New Trends in Mathematical Sciences

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Retrieval of optical solitons: Drinfeld-Sokolov-Satsuma-Hirota Equation

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Abstract: In the present study, we focus on obtaining the optical soliton solutions of the Drinfeld-Sokolov-Satsuma-Hirota (DSSH) equation using Kudryhashov methods. First, a wave transformation is applied to the DSSH equation, resulting in a nonlinear ordinary differential equation (NLODE). A balance number is deduced through the balancing of this NLODE. Subsequently, candidate solutions, inclusive of their respective derivatives and the wave transformation, are inserted into the DSSH equation. This results in an equation in polynomial form. Terms with equal power are grouped together in the new equation, and their coefficients are set to zero, leading to a system of algebraic equations. Solving this algebraic system facilitates the identification of undetermined parameters within the candidate solutions, thus yielding the solutions for the DSSH equation. Visualization of these solutions is executed through contour plots as well as two- and three-dimensional graphical representations. The contributions of this study are of substantial relevance for subsequent research in nonlinear science.

Keywords: Soliton, analytical methods, partial differential equation, nonlinearity.

1 Introduction

In numerous scientific fields, ranging from physics and engineering to biology and economics, nonlinear partial differential equations (NLPDEs) serve as indispensable tools for modeling complex systems [1,2]. Among the various solutions to NLPDEs, solitons have gained considerable scholarly interest due to their unique and stable localized wave profiles. The earliest documented instance of soliton behavior can be traced back to J. Scott Russell's observations in a Scottish canal [3]. Moreover, the utilization of solitons in optical fiber technologies attests to their applicability in contemporary studies [4,5,6].

DSSH equation is [7,8]:

$$z_{xxxxxx} - 9z_x z_{xxxx} - 18z_{xx}z_{xxx} + 18z_x^2 z_{xx} - \frac{1}{2}z_{tt} + \frac{1}{2}z_{xxxt} = 0, \tag{1}$$

in which $z = z(x,t)$ is a solution.

In the literature, the DSSH equation has garnered considerable attention and has been solved by some researchers. The DSSH equation was solved via various methods such as q-homotopy analysis transform method [9], two variable $(G'/G, 1/G)$ [10], the modified polynomial expansion method [11].

In recent scholarly contributions, various types of soliton solutions for NLPDEs have been successfully derived employing a multitude of techniques. These include the Sardar sub-equation method [12], enhanced modified extended tanh-expansion method (eMETEM) [13], the Sine-Gordon equation approach [14], Kudryashov methods [15,16],

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extended rational sine/cosine and sinh/cosh methods [17], etc.

This paper aims to obtain soliton solutions of the DSSH equation via Kudryashov and modified Kudryashov methods.

2 Wave Transformation

Inserting the following wave transformation into 1:

$$z(x,t) = \mathscr{Z}(\xi),\ \xi = x - vt, \tag{2}$$

we get:

$$\mathscr{Z}^{(6)} - \left(\frac{v}{2} + 9\mathscr{Z}'\right)\mathscr{Z}^{(4)} - 18\mathscr{Z}''\mathscr{Z}^{(3)} - \left(\frac{v^2}{2} - 18\left(\mathscr{Z}'\right)^2\right)\mathscr{Z}'' = 0, \tag{3}$$

in which $\mathscr{Z} = \mathscr{Z}(\xi)$ and the symbol $'$ and the powers in the brackets denote ordinary derivatives of $\mathscr{Z}$ with respect to ξ. The following NLODE is obtained by integrating the aforementioned equation and setting the integration constant to zero:

$$2\mathscr{Z}^{(5)} - \left(18\mathscr{Z}' + v\right)\mathscr{Z}^{(3)} - 9\left(\mathscr{Z}''\right)^2 + 12\left(\mathscr{Z}'\right)^3 - v^2\mathscr{Z}' = 0. \tag{4}$$

3 Methods

3.1 Kudryashov Method

Assume that the NLODE in 4 has the following form solutions [15]:

$$\mathscr{Z}(\xi) = \sum_{i=0}^{\mu} A_i\varphi^i(\xi), \quad A_\mu \neq 0. \tag{5}$$

From balancing between $\mathscr{Z}^{(5)}$ and $\mathscr{Z}''$ in 4, one gets $\mu = 1$. Since $\mu = 1$, 5 turns into

$$\mathscr{Z}(\xi) = A_0 + A_1\varphi(\xi), \quad A_1 \neq 0. \tag{6}$$

Here, $\varphi(\xi)$ admits 7:

$$\varphi'(\xi) = \delta\varphi(\xi)(\gamma\varphi(\xi) - 1), \tag{7}$$

where δ, γ are non-zero real constants and 7 has the following solution:

$$\varphi(\xi) = \frac{1}{\gamma + \eta\exp(\delta\xi)}, \tag{8}$$

in which η is non-zero real constant. Let us substitute 8 and its related derivatives to 1. So, the following system of equations is obtained by collecting all terms of the same power of $\varphi(\xi)$ and then setting all coefficients equal to zero:

$$\begin{aligned}
\varphi^0(\xi):&\ \delta\left(-2\delta^4+v^2+\delta^2 v\right)=0,\\
\varphi^1(\xi):&\ \delta\left(-27A_1\delta^3-\gamma\left(-62\delta^4+v^2+7\delta^2 v\right)\right)=0,\\
\varphi^2(\xi):&\ \delta^3\left(33A_1\gamma\delta-2A_1^2+2\gamma^2\left(v-30\delta^2\right)\right)=0,\\
\varphi^3(\xi):&\ \gamma\delta^3\left(3A_1\left(4A_1-51\gamma\delta\right)-2\gamma^2\left(v-130\delta^2\right)\right)=0,\\
\varphi^4(\xi):&\ \gamma^2\delta^3\left(2\gamma\delta-A_1\right)\left(10\gamma\delta-A_1\right)=0,\\
\varphi^5(\xi):&\ \gamma^3\delta^3\left(2\gamma\delta-A_1\right)\left(10\gamma\delta-A_1\right)=0.
\end{aligned}$$

The above algebraic system may be solved using a symbolic calculation software to get the set below:

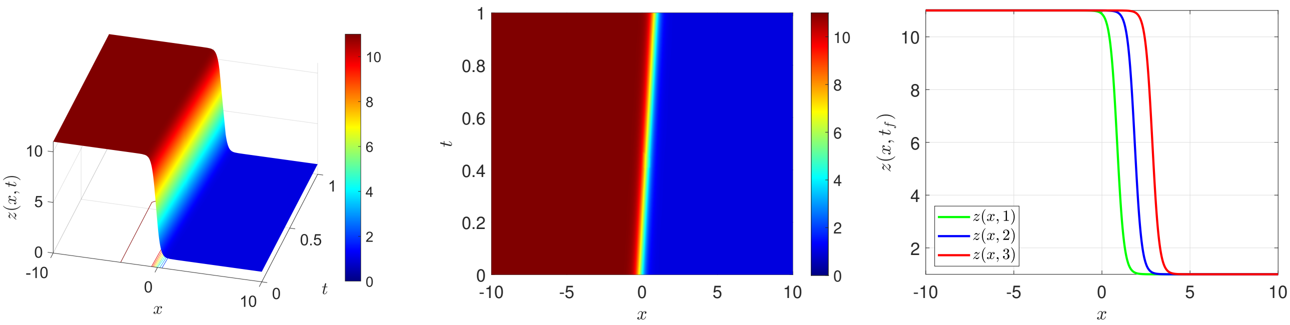
Set 1.

$$v=\delta^2, A_1=2\gamma\delta. \tag{9}$$

The following analytical solutions of 1 are obtained using Set 1 above and inserting φ and wave transformations in 2:

Using the Set 1 above and substituting φ and wave transformations in 2, we get the following analytical solutions of 1:

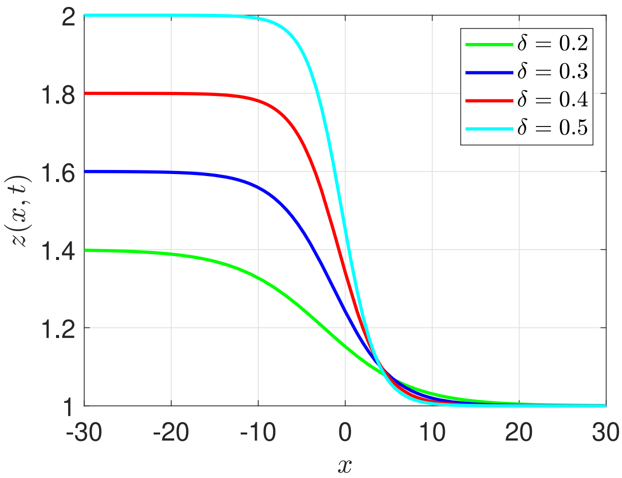
$$z(x,t)=A_0+\frac{2\gamma\delta}{\gamma+\eta e^{\delta(x-tv)}}. \tag{10}$$



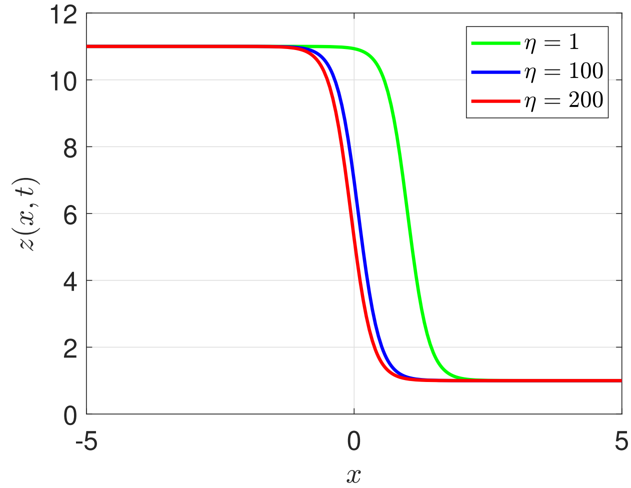
(a) $3D$ graph. **(b)** The contour graph. **(c)** The 2D graphs.

Fig. 1: The plot of $z(x,t)$ where $\gamma=1,\delta=5,\eta=2,A_0=1,v=1$.

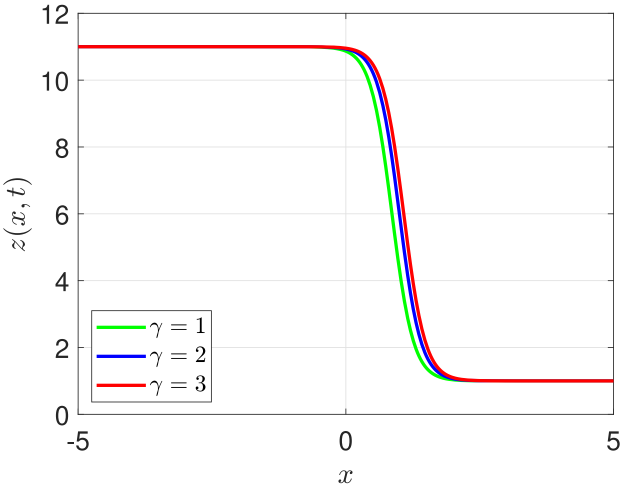
The kink soliton representation of $z(x,t)$ for $\gamma=1,\delta=5,\eta=2,A_0=1,v=1$ is demonstrated in 1.



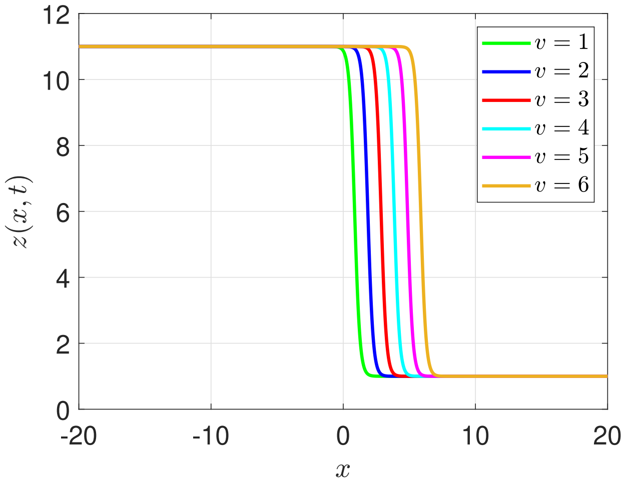
(a) Diverse δ values for $\gamma = 1, \eta = 2, A_0 = 1, v = 1$.



(b) Diverse η values for $\gamma = 1, \delta = 5, A_0 = 1, v = 1$



(c) Diverse γ values for $\delta = 5, \eta = 2, A_0 = 1, v = 1$



(d) Diverse v values for $\gamma = 1, \delta = 5, \eta = 2, A_0 = 1$

Fig. 2: The effect of the parameters α, β, σ, and μ on $z(x,1)$.

In 2, the impacts of the parameters in the trial solution function.

3.2 The Modified Kudryashov Method

Assume that the NLODE in 4 has the following form solutions [16]:

$$\mathscr{Z}(\xi) = \sum_{i=0}^{\mu} A_i \varphi^i(\xi), \quad A_\mu \neq 0. \tag{11}$$

Since $\mu = 1$, 11 turns into

$$\mathscr{Z}(\xi) = A_0 + A_1 \varphi(\xi), \quad A_1 \neq 0. \tag{12}$$

Here, $\varphi(\xi)$ admits 13:

$$\varphi'(\xi) = \delta \log(P) \varphi(\xi)(\gamma\varphi(\xi) - 1), \tag{13}$$

where δ, γ are nonzero constants, $0 < P \neq 1$ and 13 has the following solution:

$$\varphi(\xi) = \frac{1}{\gamma + \eta P^{\delta \eta}}, \tag{14}$$

where η is nonzero constant. Let us substitute 14 and its related derivatives to 1. So, the following system of equations is obtained by collecting all terms of the same power of

$$\begin{aligned}
&\varphi^0(\xi): \; v\log^2(P) - 2\log^4(P) + v^2 = 0,\\
&\varphi^1(\xi): \; 27A_1\log^3(P) - \gamma\left(7v\log^2(P) - 62\log^4(P) + v^2\right) = 0,\\
&\varphi^2(\xi): \; 33A_1\gamma\log(P) - 2A_1^2 + 2\gamma^2\left(v - 30\log^2(P)\right) = 0,\\
&\varphi^3(\xi): \; \gamma\left(3A_1(4A_1 - 51\gamma\log(P)) - 2\gamma^2\left(v - 130\log^2(P)\right)\right) = 0,\\
&\varphi^4(\xi): \; \gamma^2(2\gamma\log(P) - A_1)(10\gamma\log(P) - A_1) = 0,\\
&\varphi^5(\xi): \; \gamma^3(2\gamma\log(P) - A_1)(10\gamma\log(P) - A_1) = 0.
\end{aligned}$$

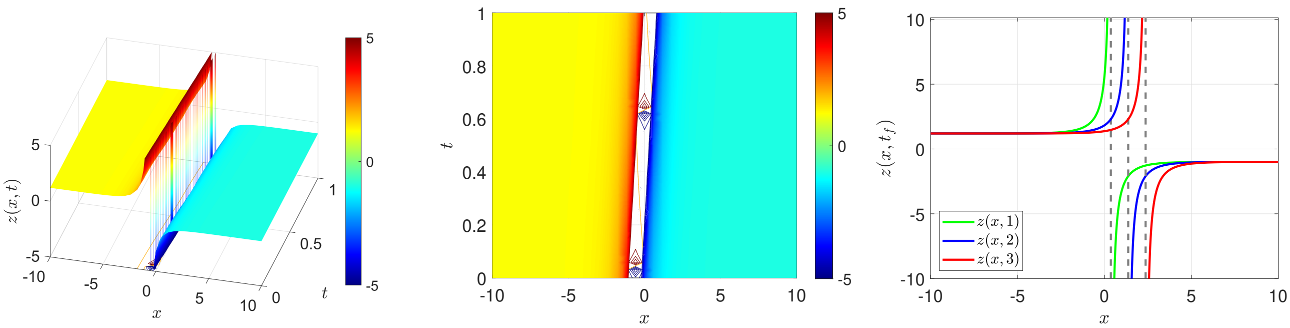
The above algebraic system may be solved using a symbolic calculation software to get the set below:

Set 1.

$$v = \delta^2\log^2(P), A_1 = 2\gamma\delta\log(P). \tag{15}$$

The following analytical solutions of 1 are obtained using Set 1 above and inserting φ and wave transformations in 2:

$$z(x,t) = A_0 + \frac{2\gamma\delta\log(P)}{\gamma + \eta P^{\delta(x-tv)}}. \tag{16}$$



(a) 3D graph. **(b)** The contour graph. **(c)** The 2D graphs.

Fig. 3: The plot of $z(x,t)$ where $\delta = 1, \gamma = -1, P = 3, \eta = 2, A_0 = -1, v = 1$.

A singular solution representation of $z(x,t)$ for $\delta = 1, \gamma = -1, P = 3, \eta = 2, A_0 = -1, v = 1$ is presented in 3.

4 Conclusion

This paper presents the successful derivation of the analytical solution of the Drinfeld-Sokolov-Satsuma-Hirota equation, a sixth-order partial differential equation. The solutions were obtained using the Kudryashov method. The kink soliton and singular solution were visualized using Matlab. The visualizations are consistent with the derived solutions and reflect their physical representations of the obtained solitons. The findings of this study are expected to be applicable to future research in the area of soliton solutions of higher-order partial differential equations.

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