significance of this study, which is expected to be helpful for future research in nonlinear partial differential equations and their applications in optics.

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