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# New Products on Undirected Graphs

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**Abstract:** In this paper we present some new products on undirected graphs, explain them by examples and introduce some properties of these products. Moreover, we define the neighborhood set of vertices in these products and examine their sizes.

**Keywords:** Coneighbor graph, product of graphs, size of product, Coneighbor set of vertices.

## 1 Introduction

A graph product is a binary operation on graphs. mainly, an operation that takes two graphs $G_1$ and $G_2$ and produces a graph $H$ with the properties that the vertex set of $H$ is the Cartesian product $V(G_1) \times V(G_2)$, in which $V(G_1)$ and $V(G_2)$ are the vertex sets of $G_1$ and $G_2$, respectively. While vertices $(u_1,u_2)$ and $(v_1,v_2)$ of $H$ are adjacent if some conditions about $u_1$, $v_1$ in $G_1$ and $u_2$, $v_2$ in $G_2$ are fulfilled. In the literature, product graph is presented to establish new graphs and investigate their properties and applications, among them, intersection, union, join, Cartesian product, Kronecker product, strong product and composition product of graphs. A graph $G$ is a finite non empty set of elements called vertices together with a set of unordered pairs of distinct vertices of $G$ called edges. The vertex set of $G$ is denoted by $V$ or $V(G)$, while the edge set is denoted by $E$ or $E(G)$. A graph $G$ with vertex set $V$ and edge set $E$ may be denoted by an ordered pair $(V,E)$. The cardinality of the vertex set of a graph $G$ is called the order of $G$ and is denoted by $p$ or $|V(G)|$, while the cardinality of its edge set is called the size of $G$ and is denoted by $q$ or $|E|$. Two vertices $u$ and $v$ are said to be neighborhood of each other if $u$ and $v$ have a common edge in $G$. The set of all neighborhoods of $u$ is said to be open neighborhood of $u$ and it is denoted by $N(u)$; the set of all neighborhoods of $u$ together with $u$ is said to be the closed neighborhood of $u$ in $G$ and it is denoted by $N[u]$. The degree of a vertex $v$ in a graph $G$, denoted by $deg_G(v)$, is the number of edges incident with $v$. The adjacency matrix $A(G)$ or $A = [a_{ij}]$ of a labeled graph $G$ of order $p$ and with vertex set $V(G)$, $V(G) = \{v_1, v_2, ..., v_p\}$ is a $p \times p$ matrix in which $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent, and 0 if they are not. For details we refer to [1], [2], [3], [5] and [6].

Harary and Wilcox [6], introduced some new Boolean operations on undirected graphs and determined the their adjacency matrices. Also, they studied some invariants and relations of Boolean operations. El-kholy et al in [9], defined some new operations on undirected graphs and examined the relations between the folding of a given pair of graphs and the folding of a graph generated from them. Shibata and Kikuchi [3], defined new two products on graphs called skew product and the converse skew product and they found relations between these two products and other products on graphs. Also, they investigated several classes of graph products. The purpose of this paper, is to construct new graphs generated from product on undirected graphs by means of the concept of coneighbor set vertices.

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## 2 Properties of Coneighbor Graphs

In this section, we introduce some definitions and results of coneighbor graphs.

**Definition 1.**[8] Two vertices (edges) are said to be coneighbor if and only if they have the same set of neighborhood vertices (edges).

**Definition 2.**[8] A graph G is said to be coneighbor graph if it contains a set of coneighbor vertices, while a graph G is said to be an edge coneighbor graph if it contains a set of coneighbor edges.

**Definition 3.**[5] The set of all coneighbor of $u$ is the open coneighbor of $u$ or the coneighbor set of $u$ and it is denoted by $CN(u)$; the set $CN[u] = CN(u) \cup \{u\}$ is the closed coneighbor of $u$ in $G$.

**Definition 4.**[8] The coneighbor matrix $con(G)$ or $con = [c_{ij}]$ of a labeled graph $G$ of order $p$ and with vertex set $V(G)$, $V(G) = \{v_1, v_2, ..., v_p\}$ is a $p \times p$ matrix in which $c_{ij} = 1$ if $u_i$ and $u_j$ are coneighbor, and 0 if they are not.

**Definition 5.**[5] The Kronecker product of two matrices $M_{(m\times n)} \otimes N_{(l\times k)}$ is the $(m\times l)\times(n\times k)$ matrix is given by

$$M_{m\times n} \otimes N_{l\times k} = \begin{bmatrix} m_{11}N & \dots & m_{1n}N \\ \vdots & \ddots & \vdots \\ m_{m1}N & \dots & m_{mn}N \end{bmatrix}$$

**Theorem 1.**[5] The sum of degrees of the vertices of a graph is equal to twice the number of its edges. That is $\sum_{u\in G} deg(u) = 2q$.

In this paper, we use the symbol $\Delta_1$ and $\Delta_2$ is the number of the sets of pairs of coneighbor vertices in a graph $G_1$ and $G_2$, respectively.

## 3 Coneighbor Products on Graphs

In this section, we define some new products on undirected graphs and present some properties and results of these new products.

**Definition 6.** *Let $G_1$ and $G_2$ be two graphs, the coneighbor Cartesian product of graphs $G_1$ and $G_2$ is the graph denoted by $G_1 \boxed{C^\times} G_2$ whose vertex set is $V_1 \times V_2$ and two vertices $u = (u_1, u_2)$ in $G_1$ and $v = (v_1, v_2)$ in $G_2$ are adjacent in $G_1 \boxed{C^\times} G_2$ if and only if either:*

*1. $u_1$ is coneighbor to $u_2$ in $G_1$ and $v_1 \equiv v_2$ in $G_2$, or*
*2. $u_1 \equiv u_2$ in $G_1$ and $v_1$ is coneighbor to $v_2$ in $G_2$.*

The following results can be proved easily.

**Proposition 1.** *Let $G_1$ and $G_2$ be two graphs with vertex sets $p_1$, $p_2$ respectively, then the following statements are fulfilled.*

1. $p(G_1 \boxed{C^\times} G_2) = p_1 \times p_2$.
2. $G_1 \boxed{C^\times} G_2 \cong G_2 \boxed{C^\times} G_1$.
3. If $G_1$ and $G_2$ are coneighbor graphs, then $G_1 \boxed{C^\times} G_2$ is coneighbor graph.

4.If $G_1$ and $G_2$ are coneighbor bipartite graphs, then $G_1 \boxed{C^{\times}} G_2$ is disconnected.

5.If one of $G_1$ or $G_2$ is non-coneighbor graph, then $G_1 \boxed{C^{\times}} G_2$ is disconnected.

6.If $G_1$ and $G_2$ are non-coneighbor graphs, then $G_1 \boxed{C^{\times}} G_2$ is null graph of order $p_1 \times p_2$.

In the following theorem, we calculate the sum of degrees of coneighbor vertices.

**Theorem 2.** *Let $G$ be a coneighbor graph and $u \in G$, then*

$$\sum_{u \in G} deg(CN(u)) = deg(u)\Delta,$$

*where $\Delta$ is the number of pairs of coneighbor sets containing $u$.*

*Proof.* The proof is obvious when $CN(u) = \phi$. Let $G$ be a graph such that $CN(u) = \{u_1, u_2, ..., u_n\}$, then they have the same neighborhoods, so the degree of each coneighbor vertex is equal to the number of neighborhoods. that is, $deg(u) = deg(u_i) = c \ 1 \leq i \leq n$. Also, we have $\Delta = n$ and hence

$$\sum_{u \in G} deg(CN(u)) = deg(u)\Delta = nc.$$

**Theorem 3.** *If $(u, v)$ in $G_1 \boxed{C^{\times}} G_2$, then*

$$N_{G_1 \boxed{C^{\times}} G_2}(u,v) = \{CN_{G_1}(u) \times \{v\}\} \cup \{\{u\} \times CN_{G_2}(v)\}.$$

*Proof.* From Definition 6, we have a vertex $(u_i, v_j) \in N_{G_1 \boxed{C^{\times}} G_2}(u,v)$, $1 \leq i \leq p_1, 1 \leq j \leq p_2$, if $u_i$ is coneighbor with $u$ in $G_1$ and $v_j \cong v$ or $u_i \cong u$ and $v_j$ is coneighbor with $v$ in $G_2$. That is, if $N_{G_1 \boxed{C^{\times}} G_2}(u,v) = \{CN_{G_1}(u) \times \{v\}\} \cup \{\{u\} \times CN_{G_2}(v)\}$.

**Theorem 4.** *The size of $G_1 \boxed{C^{\times}} G_2$ is given by*
$q(G_1 \boxed{C^{\times}} G_2) = p_1\Delta_2 + p_2\Delta_1$.

*Proof.* By Theorem 3, we have $N_{G_1 \boxed{C^{\times}} G_2}(u,v) = \{CN_{G_1}(u) \times \{v\}\} \cup \{\{u\} \times CN_{G_2}(v)\}$ Therefore,

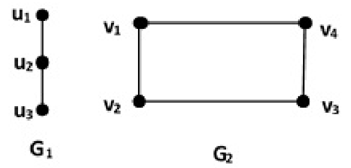
$$|N_{G_1 \boxed{C^{\times}} G_2}(u,v)| = |CN_{G_1}(u)| + |CN_{G_2}(v)|$$

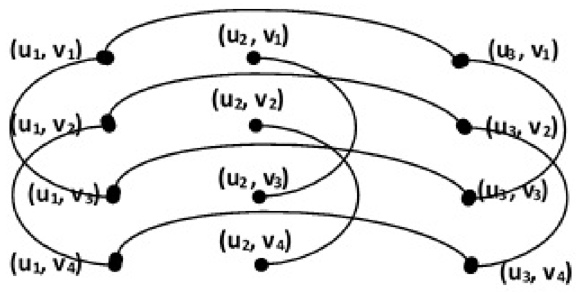
$$deg_{G_1 \boxed{C^{\times}} G_2}(u,v) = deg(CN_{G_1}(u)) + deg(CN_{G_2}(v)).$$

Hence,

$$\sum_{u \in G_1} \sum_{v \in G_2} deg_{G_1 \boxed{C^{\times}} G_2}(u,v) = \sum_{v \in G_2} \sum_{u \in G_1} deg(CN_{G_1}(u))$$

$$+ \sum_{u \in G_1} \sum_{v \in G_2} deg(CN_{G_2}(v))$$




$G_1 \boxed{C^{\times}} G_2$

$$\sum_{u\in G_1}\sum_{v\in G_2} deg_{G_1 \boxed{C^{\times}} G_2}(u,v) = p_2 \sum_{u\in G_1} deg(CN_{G_1}(u)) + p_1 \sum_{v\in G_2} deg(CN_{G_2}(v))$$

By Theorem 1 and 2, we have

$$2q(G_1 \boxed{C^{\times}} G_2) = p_2 deg_{G_1}(u)\Delta_1 + p_1 deg_{G_2}(v)\Delta_2$$

If $u$ and $v$ are coneighbor vertices in $G_1$ or $G_2$, then there exists an edge between them in $G_1 \boxed{C^{\times}} G_2$, so we have $2q(G_1 \boxed{C^{\times}} G_2) = 2p_1\Delta_2 + 2p_2\Delta_1$.
Hence, $q(G_1(C^{\times})G_2) = p_1\Delta_2 + p_2\Delta_1$.

**Proposition 2.** *The coneighbor matrix of* $G_1 \boxed{C^{\times}} G_2$ *is given by*

$$con(G_1 \boxed{C^{\times}} G_2) = (con(G_1) \otimes I_{p_2}) + (I_{p_1} \otimes con(G_2)).$$

*Proof.* Directly by Definition 6, we get the result.

Therefore, $con(G_1) \otimes I_{p_2}$ is equal to a (0, 1)-matrix of size $(p_1 \times p_2) \times (p_1 \times p_2)$ and each entry one in $con(G_1)$ replaced by $I_{p_2}$ and $I_{p_1} \otimes con(G_2)$ is equal to a matrix of size $(p_1 \times p_2) \times (p_1 \times p_2)$ and each entry one in the main diagonal of $I_{p_1}$ replaced by the $con(G_2)$. Hence, $con(G_1 \boxed{C^{\times}} G_2) = (con(G_1) \otimes I_{p_2}) + (I_{p_1} \otimes con(G_2)) = A(G_1 \boxed{C^{\times}} G_2)$ is the adjacency matrix of $G_1 \boxed{C^{\times}} G_2$ of size $(p_1 \times p_2) \times (p_1 \times p_2)$.

For example, Consider the graphs $G_1 = P_3$ and $G_2 = C_4$ are shown in below. Where $p(G_1 \boxed{C^{\times}} G_2) = p_1 \times p_2 = 3 \times 4 = 12$ and $q(G_1 \boxed{C^{\times}} G_2) = p_1\Delta_2 + p_2\Delta_1 = 3(2) + 4(1) = 10$.

The graph $G_1 \boxed{C^{\times}} G_2$ is illustrated below:

**Definition 7.** *Let $G_1$ and $G_2$ be two graphs, the coneighbor tensor product of graphs $G_1$ and $G_2$ is the graph denoted by $G_1 \boxed{C^*} G_2$ whose vertex set is $V_1 \times V_2$ and two vertices $(u_1, v_1)$ , $(u_2, v_2)$ in $G_2$ are adjacent in $G_1 \boxed{C^*} G_2$ if $[u_1$ is coneighbor to $u_2$ in $G_1$ and $v_1$ is coneighbor to $v_2$ in $G_2]$.*

The proof of the following results are obvious.

**Proposition 3.** *Let $G_1$ and $G_2$ be two graphs, then the following statements are true:*

*1.$p(G_1 \boxed{C^*} G_2) = p_1 \times p_2$.*

*2.$G_1 \boxed{C^*} G_2 \cong G_2(C^*)G_1$.*

*3.If $G_1$ and $G_2$ are coneighbor graphs, then $G_1 \boxed{C^*} G_2$ need not be a coneighbor graph.*

*4.If $G_1$ and $G_2$ are coneighbor bipartite graphs, then $G_1 \boxed{C^*} G_2$ is disconnected.*

*5.If one of $G_1$ or $G_2$ is non-coneighbor graph, then $G_1 \boxed{C^*} G_2$ is disconnected.*

*6.If $G_1$ and $G_2$ are non-coneighbor graphs, then $G_1 \boxed{C^*} G_2$ is null graph of order $p_1 \times p_2$.*

**Theorem 5.** *If $(u, v) \in G_1 \boxed{C^*} G_2$, then*

$$N_{G_1 \boxed{C^*} G_2}(u, v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\}.$$

*Proof.*From Definition 7, we have $(u_i, v_j) \in N_{G_1 \boxed{C^*} G_2}(u, v)$, $1 \le i \le p_1, 1 \le j \le p_2$, if and only if $u_i$ is coneighbor with $u$ in $G_1$ and $v_j$ is coneighbor with $v$ in $G_2$. That is, if and only if $(u_i, v_j) \in N_{G_1 \boxed{C^*} G_2}(u, v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\}$.

**Theorem 6.** *The size of $G_1 \boxed{C^*} G_2$ is given by $q(G_1 \boxed{C^*} G_2) = 2\Delta_1\Delta_2$.*

*Proof.*By Theorem 5, we have

$$N_{G_1 \boxed{C^*} G_2}(u, v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\}$$

Hence, we have

$$|N_{G_1 \boxed{C^*} G_2}(u, v)| = |CN_{G_1}(u)||CN_{G_2}(v)|.$$

Therefore,

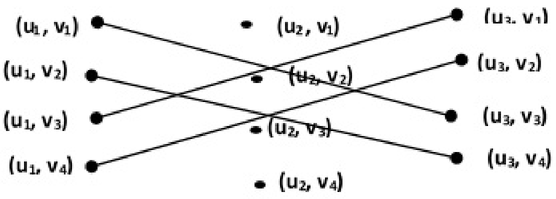
$$deg_{G_1 \boxed{C^*} G_2}(u, v) = deg(CN_{G_1}(u))deg(CN_{G_2}(v)).$$

This implies that

$$\sum_{u \in G_1} \sum_{v \in G_2} deg_{G_1 \boxed{C^*} G_2}(u, v) = \sum_{u \in G_1} deg(CN_{G_1}(u)) \sum_{v \in G_2} deg(CN_{G_2}(v))$$

By Theorem 1 and 2, we have

$$2q(G_1 \boxed{C^*} G_2) = deg_{G_1}(u)\Delta_1 deg_{G_2}(v)\Delta_2$$




$G_1 \boxed{C^*} G_2$

If $u$ and $v$ are coneighbor vertices in $G_1$ or $G_2$, then there exists an edge between them in $G_1 \boxed{C^*} G_2$, so we have $2q(G_1 \boxed{C^*} G_2) = 2\Delta_2 2\Delta_1$.
Hence, $q(G_1 \boxed{C^*} G_2) = 2\Delta_1\Delta_2$.

**Proposition 4.** *The coneighbor matrix of* $G_1 \boxed{C^*} G_2$ *is given by*

$$con(G_1 \boxed{C^*} G_2) = (con(G_1) \otimes con(G_2)).$$

*Proof.* Follows directly from Definition 7.

From Proposition 4, we deduce that $con(G_1 \boxed{C^*} G_2)$ is equal to a (0, 1)-matrix of size $(p_1 \times p_2) \times (p_1 \times p_2)$ and each entry one in $con(G_1)$ replaced by $con(G_2)$. Hence, $con(G_1 \boxed{C^*} G_2) = (con(G_1) \otimes con(G_2)) = A(G_1 \boxed{C^*} G_2)$ is the adjacency matrix of $G_1 \boxed{C^*} G_2$ of size $(p_1 \times p_2) \times (p_1 \times p_2)$. In the following example, we calculate the size of $G_1 \boxed{C^*} G_2$ using Theorem 6.

**Example 1.**Consider the graphs $G_1 = P_3$ and $G_2 = C_4$, then $G_1 \boxed{C^*} G_2$ is described below.

Where $p(G_1 \boxed{C^*} G_2) = p_1 p_2 = 3 \times 4 = 12$ and $q(G_1 \boxed{C^*} G_2) = 2\Delta_1\Delta_2 = 2(1)(2) = 4$.

**Definition 8.***Let* $G_1$ *and* $G_2$ *be two graphs, the coneighbor strong product of graphs* $G_1$ *and* $G_2$ *is the graph denoted by* $G_1 \boxed{C^\circ} G_2$ *whose vertex set is* $V_1 \times V_2$ *and two vertices* $(u_1, v_1)$ *and* $(u_2, v_2)$ *are adjacent in* $G_1 \boxed{C^\circ} G_2$ *if one of the following conditions hold:*

*1.*$u_1$ *is coneighbor to* $u_2$ *in* $G_1$ *and* $v_1 \equiv v_2$ *in* $G_2$*, or*
*2.*$u_1 \equiv u_2$ *in* $G_1$ *and* $v_1$ *is coneighbor to* $v_2$ *in* $G_2$*, or*
*3.*$u_1$ *is coneighbor to* $u_2$ *in* $G_1$ *and* $v_1$ *is coneighbor to* $v_2$ *in* $G_2$*.*

The following results are clear.

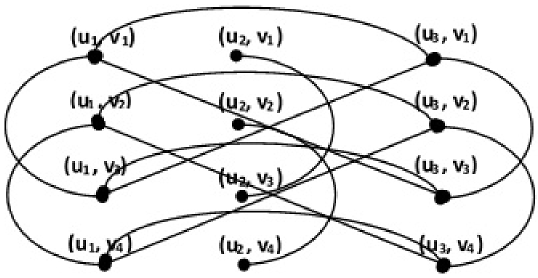
**Proposition 5.***Let* $G_1$ *and* $G_2$ *be two graphs, then:*

1.$p(G_1 \boxed{C^\circ} G_2) = p_1 \times p_2$.

2.$G_1 \boxed{C^\circ} G_2 \cong G_2 \boxed{C^\circ} G_1$.

3.If $G_1$ and $G_2$ are coneighbor graphs, then $G_1 \boxed{C^\circ} G_2$ need not be a coneighbor graph.

4.If $G_1$ and $G_2$ are coneighbor bipartite graphs, then $G_1 \boxed{C^\circ} G_2$ is disconnected.




$G_1 \boxed{C^\circ} G_2$

5.If one of $G_1$ or $G_2$ is non-coneighbor graph, then $G_1 \boxed{C^\circ} G_2$ is disconnected.

6.If $G_1$ and $G_2$ are non-coneighbor graphs, then $G_1 \boxed{C^\circ} G_2$ is null graph of order $p_1 \times p_2$.

**Theorem 7.** *For each vertex $(u,v)$ in $G_1 \boxed{C^\circ} G_2$, we have*

$$N_{G_1 \boxed{C^\circ} G_2}(u,v) = \{CN_{G_1}(u) \times \{v\}\} \cup \{\{u\} \times CN_{G_2}(v)\} \cup \{CN_{G_1}(u) \times CN_{G_2}(v)\}.$$

*Proof.* The proof follows from Theorem 3 and 5.

**Theorem 8.** *The size of $G_1 \boxed{C^\circ} G_2$ is given by $q(G_1 \boxed{C^\circ} G_2) = p_1\Delta_2 + p_2\Delta_1 + 2\Delta_1\Delta_2$.*

*Proof.* From Theorem 4 and 6, we get the result.

**Proposition 6.** *The coneighbor matrix of $G_1 \boxed{C^\circ} G_2$ is given by*

$$con(G_1 \boxed{C^\circ} G_2) = (con(G_1) \otimes I_{p_2}) + (I_{p_1} \otimes con(G_2)) + (con(G_1) \otimes con(G_2)).$$

*Proof.* Follows from Proposition 2 and 4.

The following example describes $G_1 \boxed{C^\circ} G_2$.

**Example 2.** Let $G_1 = P_3$ and $G_2 = C_4$, we have
$q(G_1 \boxed{C^\circ} G_2) = p_1\Delta_2 + p_2\Delta_1 + 2\Delta_1\Delta_2 = 3(2) + 4(1) + 2(1)(2) = 15$.
The graph $G_1 \boxed{C^\circ} G_2$ is shown in below.

**Definition 9.** *Let $G_1$ and $G_2$ be two graphs, the coneighbor composition product of graphs $G_1$ and $G_2$ is the graph denoted by $G_1 \boxed{C^c} G_2$ whose vertex set is $V_1 \times V_2$ and two vertices $(u_1,v_1)$ and $(u_2,v_2)$ are adjacent in $G_1 \boxed{C^c} G_2$ if and only if either:*

*1.$u_1$ is coneighbor to $u_2$ in $G_1$, or*
*2.$u_1 \equiv u_2$ in $G_1$ and $v_1$ is coneighbor to $v_2$ in $G_2$.*

From Definition 9, we obtain the following results:

**Proposition 7.** *Let $G_1$ and $G_2$ be two graphs, then*

1.$p(G_1 \boxed{C^c} G_2) = p_1 \times p_2$.

2.$G_1 \boxed{C^c} G_2 \ncong G_2 \boxed{C^c} G_1$.

3.If $G_1$ and $G_2$ are coneighbor graphs, then $G_1 \boxed{C^c} G_2$ need not be a coneighbor graph.

4.If one of $G_1$ or $G_2$ is non-coneighbor graph, then $G_1 \boxed{C^c} G_2$ is disconnected.

5.If one of $G_1$ or $G_2$ is non-coneighbor graph, then $G_1 \boxed{C^c} G_2$ may not be a coneighbor graph.

6.If $G_1$ and $G_2$ are non-coneighbor graphs, then $G_1 \boxed{C^c} G_2$ is null graph of order $p_1 \times p_2$.

**Theorem 9.** *For each vertex $(u,v)$ in $G_1 \boxed{C^c} G_2$, $N_{G_1 \boxed{C^c} G_2}(u,v) = \{CN_{G_1}(u) \times V(G_2)\} \cup \{\{u\} \times CN_{G_2}(v)\}$.*

*Proof.* From Definition 9, we have a vertex $(u_i, v_j) \in N_{G_1 \boxed{C^c} G_2}(u,v)$, $1 \le i \le p_1, 1 \le j \le p_2$, if and only if $u_i$ is coneighbor with $u$ in $G_1$ for each vertex in $G_2$ or $u_i \cong u$ and $v_j$ is coneighbor with $v$ in $G_2$. That is, if and only if $(u_i, v_j) \in N_{G_1 \boxed{C^c} G_2}(u,v) = \{CN_{G_1}(u) \times V(G_2)\} \cup \{\{u\} \times CN_{G_2}(v)\}$.

**Theorem 10.** *The size of $G_1 \boxed{C^c} G_2$ is given by*

$$q(G_1 \boxed{C^c} G_2) = p_2^2 \Delta_1 + p_1 \Delta_2.$$

*Proof.* By Theorem 9, we have

$$N_{G_1 \boxed{C^c} G_2}(u,v) = \{CN_{G_1}(u) \times V(G_2)\} \cup \{\{u\} \times CN_{G_2}(v)\}$$

Hence,

$$|N_{G_1 \boxed{C^c} G_2}(u,v)| = p_2 |CN_{G_1}(u)| + |CN_{G_2}(v)|.$$

Therefore,

$$deg_{G_1 \boxed{C^c} G_2}(u,v) = p_2 deg(CN_{G_1}(u)) + deg(CN_{G_2}(v))$$

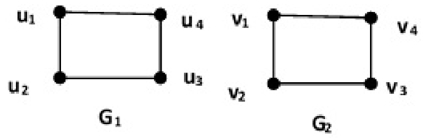
Consequently, we have

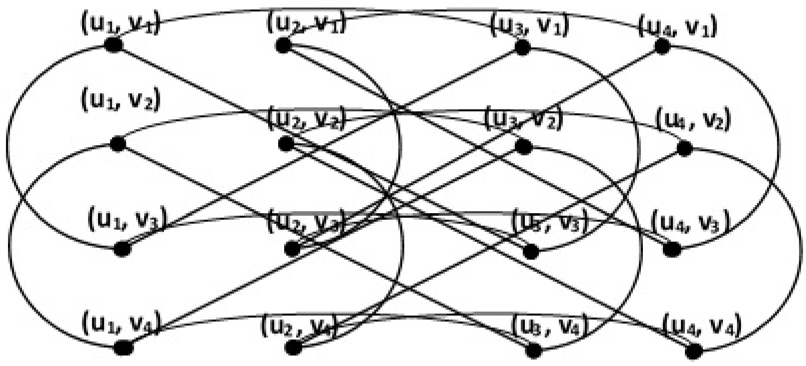
$$\sum_{u \in G_1} \sum_{v \in G_2} deg_{G_1 \boxed{C^c} G_2}(u,v) = p_2 \sum_{v \in G_2} \sum_{u \in G_1} deg(CN_{G_1}(u)) + \sum_{u \in G_1} \sum_{v \in G_2} deg(CN_{G_2}(v)).$$

By Theorem 1 and 2, we obtain that

$$2q(G_1 \boxed{C^c} G_2) = p_2^2 deg_{G_1}(u) \Delta_1 + p_1 deg_{G_2}(v) \Delta_2$$

If $u$ and $v$ are coneighbor vertices in $G_1$ or $G_2$, then there exists an edge between them in $G_1 \boxed{C^c} G_2$, so we have $2q(G_1 \boxed{C^c} G_2) = 2p_2^2 \Delta_1 + 2p_1 \Delta_2$. Hence, $q(G_1 \boxed{C^c} G_2) = p_2^2 \Delta_1 + p_1 \Delta_2$.




$G_1 \boxed{C^c} G_2$

**Proposition 8.** *The coneighbor matrix of* $G_1 \boxed{C^c} G_2$ *is given by*

$$con(G_1 \boxed{C^c} G_2) = (con(G_1) \otimes J_{p_2}) + (I_{p_1} \otimes con(G_2)).$$

*Proof.*Follows from Definition 9.

From the above proposition, we conclude that $con(G_1) \otimes J_{p_2}$ is equal to a (0, 1)-matrix of size $(p_1 \times p_2) \times (p_1 \times p_2)$ and each entry one in $con(G_1)$ replaced by $J_{p_2}$ and $I_{p_1} \otimes con(G_2)$ is equal to a matrix of size $(p_1 \times p_2) \times (p_1 \times p_2)$ and each entry one in the main diagonal of $I_{p_1}$ replaced by the $con(G_2)$. Hence, $con(G_1 \boxed{C^c} G_2) = (con(G_1) \otimes J_{p_2}) + (I_{p_1} \otimes con(G_2)) = A(G_1 \boxed{C^c} G_2)$ is the adjacency matrix of $G_1 \boxed{C^c} G_2$ of size $(p_1 \times p_2) \times (p_1 \times p_2)$.

In the following examples we describe $G_1 \boxed{C^c} G_2$ and calculate their sizes.

**Example 3.** Consider the graphs $G_1 = G_2 = C_4$.

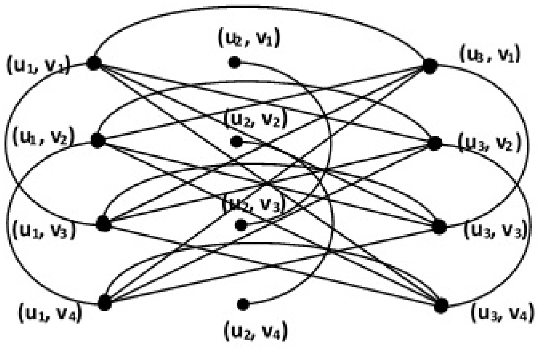
We have, $p(G_1 \boxed{C^c} G_2) = p_1 p_2 = (4)(4) = 16$ and $q(G_1 \boxed{C^c} G_2) = p_2^2 \Delta_1 + p_1 \Delta_2 = 16(2) + 4(2) = 40$. Then, $G_1 \boxed{C^c} G_2$ as shown in below: In the following example, we deduce that $P_3 \boxed{C^c} C_4 \ncong C_4 \boxed{C^c} P_3$.

**Example 4.**Consider the graphs $G_1 = P_3$ and $G_2 = C_4$ shown in below.

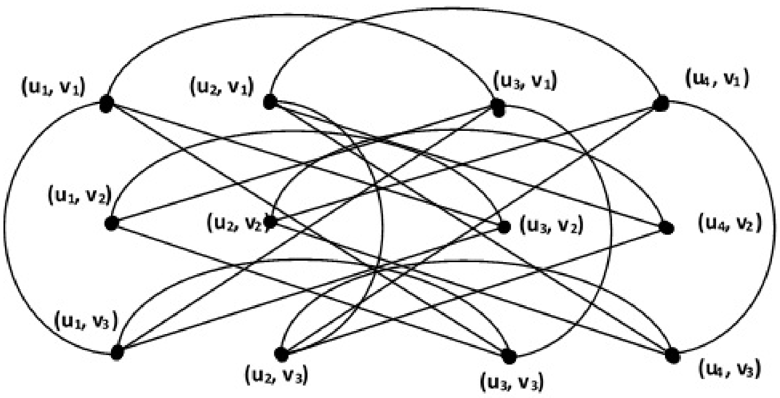
Then, $p(G_1 \boxed{C^c} G_2) = p_1 p_2 = (3)(4) = 12$ and $q(G_1 \boxed{C^c} G_2) = p_2^2 \Delta_1 + p_1 \Delta_2 = 16(1) + 3(2) = 22$. Then, $G_1 \boxed{C^c} G_2$ as shown in below:

**Example 5.**Let $G_1 = C_4$ and $G_2 = P_3$, then we have

$$p(G_1 \boxed{C^c} G_2) = p_1 p_2 = (4)(3) = 12 \text{ and } q(G_1 \boxed{C^c} G_2) = p_2^2 \Delta_1 + p_1 \Delta_2 = 2(9) + 4(1) = 22.$$




$\mathbf{G_1 \boxed{C^c} G_2}$




$\mathbf{G_1 \boxed{C^c} G_2}$

**Definition 10.** *Let $G_1$ and $G_2$ be two graphs, the coneighbor skew product of graphs $G_1$ and $G_2$ is the graph denoted by $G_1 \boxed{C^{\diamond}} G_2$ whose vertex set is $V_1 \times V_2$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G_1 \boxed{C^{\diamond}} G_2$ if and only if either:*

*1. $u_1$ is coneighbor to $u_2$ in $G_1$ and $v_1$ is coneighbor to $v_2$ in $G_2$, or*
*2. $u_1 \equiv u_2$ in $G_1$ and $v_1$ is coneighbor to $v_2$ in $G_2$.*

From Definition 10, we get the following results:

**Proposition 9.** *Let $G_1$ and $G_2$ be two graphs, then the following statements are true:*

1. $p(G_1 \boxed{C^{\diamond}} G_2) = p_1 \times p_2$.
2. $G_1 \boxed{C^{\diamond}} G_2 \ncong G_2 \boxed{C^{\diamond}} G_1$.
3. If $G_1$ and $G_2$ are coneighbor graphs, then $G_1 \boxed{C^{\diamond}} G_2$ is coneighbor graph.
4. If $G_1$ and $G_2$ are coneighbor or non-coneighbor graphs, then $G_1 \boxed{C^{\diamond}} G_2$ is disconnected.
5. If $G_2$ is non-coneighbor graph, then $G_1 \boxed{C^{\diamond}} G_2$ is null graph of order $p_1 \times p_2$.
6. If $G_1$ and $G_2$ are non-coneighbor graphs, then $G_1 \boxed{C^{\diamond}} G_2$ is null graph of order $p_1 \times p_2$.

**Theorem 11.** *For each vertex $(u, v)$ in $G_1 \boxed{C^{\diamond}} G_2$, $N_{G_1 \boxed{C^{\diamond}} G_2}(u, v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\} \cup \{\{u\} \times CN_{G_2}(v)\}$.*

*Proof.* From Definition 10, a vertex $(u_i,v_j) \in N_{G_1 \boxed{C^\diamond} G_2}(u,v)$, $1 \le i \le p_1, 1 \le j \le p_2$, if and only if $u_1$ is coneighbor to $u_2$ in $G_1$ and $v_1$ is coneighbor to $v_2$ in $G_2$ or $u_1 \equiv u_2$ in $G_1$ and $v_1$ is coneighbor to $v_2$ in $G_2$. That is, if and only if $(u_i,v_j) \in N_{G_1 \boxed{C^\diamond} G_2}(u,v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\} \cup \{\{u\} \times CN_{G_2}(v)\}$.

**Theorem 12.** *The size of* $G_1 \boxed{C^\diamond} G_2$ *is given by*

$$q(G_1 \boxed{C^\diamond} G_2) = 2\Delta_1\Delta_2 + p_1\Delta_2.$$

*Proof.* By Theorem 11, we have $N_{G_1 \boxed{C^\diamond} G_2}(u,v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\} \cup \{\{u\} \times CN_{G_2}(v)\}$

Hence,

$$|N_{G_1 \boxed{C^\diamond} G_2}(u,v)| = |CN_{G_1}(u)||CN_{G_2}| + |CN_{G_2}(v)|$$

Therefore,

$$deg_{G_1 \boxed{C^\diamond} G_2}(u,v) = deg(CN_{G_1}(v))deg(CN_{G_2}(v)) + deg(CN_{G_2}(v))$$

As a result, we have

$$\sum_{u\in G_1}\sum_{v\in G_2} deg_{G_1 \boxed{C^\diamond} G_2}(u,v) = \sum_{v\in G_2}\sum_{u\in G_1} deg(CN_{G_1}(v))deg(CN_{G_2}(v)) + \sum_{u\in G_1}\sum_{v\in G_2} deg(CN_{G_2}(v))$$

By Theorem 1 and 2, we get

$$2q(G_1 \boxed{C^\diamond} G_2) = deg_{G_1}(u)\Delta_1 deg_{G_2}(u)\Delta_2 + p_1 deg_{G_2}(v)\Delta_2$$

If $u$ and $v$ are coneighbor vertices in $G_1$ or $G_2$, then there exists an edge between them in $G_1 \boxed{C^\diamond} G_2$, so we have $2q(G_1 \boxed{C^\diamond} G_2) = 4\Delta_1\Delta_2 + 2p_1\Delta_2$. Hence, $q(G_1 \boxed{C^\diamond} G_2) = 2\Delta_1\Delta_1 + p_1\Delta_2$.

**Proposition 10.** The coneighbor matrix of $G_1 \boxed{C^\diamond} G_2$ is given by

$$con(G_1 \boxed{C^\diamond} G_2) = (con(G_1) \otimes con(G_2)) + (I_{p_1} \otimes con(G_2)).$$

*Proof.* Follows from Definition 10.

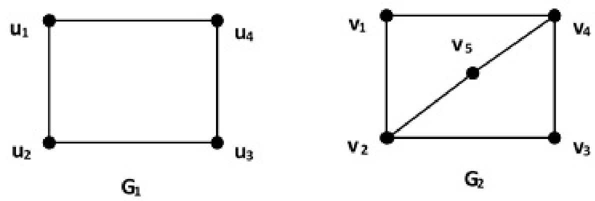
It is not difficult to show that $P_3 \boxed{C^\diamond} C_4 \ncong C_4 \boxed{C^\diamond} P_3$.

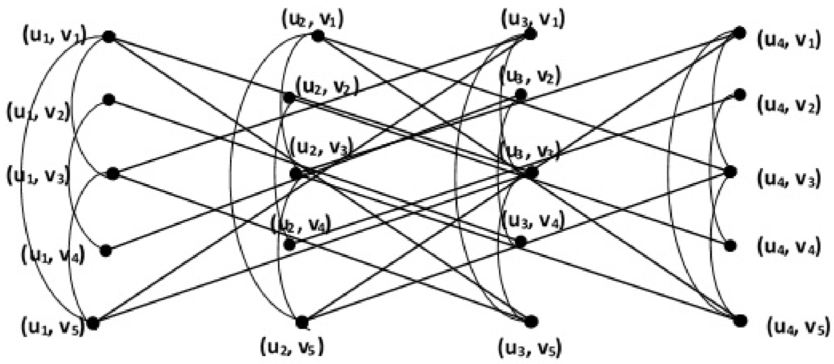
**Example 6.** consider the following graphs.

We have $p(G_1 \boxed{C^\diamond} G_2) = p_1p_2 = (4)(5) = 20$ and $q(G_1 \boxed{C^\diamond} G_2) = p_1\Delta_2 + 2\Delta_1\Delta_2 = 4(4) + 2(2)(4) = 32$.

The graph $G_1 \boxed{C^\diamond} G_2$ is given below:

**Definition 11.***Let $G_1$ and $G_2$ be two graphs, the coneighbor skew product of graphs $G_1$ and $G_2$ is the graph denoted by $G_1 \boxed{C^\blacklozenge} G_2$ whose vertex set is $V_1 \times V_2$ and two vertices $(u_1,v_1)$ and $(u_2,v_2)$ are adjacent in $G_1 \boxed{C^\blacklozenge} G_2$ if and only if either:*




$G_1 \boxed{C^{\lozenge}} G_2$

*1.$u_1$ is coneighbor to $u_2$ in $G_1$ and $v_1$ is coneighbor to $v_2$ in $G_2$, or*
*2.$u_1$ is coneighbor to $u_2$ in $G_1$ and $v_1 \equiv v_2$ in $G_2$.*

In $G_1 \boxed{C^{\blacklozenge}} G_2$ , the following statements are true:

1.$p(G_1 \boxed{C^{\blacklozenge}} G_2) = p_1 \times p_2$.

2.$G_1 \boxed{C^{\blacklozenge}} G_2 \ncong G_2 \boxed{C^{\blacklozenge}} G_1$.

3.If $G_1$ and $G_2$ are coneighbor or non-coneighbor graphs, then $G_1 \boxed{C^{\blacklozenge}} G_2$ is disconnected.

4.If $G_1$ is non-coneighbor graph, then $G_1 \boxed{C^{\blacklozenge}} G_2$ is null graph of order $p_1 \times p_2$.

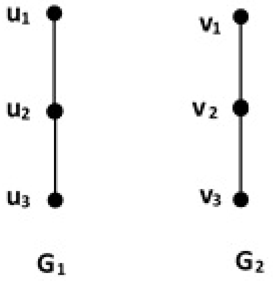
5.If $G_1$ and $G_2$ are non-coneighbor graphs, then $G_1 \boxed{C^{\blacklozenge}} G_2$ is null graph of order $p_1 \times p_2$.

**Theorem 13.** *For each vertex $(u,v)$ in $G_1 \boxed{C^{\blacklozenge}} G_2$, $N_{G_1 \boxed{C^{\blacklozenge}} G_2}(u,v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\} \cup \{CN_{G_1}(u) \times \{v\}\}$.*

*Proof.* From Definition 11, a vertex $(u_i, v_j) \in N_{G_1 \boxed{C^{\blacklozenge}} G_2}(u,v)$, $1 \le i \le p_1, 1 \le j \le p_2$, if and only if $u_i$ is coneighbor to $u$ in $G_1$ and $v_j$ is coneighbor to $v$ in $G_2$ or $u_i$ is coneighbor to $u$ in $G_1$ and $v_j \equiv v_2$ in $G_2$. That is, if and only if $(u_i, v_j) \in N_{G_1 \boxed{C^{\blacklozenge}} G_2}(u,v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\} \cup \{CN_{G_1}(u) \times \{v\}\}$.

**Theorem 14.** *The size of $G_1 \boxed{C^{\blacklozenge}} G_2$ is given by*

$q(G_1 \boxed{C^{\blacklozenge}} G_2) = 2\Delta_1\Delta_2 + p_2\Delta_1$.




*Proof.* By Theorem 13, we have

$$N_{G_1 \boxed{C^{\blacklozenge}} G_2}(u,v) = \{CN_{G_1}(u) \times CN_{G_2}(v)\} \cup \{CN_{G_1}(u) \times \{v\}\}.$$

Hence,

$$|N_{G_1 \boxed{C^{\blacklozenge}} G_2}(u,v)| = |CN_{G_1}(u)||CN_{G_2}| + |CN_{G_1}(u)|.$$

Consequently,

$$deg_{G_1 \boxed{C^{\blacklozenge}} G_2}(u,v) = deg(CN_{G_1}(v))deg(CN_{G_2}(v)) + deg(CN_{G_1}(u))$$

Therefore, we have

$$\sum_{u \in G_1} \sum_{v \in G_2} deg_{G_1 \boxed{C^{\blacklozenge}} G_2}(u,v) = \sum_{v \in G_2} \sum_{u \in G_1} deg(CN_{G_1}(v))deg(CN_{G_2}(v)) + \sum_{u \in G_1} \sum_{v \in G_2} deg(CN_{G_1}(u))$$

By Theorem 1 and 2, we have

$$2q(G_1 \boxed{C^{\blacklozenge}} G_2) = deg_{G_1}(u)\Delta_1 deg_{G_2}(u)\Delta_2 + p_2 deg_{G_1}(u)\Delta_1$$

If $u$ and $v$ are coneighbor vertices in $G_1$ or $G_2$, then there exists an edge between them in $G_1 \boxed{C^{\blacklozenge}} G_2$, so we have $2q(G_1 \boxed{C^{\blacklozenge}} G_2) = 4\Delta_1\Delta_2 + 2p_2\Delta_1$. Hence, $q(G_1 \boxed{C^{\blacklozenge}} G_2) = 2\Delta_1\Delta_1 + p_2\Delta_1$.

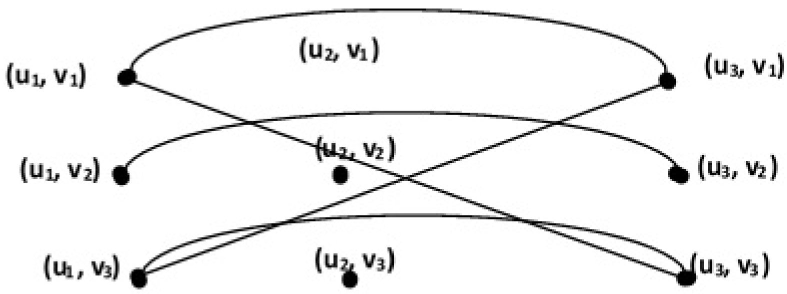
**Proposition 11.** *The coneighbor matrix of* $G_1 \boxed{C^{\blacklozenge}} G_2$ *is given by*

$con(G_1 \boxed{C^{\blacklozenge}} G_2) = (con(G_1) \otimes con(G_2)) + (con(G_1) \otimes I_{p_2})$.

*Proof.* Follows directly from Definition 11.

It is easy to show that $P_3 \boxed{C^{\blacklozenge}} C_4 \ncong C_4 \boxed{C^{\blacklozenge}} P_3$.

**Example 7.** Consider the graphs $G_1 = G_2 = P_3$ as shown below




$G_1 \boxed{C^{\blacklozenge}} G_2$

Then $p(G_1 \boxed{C^{\blacklozenge}} G_2) = p_1 p_2 = (3)(3) = 9$ and $q((G_1 \boxed{C^{\blacklozenge}} G_2) = p_2 \Delta_1 + 2\Delta_1 \Delta_2 = 3(1) + 2(1)(1) = 5$. The graph $G_1 \boxed{C^{\blacklozenge}} G_2$ is shown below:

## 4 Conclusion

In this paper, we introduced and studied some new products in theory of undirected graphs and presented some properties of new products, also showed that the size of their new products. Some examples are explained.

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