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# On the Maximum Apollonian Sets

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**Abstract:** In this study, the Apollonian sets are analytically examined in the plane equipped with the maximum distance function. It is determined that the maximum Apollonian set is not a maximum circle, but the closed simple rectilinear figure. Also, the classifications of the maximum Apollonian sets are given. In the case that the pair of points is on the gradual line or the steep line, the maximum Apollonian set is classified with respect to the condition.

**Keywords:** Apollonian set, maximum metric, maximum plane

## 1 Introduction

The Minkowski metric of order $p$ $(p \geq 1)$ is for distance between two points $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ determined by

$$d(X,Y) = \left(|x_2 - x_1|^p + |y_2 - y_1|^p\right)^{\frac{1}{p}}.$$

In the cases of $p = 1,\ 2$ and $\infty$, Minkowski metric is the taxicab metric, the Euclidean metric and the maximum metric, respectively. The maximum plane is the analytical plane equipped with the maximum metric and symbolized by $\mathbb{R}^2_M$. The plane and space geometries defined by different metrics have been studied and developed by some mathematicians $[5,6,7,8,9,10,11]$. The maximum plane is almost the same as the Euclidean plane except the distance function. The points and lines of the Maximum plane are the same as the Euclidean plane, and the angles are measured the same way as the Euclidean plane, but the concepts included the distance function are different. For example, the maximum unit circle with the equation $\max\{|x|, |y|\} = 1$ is a square where $(x,y) \in \mathbb{R}^2_M$.

The Apollonian set consists of points that the ratio of whose distances from two points is a constant $k$. The Apollonian set in Euclidean plane is a circle except the cases $k = 0,\ k = 1$. The Apollonian sets are stated as the degenerate cases for $k = 0$ or $k = 1$. It is possible to apply the definition of Apollonian set in different metric spaces. The Apollonian sets are studied and characterized in the taxicab plane, [1]. And it is obtained that the Apollonian set in taxicab plane is not a taxicab circle. Inspired by [1], we studied and developed analytically the Apollonian sets in the maximum plane in [4]. We introduce the results obtained in [4]. In the present paper, the Apollonian sets are examined in the maximum plane and named as the maximum Apollonian sets. It is determined that the maximum Apollonian set is not a maximum circle, but the simple closed rectilinear figure. Then, the classifications of the maximum Apollonian sets are given. Also, in the case that two distinct points are on the gradual line or the steep line, the maximum Apollonian sets are characterized on the conditions.

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The paper is organized as follows. In section 2, we consider the preliminaries of the maximum plane which includes the some definitions, concepts and theorems. In section 3, we introduce the Apollonian sets in maximum plane and mention their basic properties. Since the maximum Apollonian sets have different figures and properties when the two given points are on a gradual line or a steep line, we classify them according to the slope of the line and the value $k$ . Besides, in the case that the given two points are either on the same coordinate axis or on the same separator line, we determine the properties of the maximum Apollonian set without any conditions.

## 2 Preliminary

In sequel, the some definitions, concepts and theorems required for this study are summarized as the following:

**Definition 1.** *The maximum distance between points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ in the analytical plane is*

$$d_M(A_1, A_2) = \max\{|x_2 - x_1|, |y_2 - y_1|\}.$$

It is seen from definition that the maximum distance between the points $A_1$ and $A_2$ is equal to the greatest of the lengths of the line segments parallel to the coordinates axes in the right triangle with the hypotenuse $A_1A_2$.

**Definition 2.** *Let $m$ be the slope of the line $l$ in maximum plane. The line $l$ is called the steep line, the gradual line and the separator (guide) line in the cases of $|m| > 1$, $|m| < 1$ and $|m| = 1$, respectively. In the special cases that the line $l$ is parallel to x-axis or y-axis, $l$ is named as the horizontal line or the vertical line, respectively,* [2].

**Theorem 1.** *All translations are isometries in the maximum plane,* [3].

**Theorem 2.** *The reflections in the lines of the set $\{x = 0, y = 0, y = \pm x\}$ are isometries in the maximum plane,* [3].

**Theorem 3.** *The rotations by $\theta$-angle in the set $\left\{\theta \mid \theta = k\frac{\pi}{2}, k = 0, 1, 2, 3\right\}$ are isometries in the maximum plane,* [3].

**Definition 3.** *The minimum distance set (the shortest distance set) of the points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ in the maximum plane is the set*

$$\{X \mid d_M(A_1, X) + d_M(X, A_2) = d_M(A_1, A_2)\}.$$

Notice that this set is a filled rectangle whose sides are on the separator lines through the points $A_1$ and $A_2$. If the points $A_1$ and $A_2$ is on a separator line, it is the line segment $A_1A_2$ , (Fig.1).

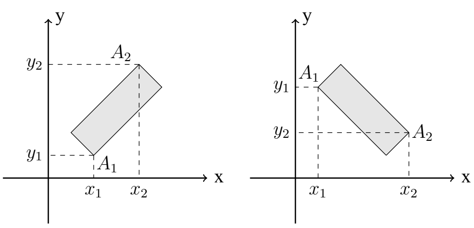


**Fig. 1:** Minimum distance set

**Definition 4.** *The maximum midset of the points $A_1$ and $A_2$ in the maximum plane is the set*

$$\{X \mid d_M(A_1, X) = d_M(X, A_2)\}.$$

It is well known that the Euclidean midset is the line through the midpoint of $A_1A_2$ that is the perpendicular to $A_1A_2$, whereas the maximum midset has the different figures, (Fig.2). The maximum midset depending on the position of the points $A_1$ and $A_2$ has the following properties:
(1) If the line segment $A_1A_2$ is parallel to the coordinate axis, the maximum midset consists of two planar regions connected by a line segment through the midpoint of $A_1$ and $A_2$ perpendicular to $A_1A_2$ inside the minimum distance set such that the planar regions are formed by the separator lines of $A_1$ and $A_2$.
(2) If the completion of the line segment $A_1A_2$ is a separator line, the maximum midset is the other separator line through the midpoint of $A_1$ and $A_2$.
(3) If the completion of the line segment $A_1A_2$ is a gradual line or a steep line , the maximum midset consists of two rays connected by a line segment through the midpoint of $A_1$ and $A_2$ parallel to the coordinate axis inside the minimum distance set of the points $A_1$ and $A_2$, such that the rays are on the separator lines through the intersection points of the line segment and the boundary of the minimum distance set.

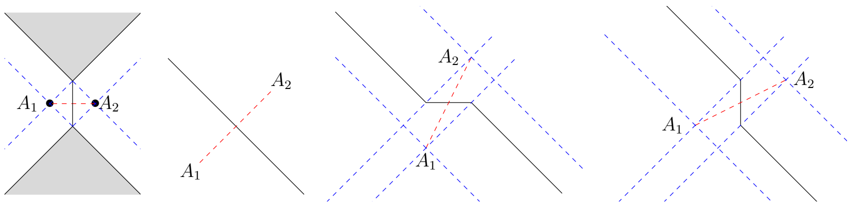


**Fig. 2:** Maximum midset

## 3 The Maximum Apollonian Sets

The Apollonian set in Euclidean plane is the set of points that the ratio of whose the Euclidean distances from two given points is a constant $k \geq 0$, $k \in \mathbb{R}$. Let $A_1$ and $A_2$ be two distinct points in Euclidean plane and $k \geq 0$. The Apollonian set is

$$\left\{X \in \mathbb{R}^2 \mid \frac{d_E(A_1,X)}{d_E(A_2,X)} = k\right\}$$

and it is denoted by $\mathscr{A}(A_1,A_2;k)$. Analogously, the Apollonian set in the maximum plane is defined as the following:

**Definition 5.** *Let $A_1$ and $A_2$ be two distinct points in maximum plane and $k \geq 0$, $k \in \mathbb{R}$. The set*

$$\left\{X \in \mathbb{R}^2_M \mid \frac{d_M(A_1,X)}{d_M(A_2,X)} = k\right\}$$

*is called the maximum Apollonian set and denoted by $\mathscr{A}(A_1,A_2;k)_M$.*

The maximum Apollonian set is the set of points that the ratio of whose the maximum distances from two given points is a constant $k \geq 0$. It is immediately seen from the definition that $\mathscr{A}(A_1,A_2;1)_M$ is the maximum midset of the points $A_1$ and $A_2$, $\mathscr{A}(A_1,A_2;0)_M = \{A_1\}$ ve $\mathscr{A}(A_1,A_2;\infty)_M = \{A_2\}$. In the Euclidean plane, the Apollonian set is a circle except the cases $k \in \{0,1,\infty\}$. Despite that the maximum Apollonian sets have different figures as the positions of the points and the constant $k$ are changed. In this study, we emphasize that the conditions on the constant $k$ and the slope of the line through two given points are used to characterize the maximum Apollonian sets.

The following proposition illustrate that no generality is lost to take $k>1$ in the maximum Apollonian sets.

**Proposition 1.** *Let any two distinct points be $A_1$ and $A_2$ in $\mathbb{R}^2_M$ and $k\in\mathbb{R}^+$. Then $\mathscr{A}(A_1,A_2;k)_M$ is equal to $\mathscr{A}(A_2,A_1;\frac{1}{k})_M$.*

*Proof.* For $A_1,A_2\in\mathbb{R}^2_M$, $A_1\neq A_2$ and $k>0$ ,

$$\begin{aligned}\mathscr{A}(A_1,A_2;k)_M &= \left\{X\in\mathbb{R}^2_M \mid \frac{d_M(A_1,X)}{d_M(A_2,X)}=k\right\}\\ &= \left\{X\in\mathbb{R}^2_M \mid \frac{d_M(A_2,X)}{d_M(A_1,X)}=\frac{1}{k}\right\}=\mathscr{A}(A_2,A_1;\frac{1}{k})_M.\end{aligned}$$

In special case, $\mathscr{A}(A_1,A_2;0)_M=\mathscr{A}(A_2,A_1;\infty)_M$ and $\mathscr{A}(A_1,A_2;\infty)_M=\mathscr{A}(A_2,A_1;0)_M$.

The following proposition state that the isometry in the maximum plane maps a maximum Apollonian set to another maximum Apollonian set.

**Proposition 2.** *Let $A_1$ and $A_2$ be any two points in $\mathbb{R}^2_M$ and $k\in\mathbb{R}^+$. If the map $\varphi$ is an isometry in the maximum plane, then*

$$\varphi(\mathscr{A}(A_1,A_2;k)_M)=\mathscr{A}(\varphi(A_1),\varphi(A_2);k)_M\ .$$

*Proof.* Suppose first that $Y\in\mathscr{A}(\varphi(A_1),\varphi(A_2);k)_M$. Then the equality

$$\frac{d_M(\varphi(A_1),Y)}{d_M(\varphi(A_2),Y)}=k$$

is valid. It is well known that the isometric map $\varphi$ is bijection in $\mathbb{R}^2_M$. Thus, there is a unique point $X$ in $\mathbb{R}^2_M$ such that $\varphi(X)=Y$ and the equality

$$\frac{d_M(\varphi(A_1),\varphi(X))}{d_M(\varphi(A_2),\varphi(X))}=k$$

is obtained. It is

$$\frac{d_M(A_1,X)}{d_M(A_2,X)}=k.$$

So, $X\in\mathscr{A}(A_1,A_2;k)_M$ and $Y=\varphi(X)\in\varphi(\mathscr{A}(A_1,A_2;k)_M)$. Hence, $\mathscr{A}(\varphi(A_1),\varphi(A_2);k)_M\subset\varphi(\mathscr{A}(A_1,A_2;k)_M)$. Conversely, suppose that $Y=\varphi(X)\in\varphi(\mathscr{A}(A_1,A_2;k)_M)$. Then,

$$k=\frac{d_M(A_1,X)}{d_M(A_2,X)}=\frac{d_M(\varphi(A_1),\varphi(X))}{d_M(\varphi(A_2),\varphi(X))}=\frac{d_M(\varphi(A_1),Y)}{d_M(\varphi(A_2),Y)}$$

and $Y\in\mathscr{A}(\varphi(A_1),\varphi(A_2);k)_M,\ \varphi(\mathscr{A}(A_1,A_2;k)_M)\subset\mathscr{A}(\varphi(A_1),\varphi(A_2);k)_M$.

For the points $A_1$ and $A_2$ in $\mathbb{R}^2_M$, $M=\frac{1}{2}(A_1+A_2)$ is the midpoint of $A_1$ and $A_2$ satisfying

$$d_M(A_1,M)=d_M(M,A_2)=\frac{1}{2}d_M(A_1,A_2).$$

**Proposition 3.** *Let $A_1$ and $A_2$ be any two points in $\mathbb{R}^2_M$ and $k\in\mathbb{R}^+$. If $\varphi$ is the rotation by $\pi$-angle about the midpoint $M$ of $A_1$ and $A_2$, then*

$$\varphi(\mathscr{A}(A_1,A_2;k)_M)=\mathscr{A}(A_1,A_2;\frac{1}{k})_M\ .$$

*Proof.* Without loss of generality, let $\varphi$ be the rotation by $\pi$-angle about the midpoint $M$ of the points $A_1=(0,0)$ and $A_2=(x_0,y_0)$. Then the map $\varphi=\tau_M rot\pi\tau_{-M}$ is a rotation about $M$ where $rot\pi$ is rotation about the origin and $\tau_M,\ \tau_{-M}$

are translations. Since $\varphi$ maps the point $(x,y)$ to the point with the coordinates $(-x+x_0,-y+y_0)$, then $\varphi(A_1)=A_2$ and $\varphi(A_2)=A_1$. From the proposition 1 and 2

$$\varphi(\mathscr{A}(A_1,A_2;k)_M)=\mathscr{A}(A_2,A_1;k)_M=\mathscr{A}(A_1,A_2;\frac{1}{k})_M.$$

Now, we give the maximum Apollonian sets by expressing their properties as follows. Since translations preserve the maximum distance, no generality is lost to take one of two points as origin in proofs.

**Theorem 4.** *Let $A_1$ and $A_2$ be two distinct points on the same coordinate axis in the maximum plane and $k>1,\ k\in\mathbb{R}^+$. The maximum Apollonian set $\mathscr{A}(A_1,A_2;k)_M$ has the following properties :*
*(1) It consists of four line segments. The completions of the two line segments intersect at the point $A_1$ and the other two line segments are on lines perpendicular to the line $A_1A_2$.*
*(2) Its vertices are on the separator lines through the point $A_2$.*
*(3) It has two parallel sides.*
*(4) Two sides of $\mathscr{A}(A_1,A_2;k)_M$ have equal maximum length.*
*(5) It is symmetric about the line $A_1A_2$.*

*Proof.* Suppose that the points $A_1$ and $A_2$ are on the $x$-axis. Let the coordinates of the points $A_1$ and $A_2$ be $(0,0)$ and $(x_0,0)$. The maximum Apollonian set $\mathscr{A}(A_1,A_2;k)_M$ has the equality

$$\frac{\max\{|x|,|y|\}}{\max\{|x-x_0|,|y|\}}=k \tag{1}$$

where $k>1$. In the region determined by the inequalities $|x|>|y|$ and $|x-x_0|>|y|$, the solutions of the equality (1) are the line segment with the equation $x=\frac{k}{k+1}x_0$ joining $V_1=(\frac{k}{k+1}x_0,\frac{x_0}{k+1})$ and $V_2=(\frac{k}{k+1}x_0,-\frac{x_0}{k+1})$ for the case $0\leq x<x_0$ and the line segment with the equation $x=\frac{k}{k-1}x_0$ joining $V_3=(\frac{k}{k-1}x_0,\frac{x_0}{k-1})$ and $V_4=(\frac{k}{k-1}x_0,-\frac{x_0}{k-1})$ for the case $x\geq x_0$. These two line segments are parallel and the completions of them are perpendicular to the line $A_1A_2$. The other solutions of the equality (1) are the line segment $x+ky=0$ joining $V_2$ and $V_4$ and the line segment with the equation $x-ky=0$ joining $V_1$ and $V_3$. The completions of these two line segments intersect at the point $A_1=(0,0)$. Also, the maximum lengths of the sides $V_1V_3$ and $V_2V_4$ are equal to $\frac{2k|x_0|}{k^2-1}$. Considering the reflection $\Omega$ in the $x$-axis, $\Omega$ leaves the points $A_1$ and $A_2$ fixed. And from Proposition 2, $\Omega(\mathscr{A}(A_1,A_2;k)_M)=\mathscr{A}(A_1,A_2;k)_M$. Thus, $\mathscr{A}(A_1,A_2;k)_M$ is symmetric about the line $A_1A_2$. It is seen that $\mathscr{A}(A_1,A_2;k)_M$ a isosceles trapezoid.

If the points $A_1$ and $A_2$ are on the $y$-axis, then the reflection $\varphi$ in the separator line $y=x$ is considered. $\varphi$ leaves $A_1$ fixed and takes $A_2$ to a point $A_3$ on the $x$-axis. From Proposition 2, $\mathscr{A}(A_1,A_2;k)_M=\varphi(\mathscr{A}(A_1,A_3;k)_M)$. Thus, it is seen that $\mathscr{A}(A_1,A_2;k)_M$ has the aforementioned properties.

In case that the points $A_1$ and $A_2$ are on the line parallel to the coordinate axis, using a suitable translation, it is observed that $\mathscr{A}(A_1,A_2;k)_M$ has the stated properties in Theorem 4.

**Theorem 5.** *Let $A_1$ and $A_2$ be two distinct points on the same separator line in the maximum plane and $k>1,\ k\in\mathbb{R}^+$. The maximum Apollonian set $\mathscr{A}(A_1,A_2;k)_M$ has the following properties :*
*(1) It consists of four line segments. The slopes of the two line segments are $-m\frac{1}{k}$ and $-mk$ where $m$ is the slope of the line $A_1A_2$ and the others are parallel to the coordinate axes.*
*(2) Its vertices are on the separator lines passing through the point $A_2$.*
*(3) $\mathscr{A}(A_1,A_2;k)_M$ is symmetric about the separator line $A_1A_2$.*

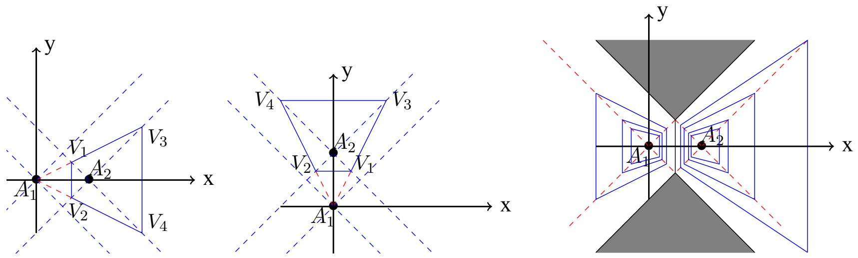


**Fig. 3:** Maximum Apollonian sets for pair of the points on the same coordinate axis

*Proof.* Suppose that the points $A_1$ and $A_2$ are on the separator line $y = x$. Let the abscissas of the points $A_1$ and $A_2$ be 0 and $x_0$, $x_0 > 0$. The maximum Apollonian set $\mathscr{A}(A_1, A_2; k)_M$ has the equality

$$\frac{\max\{|x|, |y|\}}{\max\{|x - x_0|, |y - x_0|\}} = k \tag{2}$$

where $k > 1$. In the region of inequalities $|x| > |y|$ and $|x - x_0| > |y - x_0|$, the solution of the equality (2) is the line segment with the equation $x = \frac{k}{k-1}x_0$ joining $V_2 = (\frac{k}{k-1}x_0, \frac{k-2}{k-1}x_0)$ and $V_4 = (\frac{k}{k-1}x_0, \frac{k}{k-1}x_0)$. Other solutions are the line segment $x + ky = kx_0$ joining $V_1 = (\frac{k}{k+1}x_0, \frac{k}{k+1}x_0)$ and $V_2$, the line segment with the equation $kx + y = kx_0$ joining $V_1$ and $V_3 = (\frac{k-2}{k-1}x_0, \frac{k}{k-1}x_0)$ and the line segment with the equation $y = \frac{k}{k-1}x_0$ joining $V_3$ and $V_4$. It is seen that all the vertices $V_i$ are on the separator lines $y = x$ and $y = -x + 2x_0$ through $A_2$. Since the points $A_1$ and $A_2$ are fixed under the reflection in the separator line $y = x$ and from proposition 2, $\mathscr{A}(A_1, A_2; k)_M$ is symmetric about the separator line $A_1A_2$.

If the points $A_1$ and $A_2$ be on the separator line $y = -x$ where the abscissa of the point $A_2$ is positive, then the reflection $\varphi$ in $x$-axis takes the point $A_2$ to a point $A_3$ on the line $y = x$. From Proposition 2, $\mathscr{A}(A_1, A_2; k)_M = \varphi(\mathscr{A}(A_1, A_3; k)_M)$. Thus, it is seen that $\mathscr{A}(A_1, A_2; k)_M$ and $\mathscr{A}(A_1, A_3; k)_M$ are congruent and have the same properties.

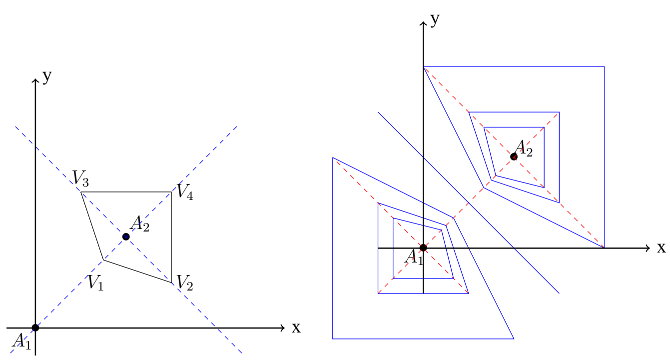


**Fig. 4:** Maximum Apollonian sets for pair of the points on the same separator line

Let the other two vertices of the rectangle with the diagonal $A_1A_2$ be denoted by $B_{12} = (x_1, y_2)$ and $B_{21} = (x_2, y_1)$ where the coordinates of the points $A_1$ and $A_2$ are $(x_1, y_1)$ and $(x_2, y_2)$. Since the points $A_2$ and $B_{12}$ are on the line parallel to the coordinate axis, $\mathscr{A}(B_{12}, A_2; k)_M$ is a isosceles trapezoid. Similarly, $\mathscr{A}(B_{21}, A_2; k)_M$ is a isosceles trapezoid. We observe that $\mathscr{A}(A_1, A_2; k)_M$ is the boundary of the region covered by the union of $\mathscr{A}(B_{12}, A_2; k)_M$ and $\mathscr{A}(B_{21}, A_2; k)_M$.

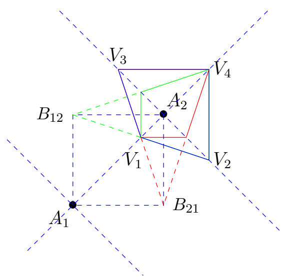


**Fig. 5**

When two given points are on the gradual line, the maximum Apollonian sets are expressed according to the value $k$ and the slope of the line in the following theorems.

**Theorem 6.** *Let $A_1$ and $A_2$ be two distinct points on a gradual line in the maximum plane . If $k < \frac{|m|+1}{1-|m|}$ where $m$ is the slope of the line segment $A_1A_2$, then the maximum Apollonian set $\mathscr{A}(A_1,A_2;k)_M$ has the following properties :*
*(1) It consists of six line segments. The slopes of the three line segments are $\pm sgn(m)\frac{1}{k}$, $sgn(-m)k$ and the others are parallel to the coordinate axes.*
*(2) Four of the six vertices are on the separator lines passing through the point $A_2$ and the other two are on the same separator line through the point $A_1$.*

*Proof.* Suppose that the coordinates of the points $A_1$ and $A_2$ are $(0,0)$ and $(x_0,y_0)$, where $x_0 > 0$. The maximum Apollonian set consists of points satisfying the equality

$$\frac{\max\{|x|,|y|\}}{\max\{|x-x_0|,|y-y_0|\}} = k \tag{3}$$

where $k > 1$ and $(x,y) \in \mathbb{R}^2_M$. The solutions of the equality (3) are obtained depending on the sign of $m$ as follows:

For the inequalities $|x| > |y|$ and $|x-x_0| > |y-y_0|$, the solutions are the line segment with the equation $x = \frac{k}{k+1}x_0$ joining the points $V_1 = (\frac{k}{k+1}x_0, sgn(-m)\frac{x_0}{k+1}+y_0)$ and $V_2 = (\frac{k}{k+1}x_0, sgn(m)\frac{k}{k+1}x_0)$ and the line segment with the equation $x = \frac{k}{k-1}x_0$ joining the points $V_3 = (\frac{k}{k-1}x_0, sgn(m)\frac{x_0}{k-1}+y_0)$ and $V_4 = (\frac{k}{k-1}x_0, sgn(-m)\frac{x_0}{k-1}+y_0)$. Other solutions are the line segment $x + sgn(m)ky = sgn(m)ky_0$ joining the points $V_1$ and $V_4$, the line segment with the equation $x - sgn(m)ky = -sgn(m)ky_0$ joining the points $V_3$ and $V_5 = (sgn(m)\frac{k}{k-1}y_0, \frac{k}{k-1}y_0)$, the line segment with the equation $kx + sgn(m)y = kx_0$ joining the points $V_2$ and $V_6 = (x_0 + sgn(-m)\frac{y_0}{k-1}, \frac{k}{k-1}y_0)$ and the line segment with the equation $y = \frac{k}{k-1}y_0$ joining the points $V_5$ and $V_6$. Thus, it is seen that the stated properties are obtained. In the case that $x_0 < 0$, all the vertices and sides of $\mathscr{A}(A_1,A_2;k)_M$ are determined immediately by using the reflection in $y$-axis and it is seen that it has the same properties.

On the other hand, it is observed that $\mathscr{A}(A_1,A_2;k)_M$ is the boundary of the region covered by the union of $\mathscr{A}(B_{12},A_2;k)_M$ and $\mathscr{A}(B_{21},A_2;k)_M$.

**Theorem 7.** *Let $A_1$ and $A_2$ be two distinct points on a gradual line in the maximum plane . If $k \geq \frac{|m|+1}{1-|m|}$ where $m$ is the slope of the line segment $A_1A_2$, then the maximum Apollonian set $\mathscr{A}(A_1,A_2;k)_M$ has the following properties :*
*(1) It consists of four line segments. The slopes of the two line segments are $\pm\frac{1}{k}$ and the others are on lines perpendicular to the same coordinate axis.*
*(2) Four vertices are on the separator lines passing through the point $A_2$.*

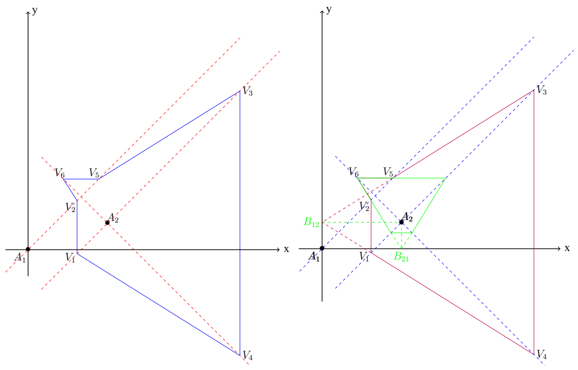


**Fig. 6:** Maximum Apollonian sets for pair of the points on the same gradual line

*(3) The intersection points of the completions of two non-parallel sides is on the line through $A_2$ parallel to the coordinate axis.*
*(4) Two sides of $\mathscr{A}(A_1,A_2;k)_M$ are of equal maximum length.*

*Proof.* Suppose that the coordinates of the points $A_1$ and $A_2$ are $(0,0)$ and $(x_0,y_0)$ where $x_0>0$. Examining all cases in the equality (3), the sides and vertices of $\mathscr{A}(A_1,A_2;k)_M$ are obtained as the following:

For $|x|>|y|$ and $|x-x_0|>|y-y_0|$, the solutions are the line segment with the equation $x=\frac{k}{k+1}x_0$ joining $V_1=(\frac{k}{k+1}x_0, sgn(-m)\frac{x_0}{k+1}+y_0)$ and $V_2=(\frac{k}{k+1}x_0, sgn(m)\frac{x_0}{k+1}+y_0)$ and the line segment with the equation $x=\frac{k}{k-1}x_0$ joining $V_3=(\frac{k}{k-1}x_0, sgn(m)\frac{x_0}{k-1}+y_0)$ and $V_4=(\frac{k}{k-1}x_0, sgn(-m)\frac{x_0}{k-1}+y_0)$ . Other solutions of the equality (3) are the line segment $x+sgn(m)ky=sgn(m)ky_0$ joining $V_1$ and $V_4$ and the line segment with the equation $x+sgn(-m)ky=sgn(-m)ky_0$ joining $V_2$ and $V_3$ . For $|x|\leq|y|$, $|x-x_0|>|y-y_0|$ and $|x|\leq|y|$, $|x-x_0|\leq|y-y_0|$, since $k\geq\frac{|m|+1}{1-|m|}$, there is no solution. Also, $|V_2V_3|_M=|V_1V_4|_M=\frac{2k}{k^2-1}|x_0|$. Considering the reflection in $y$-axis for the case $x_0<0$, it is seen that $\mathscr{A}(A_1,A_2;k)_M$ satisfies the stated properties.

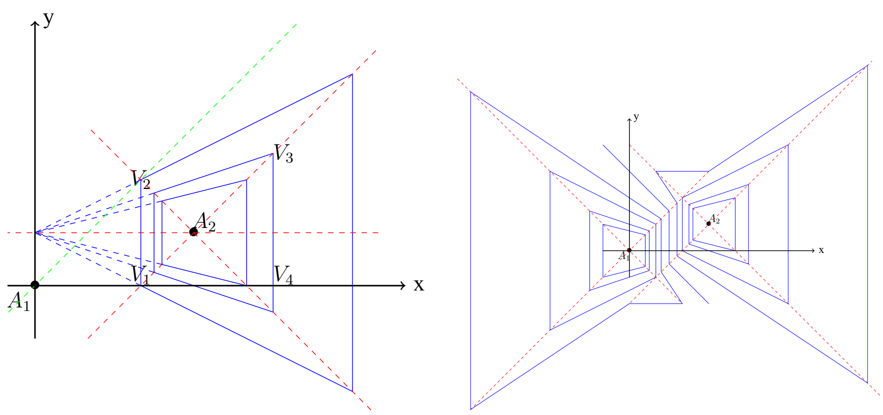


**Fig. 7:** Maximum Apollonian sets for pair of the points on the same gradual line

On the other hand, considering the reflection $\Omega$ in the line $y=y_0$ through the point $A_2=(x_0,y_0)$, $\Omega$ maps the point $(x,y)$ to the point with the coordinates $(x,2y_0-y)$ in the maximum plane. In $\mathscr{A}(A_1,A_2;k)_M$ mentioned in theorem 6, the points

$V_1$, $V_4$ and the side with the equation $x+ky=ky_0$ transform the points $V_2$, $V_3$ and the side with the equation $x-ky=-ky_0$ under $\Omega$ , respectively. Since the other two sides are perpendicular to the axis of $\Omega$, they are invariant under $\Omega$. Thus, $\mathscr{A}(A_1,A_2;k)_M$ is symmetric about line $y=y_0$. And, it is observed that $\mathscr{A}(A_1,A_2;k)_M$ a isosceles trapezoid.

In the case that two given points are on the steep line, the maximum Apollonian sets are given as follows. Since these sets are obtained as a result of applying the reflection in the separator line in theorems 6 and 7, we give the following corollaries without proof.

**Corollary 1.** *Let $A_1$ and $A_2$ be two distinct points on a steep line in the maximum plane . If $k<\frac{|m|+1}{|m|-1}$ where m is the slope of the line segment $A_1A_2$, then the maximum Apollonian set $\mathscr{A}(A_1,A_2;k)_M$ has the following properties :*
*(1) It consists of six line segments. The slopes of the three line segments are $\pm k$, $sgn(-m)\frac{1}{k}$ and the others are parallel to the coordinate axes.*
*(2) Four of the six vertices are on the separator lines passing through the point $A_2$ and the others are on the same separator line through the point $A_1$.*

**Corollary 2.** *Let $A_1$ and $A_2$ be two distinct points on a steep in the maximum plane . If $k\geq\frac{|m|+1}{|m|-1}$ where m is the slope of the line segment $A_1A_2$, then the maximum Apollonian set $\mathscr{A}(A_1,A_2;k)_M$ has the following properties :*
*(1) It consists of four line segments. The slopes of the two line segments are $\pm k$ and the others are on lines perpendicular to the same coordinate axis.*
*(2) Four vertices are on the separator lines passing through the point $A_2$.*
*(3) The intersection points of the completions of two non-parallel sides are on the line through $A_2$ parallel to the coordinate axis.*
*(4) Two sides of $\mathscr{A}(A_1,A_2;k)_M$ are of equal maximum length.*

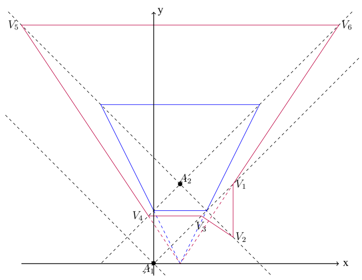


**Fig. 8:** Maximum Apollonian sets for pair of the points on the same steep line

## 4 Conclusion

In this study, the Apollonian sets in the maximum plane have been analytically examined by determining the properties provided by their sides and their vertices. It has been seen that these sets are the simple closed rectilinear figures. It has been observed that the sides and vertices have different properties according to the position of the two points and the value $k$ that determine the sets. The sets have been classified according to these differences. For a pair of points on a gradual line or a steep line , there have been differences in the maximum Apollonian sets corresponding to different values $k$. These differences have been given according to the condition on the value $k$ and the slope of the line through the pair of points.

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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