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New results on delay dependent stability for a class of nonlinear systems with additive time delay

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Abstract: In this manuscript, the issue of delay dependent stability for a class of additive delayed nonlinear systems has been searched. By constructing an appropriate Lyapunov-Krasovskii functional (LKF), some new stability criteria are established in terms of linear matrix inequalities (LMIs) which can be easily tested. This manuscript also presents a new result of stability analysis for a class of nonlinear systems with two additive time delays. Finally, two illustrative examples with numerical simulations have been given to show the applicability of the obtained theoretical results by using the MATLAB-Simulink.

Keywords: Dependent stability, nonlinear systems, additive time delay.

1 Introduction

During the last several decades, time delays arising for different areas of dynamical systems are commonly appear in many practical systems such as neural networks, economic systems, biology, chemical processes, networked control systems, and so on [1,2,3,4,5,6,7,8,9,10,11]. It is generally believed to be one of the important reasons of instability and underperforming of the system. Therefore, there has been tremendous interest in developing the stability of delayed linear or nonlinear systems both theoretically and practically in terms of calculation. For more details, see the studies [2, 8,12,13,14,15,16,17,18,19,20] and related references therein.

Due to its successful applications in the above mentioned practical areas, the interest in delay systems and the qualitative behavior of these systems have been increasing quickly in lately. When the studies in the related literature are analyzed, it is seen that most of the stability of time delay systems are delay dependent stability. Delay-independent stability states are conservative particularly for small dimensions delays. As we discussed in this study, a few studies have been conducted on the stability of additive delayed systems [21,22,23,24,25,26,27]. Based on LKF, a new stability criterion for the uncertain systems with two additive time delays was investigated using free matrix variables to approach certain integral terms [28]. An improved stability criterion was proposed for successive time delay systems, by constructing a new LKF and by making use of some new techniques [29]. Some new sufficient conditions for the delay-range-dependent stability of delayed systems were proposed on the basis of LKF [24]. Similarly, based on LKF, some new stability criterions in terms of LMIs have been presented for linear continuous systems with two additive time delays [23,27]. In the studies in the literature, researchers have generally studied the stability of additive time delay linear systems. On the other hand, in this study we examine the stability of additive time delay nonlinear systems. In this sense, the question of how it can further improve the stability criteria is very important for further investigation of delayed nonlinear systems.

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Motivated by the above discussions, in this manuscript we deal firstly with the delay dependent stability for a class of delayed nonlinear systems by choosing an appropriate LKF. Secondly, in the current study we deal with the delay dependent stability analysis for nonlinear systems with two additive time delay components by choosing an appropriate LKF. The contribution and aim of this manuscript can be listed as follows:

(a) This study on the stability of nonlinear systems with time delay and additive time delay components is still open to development. Therefore, we propose novel stability criteria for further improvements.

(b) Compared to the studies in [7,12,17,18,19,20,21,22,23,24,25,27,28], the theoretical results of this manuscript appear to be more general. So, the present study may be helpful for researchers investigating the qualitative behaviors of nonlinear systems.

(c) Although there are many studies about the stability of linear and nonlinear systems with time varying delays in the related literature, there are very few studies about the stability of nonlinear systems with additive time delay components. The existing manuscript is an extended version of the studies in [12,17,18,19,20,21,22,23,24,25,26,27,28] to ensure the stability of nonlinear systems with time delay and additive time delays.

(d) Based on the above discussions, in this manuscript, our aim is to introduce some sufficient conditions for delay dependent stability of nonlinear systems with time delay and two additive time delays.

The rest of this manuscript is summarized as follows. In Section 2, the asymptotically stability of the considered nonlinear system with time delay and the necessary assumptions are presented. In Section 3, some sufficient conditions for the asymptotically stability of the considered nonlinear system with two additive time delays are obtained. In Section 4, two illustrative examples with numerical simulations are given to show the applicability of the proposed theoretical results. In last section, the manuscript is finalized.

Notations: Throughout this manuscript, the symbol $\|\,.\,\|$ means the Euclidean norm for vectors; R^n denotes the $n-$dimensional Euclidean space and $R^{n\times m}$ is the set of all $n\times m$ real matrices; $C=C([-h,\ 0],\ R^n)$ denotes the space of piecewise continuous differentiable functions from $[-h,\ 0]$to R^n; K^T means the transpose of the matrix K; $\lambda_{\min}(K)$ (respectively, $\lambda_{\max}(K)$)is the minimal eigenvalue of the matrix K (respectively, maximal eigenvalue of the matrix K); I represents identity matrix; for real symmetric matrices X_1 and X_2, $''X_1 > X_2''$ denotes that the matrix $X_1 - X_2$ is positive-definite; the symmetric term in the symmetric matrix is denoted by $''*''$ as for example $\begin{bmatrix} X_1 & X_2 \\ * & X_3 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$.

2 Stability

In this section, we consider nonlinear time-delay system as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bx(t-u(t))+Cf(x(t))+Df(x(t-u(t))),\\ x(t) &= \varphi(t),\ t\in[-u,\ 0], \end{aligned} \tag{1}$$

where $x(t)\in R^n$,$\varphi(.)\in C^1([-u\ ,0],\ R^n)$ is the initial function, A, B, C and $D\in R^{n\times n}$ are constant matrix functions. $f(x(t))\in R^n$ are $f(x(t-u(t)))\in R^n$ represent the nonlinear terms of system (1) with respect to $x(t)$ and $x(t-u(t))$, respectively, assumed as

$$\|f(x(t))\|\le\alpha\,\|x(t)\|\,,\quad \|f(x(t-u(t)))\|\le\beta\,\|x(t-u(t))\|\,, \tag{2}$$

where $\alpha \geq 0$ and $\beta \geq 0$ are given positive constants.

Constraints described by (2) can be rewritten as follows

$$\begin{aligned} f^T(x(t))f(x(t)) &\leq \alpha^2 x^T(t)x(t), \\ f^T(x(t-u(t)))f(x(t-u(t))) &\leq \beta^2 x^T(t-u(t))x(t-u(t)). \end{aligned} \tag{3}$$

Moreover, the delay $d(t)$ is a continuous time varying function which satisfies

$$0 \leq u(t) \leq u,\ \dot{u}(t) \leq \delta < 1\ , \tag{4}$$

where u and δ are some positive constants.

Before state our main result, the following lemmas are needed in deriving the proposed stability criteria.

Lemma 1 *([5]). For any symmetric positive-definite matrix $\Sigma \in D^{n\times n}$, scalar $\gamma_2 > \gamma_1 > 0$, such that the integrations in the following inequality are well defined, then the following inequality holds:*

$$(\gamma_2-\gamma_1)\int_{t-\gamma_2}^{t-\gamma_1} x^T(s)Sx(s)ds \geq \left[\int_{t-\gamma_2}^{t-\gamma_1} x(s)ds\right]^T \Sigma \left[\int_{t-\gamma_2}^{t-\gamma_1} x(s)ds\right].$$

Lemma 2 *([3]). Given matrices $\Phi_1(x)$, $\Phi_2(x)$ and $\Phi_3(x)$ depend on affine on xwith appropriate dimensions. Then, the following matrix inequality*

$$\begin{pmatrix} \Phi_1(x) & \Phi_2(x) \\ \Phi_2^T(x) & \Phi_3(x) \end{pmatrix} < 0,$$

where $\Phi_1(x) = \Phi_1^T(x)$, $\Phi_3(x) = \Phi_3^T(x)$ and $\Phi_2(x)$ is equivalent to

$$\Phi_1(x) < 0,\ \Phi_3(x) < 0,$$

and

$$\Phi_1(x) - \Phi_2(x)\Phi_3^{-1}(x)\Phi_2^T(x) < 0.$$

Theorem 1 *Given scalars $u > 0$ and $\delta > 0$, system (1) with time-varying satisfying (4) is asymptotically stable if there exist symmetric positive-definite matrices $K = K^T > 0$, $L = L^T > 0$,$M = M^T > 0$ and $N = N^T > 0$ such that the following LMI holds:*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} \\ * & \Xi_{22} & 0 & 0 & \Xi_{25} \\ * & * & \Xi_{33} & 0 & \Xi_{35} \\ * & * & * & \Xi_{44} & \Xi_{45} \\ * & * & * & * & \Xi_{55} \end{bmatrix} < 0, \tag{5}$$

where $\Xi_{11} = KA + A^TK + L + \varepsilon_1\alpha^2 I$, $\Xi_{12} = KB$, $\Xi_{13} = KC$, $\Xi_{14} = KD$, $\Xi_{15} = A^T(uN+M)$, $\Xi_{22} = -(1-\delta)L + \varepsilon_2\beta^2 I$, $\Xi_{25} = B^T(uN+M)$, $\Xi_{33} = -\varepsilon_1 I$, $\Xi_{35} = C^T(uN+M)$, $\Xi_{44} = -\varepsilon_2 I$, $\Xi_{45} = D^T(uN+M)$, $\Xi_{55} = -(uN+M)$.

Proof. For system (1), we construct a LKF candidate as

$$V(t)=x^T(t)Kx(t)+\int_{t-u(t)}^{t} x^T(s)Lx(s)ds+\int_{t-u(t)}^{t} \dot{x}^T(s)M\dot{x}(s)ds+\int_{-u}^{0}\int_{t+\phi}^{t} \dot{x}^T(s)N\dot{x}(s)dsd\phi. \tag{6}$$

It is clear that, V is positive-definite functions. Calculating the time derivative of $V(t)$ along the trajectory of system (1) for $t\geq 0$, we have

$$\begin{aligned}
\dot{V}(t)=&x^T(t)(KA+A^TK)x(t)+x^T(t)KBx(t-u(t))+x^T(t)KCf(x(t))\\
&+x^T(t)KDf(x(t-u(t))+x^T(t-u(t))B^TKx(t)+f^T(x(t))C^TKx(t)\\
&+f^T(x(t-u(t)))D^TKx(t)+x^T(t)Lx(t)-(1-\dot{u}(t))x^T(t-u(t))\\
&\times Lx(t-u(t))+\dot{x}^T(t)M\dot{x}(t)-(1-\dot{u}(t))\dot{x}^T(t-u(t))M\dot{x}(t-u(t))\\
&+\dot{x}^T(t)uN\dot{x}(t)-\int_{t-u}^{t}\dot{x}^T(s)N\dot{x}(s)ds\\
\leq &x^T(t)(KA+A^TK+L)x(t)+x^T(t)KBx(t-u(t))+x^T(t)KCf(x(t))\\
&+x^T(t)KDf(x(t-u(t))+x^T(t-u(t))B^TKx(t)+f^T(x(t))C^TKx(t)\\
&+f^T(x(t-u(t)))D^TKx(t)-(1-\delta)x^T(t-u(t))Lx(t-u(t))\\
&+f^T(x(t))Mf(x(t))-(1-\delta)f^T(x(t-u(t)))Mf(x(t-u(t)))\\
&+\dot{x}^T(t)(uN+M)\dot{x}(t)-\int_{t-u}^{t}\dot{x}^T(s)N\dot{x}(s)ds.
\end{aligned} \tag{7}$$

By utilizing Lemma 1, we have

$$-\int_{t-u}^{t}\dot{x}^T(s)N\dot{x}(s)ds=-\int_{t-u(t)}^{t}\dot{x}^T(s)N\dot{x}(s)ds-\int_{t-u}^{t-u(t)}\dot{x}^T(s)N\dot{x}(s)ds\leq -u(t)v_1^T(t)Nv_1(t)-(u-u(t))v_2^T(t)Nv_2(t), \tag{8}$$

where

$$v_1(t)=\frac{1}{u(t)}\int_{t-u(t)}^{t}\dot{x}(s)ds, v_2(t)=\frac{1}{u-u(t)}\int_{t-u}^{t-u(t)}\dot{x}(s)ds,$$

and

$$\lim_{u(t)\to 0}\frac{1}{u(t)}\int_{t-u(t)}^{t}\dot{x}(s)ds=\dot{x}(t), \lim_{u(t)\to u}\frac{1}{u-u(t)}\int_{t-u}^{t-u(t)}\dot{x}(s)ds=\dot{x}(t-u).$$

By substituting (8) into (7), we obtain

$$\begin{aligned}\dot{V}(t) \leq & x^T(t)(KA+A^TK)x(t)+x^T(t)KBx(t-u(t))+x^T(t)KCf(x(t))\\ &+x^T(t)KDf(x(t-u(t))+x^T(t-u(t))B^TKx(t)+f^T(x(t))C^TKx(t)\\ &+f^T(x(t-u(t)))D^TKx(t)+x^T(t)Lx(t)-(1-\delta)x^T(t-u(t))\\ &\times Lx(t-u(t))+\dot{x}^T(t)(uN+M)\dot{x}(t)-u(t)v_1^T(t)Nv_1(t)\\ &-(u-u(t))v_2^T(t)Nv_2(t). \end{aligned} \tag{9}$$

The operator for term $\dot{x}^T(t)(uN+M)\dot{x}(t)$ is as follows:

$$\begin{aligned}\dot{x}^T(t)uN\dot{x}(t) =& [Ax(t)+Bx(t-u(t))+Cf(x(t))+Df(x(t-u(t)))]^T(uN+M)\\ &\times[Ax(t)+Bx(t-u(t))+Cf(x(t))+Df(x(t-u(t)))]\\ =& x^T(t)A^T(uN+M)Ax(t)+x^T(t)A^T(uN+M)Bx(t-u(t))\\ &+x^T(t)A^T(uN+M)Cf(x(t))+x^T(t)A^T(uN+M)Df(x(t-u(t)))\\ &+x^T(t-u(t))B^T(uN+M)Ax(t)+x^T(t-u(t))B^T(uN+M)\\ &\times Bx(t-u(t))+x^T(t-u(t))B^T(uN+M)Cf(x(t))+x^T(t-u(t))\\ &\times B^T(uN+M)Df(x(t-u(t)))+f^T(x(t))C^T(uN+M)Ax(t)\\ &+f^T(x(t))C^T(uN+M)Bx(t-u(t))+f^T(x(t))C^T(uN+M)\\ &\times Cf(x(t))+f^T(x(t))C^T(uN+M)Df(x(t-u(t)))\\ &+f^T(x(t-u(t)))D^T(uN+M)Ax(t)+f^T(x(t-u(t)))\\ &\times D^T(uN+M)Bx(t-u(t))+f^T(x(t-u(t)))D^T(uN+M)\\ &\times Cf(x(t))+f^T(x(t-u(t)))D^T(uN+M)Df(x(t-u(t))). \end{aligned} \tag{10}$$

Note that for any $\varepsilon_1 \geq 0,\ \varepsilon_2 \geq 0$, it follows from (2) and (3) that

$$\varepsilon_1[\alpha^2 x^T(t)x(t)-f^T(x(t))f(x(t))] \geq 0, \tag{11}$$

and

$$\varepsilon_2[\beta^2 x^T(t-u(t))x(t-u(t))-f^T(x(t-u(t)))f(x(t-u(t)))] \geq 0. \tag{12}$$

Combining (7-12) yields

$$\dot{V}(t) \leq \eta^T(t)\Pi\eta(t)-u(t)v_1^T(t)Nv_1(t)-(u-u(t))v_2^T(t)Nv_2(t), \tag{13}$$

where $\eta^T(t) = [x^T(t)\ x^T(t-u(t))\ f^T(x(t))\ f^T(x(t-u(t)))]$

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ * & * & \Pi_{33} & \Pi_{34} \\ * & * & * & \Pi_{44} \end{bmatrix},$$

with $\Pi_{11} = KA+A^TK+L+\varepsilon_1\alpha^2 I+A^T(uN+M)A, \Pi_{12} = KB+A^T(uN+M)B,\ \Pi_{13} = KC+A^T(uN+M)C, \Pi_{14} = KD+A^T(uN+M)D,\ \Pi_{22} = -(1-\delta)L+\varepsilon_2\beta^2 I+B^T(uN+M)B, \Pi_{23} = B^T(uN+M)C, \Pi_{24} = B^T(uN+M)D,\ \Pi_{33} =$

$-\varepsilon_1 I + C^T(uN+M)C, \Pi_{34} = C^T(uN+M)D, \Pi_{44} = -\varepsilon_2 I + D^T(uN+M)D$. To provide $\dot{V}(t) < 0$, one needs to insure that $\Pi < 0$. From the LK stability theorem and Lemma 2, if condition LMI described by (5) holds, system (1) is asymptotically stable.

3 Stability of nonlinear systems with two additive time delays

Most of the investigated results in delayed systems are based on the basic mathematical model cited in [25,26,27,28, 29] and the references therein. It should be noted in this mathematical model that the time delay in the state variable is assumed to be singular or simple. In this section, we deal the nonlinear system with two additive delays as follows:

$$\dot{x}(t) = Ax(t) + Bx(t-u_1(t)-u_2(t)) + Cf(x(t)) + Df(x(t-u_1(t)-u_2(t))), x(t) = \varphi(t),\ t \in [-u,\ 0], \tag{14}$$

where $x(t) \in R^n, \varphi(.) \in C^1([-u\ ,0],\ R^n)$ is the initial function and $A,\ B,\ C,\ D \in R^{n\times n}$ are constant matrix functions with appropriate dimensions. $f(x(t)) \in R^n$ are $f(x(t-u(t))) \in R^n$ represent the nonlinear terms of system (14) with respect to $x(t)$ and $x(t-u_1(t)-u_2(t))$, respectively, assumed as (2) and (3). $u_1(t)$ and $u_2(t)$ represent two additive delay components in the state and we denote $u(t) = u_1(t) + u_2(t)$.

It is assumed that

$$0 \le u_1(t) \le u_1,\ \ \dot{u}_1(t) \le \delta_1 < \infty\ ,\ 0 \le u_2(t) \le u_2,\ \ \dot{u}_2(t) \le \delta_2 < \infty\ . \tag{15}$$

Let $u = u_1 + u_2$ and $\delta = \delta_1 + \delta_2$. In this section, our goal is to show that system (14) with (15) is asymptotically stable.

Theorem 2 *Given scalars $u_1 > 0,\ u_2 > 0,\ \delta_1 > 0$ and $\delta_2 > 0$, system (14) with time delays satisfying (15) is asymptotically stable if there exist symmetric positive-definite matrices $K = K^T > 0,\ L_i = L_i^T \ge 0$ and $M_i = M_i^T > 0,\ (i = 1,\ 2,\ 3,4)$ such that the following LMI holds*

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & 0 & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} \\ * & \Omega_{22} & 0 & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & \Omega_{47} \\ * & * & * & * & \Omega_{55} & 0 & \Omega_{57} \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} \\ * & * & * & * & * & * & \Omega_{77} \end{bmatrix} < 0, \tag{16}$$

where $\Omega_{11} = KA + A^TK + L_1 + L_3 + \varepsilon_1\alpha^2 I,\ \Omega_{14} = KB,\ \Omega_{15} = KC,\ \Omega_{16} = KD, \Omega_{17} = A^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)A,$ $\Omega_{22} = -(1-\delta_1)(L_1 - L_2), \Omega_{33} = -(1-\delta_2)(L_3 - L_4),\ \Omega_{44} = -(1-\delta_1-\delta_2)(L_2 + L_4) + \varepsilon_2\beta^2 I, \Omega_{47} = B^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3), \Omega_{55} = -\varepsilon_1 I, \Omega_{57} = C^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3),\ \Omega_{66} = -\varepsilon_2 I,\ \Omega_{67} = D^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3),\ \Omega_{77} = -(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3).$

Proof. For system (14), we construct a LKF candidate as

$$
\begin{aligned}
V(t) =& x^T(t)Kx(t) + \int_{t-u_1(t)}^{t} x^T(s)L_1x(s)ds + \int_{t-u_1(t)-u_2(t)}^{t-u_1(t)} x^T(s)L_2x(s)ds \\
&+ \int_{t-u_2(t)}^{t} x^T(s)L_3x(s)ds + \int_{t-u_1(t)-u_2(t)}^{t-u_2(t)} x^T(s)L_4x(s)ds \\
&+ \int_{-u_1}^{0}\int_{t+\phi}^{t} \dot{x}^T(s)M_1\dot{x}(s)dsd\phi + \int_{-u_1-u_2}^{-u_1}\int_{t+\phi}^{t} \dot{x}^T(s)M_2\dot{x}(s)dsd\phi \\
&+ \int_{-u_2}^{0}\int_{t+\phi}^{t} \dot{x}^T(s)M_3\dot{x}(s)dsd\phi + \int_{-u_1-u_2}^{-u_2}\int_{t+\phi}^{t} \dot{x}^T(s)M_4\dot{x}(s)dsd\phi.
\end{aligned} \tag{17}
$$

It is clear that, V is positive-definite functions. Calculating time derivative of $V(t)$ along the trajectory of system (14) for $t \geq 0$, we get

$$
\begin{aligned}
\dot{V}(t) =& x^T(t)(KA + A^TK)x(t) + x^T(t)KBx(t-u_1(t)-u_2(t)) + x^T(t)KCf(x(t)) \\
&+ x^T(t)KDf(x(t-u_1(t)-u_2(t))) + x^T(t-u_1(t)-u_2(t))B^TKx(t) \\
&+ f^T(x(t))C^TKx(t) + f^T(x(t-u_1(t)-u_2(t)))D^TKx(t) + x^T(t)L_1x(t) \\
&- (1-\dot{u}_1(t))x^T(t-u_1(t))(L_1-L_2)x(t-u_1(t)) \\
&- (1-\dot{u}_1(t)-\dot{u}_2(t))x^T(t-u_1(t)-u_2(t))(L_2+L_4)x(t-u_1(t)-u_2(t)) \\
&+ x^T(t)L_3x(t) - (1-\dot{u}_2(t))x^T(t-u_2(t))(L_3-L_4)x(t-u_2(t)) \\
&+ \dot{x}^T(t)u_1M_1\dot{x}(t) - \int_{t-u_1}^{t} \dot{x}^T(s)M_1\dot{x}(s)ds + \dot{x}^T(t)u_2M_2\dot{x}(t) \\
&- \int_{t-u_1-u_2}^{t-u_1} \dot{x}^T(s)M_2\dot{x}(s)ds - \int_{t-u_2}^{t} \dot{x}^T(s)M_3\dot{x}(s)ds + \dot{x}^T(t)u_1M_4\dot{x}(t) \\
&- \int_{t-u_1-u_2}^{t-u_2} \dot{x}^T(s)M_4\dot{x}(s)ds + \dot{x}^T(t)u_2M_3\dot{x}(t) \\
\leq& x^T(t)(KA + A^TK + L_1 + L_3)x(t) + x^T(t)KBx(t-u_1(t)-u_2(t)) \\
&+ x^T(t)KCf(x(t)) + x^T(t)KDf(x(t-u_1(t)-u_2(t))) \\
&+ x^T(t-u_1(t)-u_2(t))B^TKx(t) + f^T(x(t))C^TKx(t) \\
&+ f^T(x(t-u_1(t)-u_2(t)))D^TKx(t) - (1-\delta_1)x^T(t-u_1(t)) \\
&\times (L_1-L_2)x(t-u_1(t)) - (1-\delta_1-\delta_2)x^T(t-u_1(t)-u_2(t)) \\
&\times (L_2+L_4)x(t-u_1(t)-u_2(t)) - (1-\delta_2)x^T(t-u_2(t)) \\
&\times (L_3-L_4)x(t-u_2(t)) + \dot{x}^T(t)(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)\dot{x}(t) \\
&- \int_{t-u_1}^{t} \dot{x}^T(s)M_1\dot{x}(s)ds - \int_{t-u_1-u_2}^{t-u_1} \dot{x}^T(s)M_2\dot{x}(s)ds - \int_{t-u_2}^{t} \dot{x}^T(s)M_3\dot{x}(s)ds - \int_{t-u_1-u_2}^{t-u_2} \dot{x}^T(s)M_4\dot{x}(s)ds.
\end{aligned} \tag{18}
$$

By utilizing Lemma 1, we obtain

$$
\begin{aligned}
&-\int_{t-u_1}^{t}\dot{x}^T(s)M_1\dot{x}(s)ds-\int_{t-u_1-u_2}^{t-u_1}\dot{x}^T(s)M_2\dot{x}(s)ds-\int_{t-u_2}^{t}\dot{x}^T(s)M_3\dot{x}(s)ds-\int_{t-u_1-u_2}^{t-u_2}\dot{x}^T(s)M_4\dot{x}(s)ds\\
&\leq-\int_{t-u_1(t)}^{t}\dot{x}^T(s)M_1\dot{x}(s)ds-\int_{t-u_1(t)-u_2(t)}^{t-u_1(t)}\dot{x}^T(s)M_2\dot{x}(s)ds-\int_{t-u_2(t)}^{t}\dot{x}^T(s)M_3\dot{x}(s)ds-\int_{t-u_1(t)-u_2(t)}^{t-u_2(t)}\dot{x}^T(s)M_4\dot{x}(s)ds\\
&\leq-u_1(t)\tilde{v}_1^T(t)M_1\tilde{v}_1(t)-u_2(t)\tilde{v}_2^T(t)M_2\tilde{v}_2(t)-u_2(t)\tilde{v}_3^T(t)M_3\tilde{v}_3(t)-u_1(t)\tilde{v}_4^T(t)M_4\tilde{v}_4(t)
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
\tilde{v}_1(t)&=\frac{1}{u_1(t)}\int_{t-u_1(t)}^{t}\dot{x}(s)ds,\ \tilde{v}_2(t)=\frac{1}{u_2(t)}\int_{t-u_1(t)-u_2(t)}^{t-u_1(t)}\dot{x}(s)ds,\\
\tilde{v}_3(t)&=\frac{1}{u_2(t)}\int_{t-u_2(t)}^{t}\dot{x}(s)ds,\ \tilde{v}_4(t)=\frac{1}{u_1(t)}\int_{t-u_1(t)-u_2(t)}^{t-u_2(t)}\dot{x}(s)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{u_1(t)\to 0}\frac{1}{u_1(t)}\int_{t-u_1(t)}^{t}\dot{x}(s)ds=\dot{x}(t),\\
&\lim_{u_2(t)\to 0}\frac{1}{u_2(t)}\int_{t-u_1(t)-u_2(t)}^{t-u_1(t)}\dot{x}(s)ds=\dot{x}(t-u_1(t)),\\
&\lim_{u_2(t)\to 0}\frac{1}{u_2(t)}\int_{t-u_2(t)}^{t}\dot{x}(s)ds=\dot{x}(t),\\
&\lim_{u_1(t)\to 0}\frac{1}{u_1(t)}\int_{t-u_1(t)-u_2(t)}^{t-u_2(t)}\dot{x}(s)ds=\dot{x}(t-u_2(t)).
\end{aligned}
$$

By substituting (19) into (18), we obtain

$$
\begin{aligned}
\dot{V}(t)\leq\,&x^T(t)(KA+A^TK+L_1+L_3)x(t)+x^T(t)KBx(t-u_1(t)-u_2(t))\\
&+x^T(t)KCf(x(t))+x^T(t)KDf(x(t-u_1(t)-u_2(t)))\\
&+x^T(t-u_1(t)-u_2(t))B^TKx(t)+f^T(x(t))C^TKx(t)\\
&+f^T(x(t-u_1(t)-u_2(t)))D^TKx(t)-(1-\delta_1)x^T(t-u_1(t))\\
&\times(L_1-L_2)x(t-u_1(t))-(1-\delta_1-\delta_2)x^T(t-u_1(t)-u_2(t))\\
&\times(L_2+L_4)x(t-u_1(t)-u_2(t))-(1-\delta_2)x^T(t-u_2(t))\\
&\times(L_3-L_4)x(t-u_2(t))+\dot{x}^T(t)(u_1M_1+u_2M_2+u_1M_4+u_2M_3)\dot{x}(t)\\
&-u_1(t)\tilde{v}_1^T(t)M_1\tilde{v}_1(t)-u_2(t)\tilde{v}_2^T(t)M_2\tilde{v}_2(t)\\
&-u_2(t)\tilde{v}_3^T(t)M_3\tilde{v}_3(t)-u_1(t)\tilde{v}_4^T(t)M_4\tilde{v}_4(t).
\end{aligned}
\tag{20}
$$

The operator for the term $\dot{x}^T(t)(u_1M_1+u_2M_2+u_1M_4+u_2M_3)\dot{x}(t)$ as

$$\begin{aligned}
&\dot{x}^T(t)(u_1M_1+u_2M_2+u_1M_4+u_2M_3)\dot{x}(t)\\
=&[Ax(t)+Bx(t-u_1(t)-u_2(t))+Cf(x(t))+Df(x(t-u_1(t)-u_2(t)))]^T\\
&\times(u_1M_1+u_2M_2+u_1M_4+u_2M_3)[Ax(t)+Bx(t-u_1(t)-u_2(t))\\
&+Cf(x(t))+Df(x(t-u_1(t)-u_2(t)))]\\
=&x^T(t)A^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)Ax(t)\\
&+x^T(t)A^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)Bx(t-u_1(t)-u_2(t))\\
&+x^T(t)A^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)Cf(x(t))\\
&+x^T(t)A^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)Df(x(t-u_1(t)-u_2(t)))\\
&+x^T(t-u_1(t)-u_2(t))B^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)Ax(t)\\
&+x^T(t-u_1(t)-u_2(t))B^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)\\
&\times Bx(t-u_1(t)-u_2(t))+x^T(t-u_1(t)-u_2(t))B^T(u_1M_1+u_2M_2\\
&+u_1M_4+u_2M_3)Cf(x(t))+x^T(t-u_1(t)-u_2(t))B^T(u_1M_1+u_2M_2\\
&+u_1M_4+u_2M_3)Df(x(t-u_1(t)-u_2(t)))+f^T(x(t))C^T(u_1M_1+u_2M_2\\
&+u_1M_4+u_2M_3)Ax(t)+f^T(x(t))C^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)\\
&\times Bx(t-u_1(t)-u_2(t))+f^T(x(t))C^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)\\
&\times Cf(x(t))+f^T(x(t))C^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)\\
&\times Df(x(t-u_1(t)-u_2(t)))+f^T(x(t-u_1(t)-u_2(t)))D^T(u_1M_1+u_2M_2\\
&+u_1M_4+u_2M_3)Ax(t)+f^T(x(t-u_1(t)-u_2(t)))D^T(u_1M_1+u_2M_2\\
&+u_1M_4+u_2M_3)Bx(t-u_1(t)-u_2(t))+f^T(x(t-u_1(t)-u_2(t)))\\
&\times D^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)Cf(x(t))+f^T(x(t-u_1(t)-u_2(t)))\\
&D^T(u_1M_1+u_2M_2+u_1M_4+u_2M_3)Df(x(t-u_1(t)-u_2(t))).
\end{aligned} \tag{21}$$

Note that for any $\varepsilon_1\geq 0,\ \varepsilon_2\geq 0$, it follows from (2) and (3) that

$$\varepsilon_1[\alpha^2x^T(t)x(t)-f^T(x(t))f(x(t))]\geq 0, \tag{22}$$

and

$$\begin{aligned}
&\varepsilon_2[\beta^2x^T(t-u_1(t)-u_2(t))x(t-u_1(t)-u_2(t))\\
&-f^T(x(t-u_1(t)-u_2(t)))f(x(t-u_1(t)-u_2(t)))]\geq 0.
\end{aligned} \tag{23}$$

Combining (18)-(23) yields

$$\begin{aligned}
\dot{V}(t)\leq&\xi^T(t)\Psi\xi(t)-u_1(t)\tilde{v}_1^T(t)M_1\tilde{v}_1(t)-u_2(t)\tilde{v}_2^T(t)M_2\tilde{v}_2(t)\\
&-u_2(t)\tilde{v}_3^T(t)M_3\tilde{v}_3(t)-u_1(t)\tilde{v}_4^T(t)M_4\tilde{v}_4(t)
\end{aligned} \tag{24}$$

where $\xi^T(t)=[x^T(t)\ \ x^T(t-u_1(t))\ \ x^T(t-u_2(t))\ \ x^T(t-u_1(t)-u_2(t))$
$f^T(x(t))\ \ f^T(x(t-u_1(t)-u_2(t)))]$,

and

$$\Psi = \begin{bmatrix} \Psi_{11} & 0 & 0 & \Psi_{14} & \Psi_{15} & \Psi_{16} \\ * & \Psi_{22} & 0 & 0 & 0 & 0 \\ * & * & \Psi_{33} & 0 & 0 & 0 \\ * & * & * & \Psi_{44} & \Psi_{45} & \Psi_{46} \\ * & * & * & * & \Psi_{55} & \Psi_{56} \\ * & * & * & * & * & \Psi_{66} \end{bmatrix},$$

with

$$\begin{aligned}
\Psi_{11} =& KA + A^T K + L_1 + L_3 + \varepsilon_1 \alpha^2 I + A^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)A, \\
\Psi_{14} =& KB + A^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)B, \\
\Psi_{15} =& KC + A^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)C, \\
\Psi_{16} =& KD + A^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)D, \\
\Psi_{22} =& -(1-\delta_1)(L_1 - L_2), \\
\Psi_{33} =& -(1-\delta_2)(L_3 - L_4), \\
\Psi_{44} =& -(1-\delta_1-\delta_2)(L_2 + L_4) + \varepsilon_2 \beta^2 I + B^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)B, \\
\Psi_{45} =& B^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)C, \\
\Psi_{46} =& B^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)D, \\
\Psi_{55} =& -\varepsilon_1 I + C^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)C, \\
\Psi_{56} =& C^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)D, \\
\Psi_{66} =& -\varepsilon_2 I + D^T(u_1M_1 + u_2M_2 + u_1M_4 + u_2M_3)D.
\end{aligned}$$

To provide $\dot{V}(t) < 0$, one needs to insure that $\Psi < 0$. From the LK stability theorem and Lemma 2, if condition LMI described by (16) holds, system (14) is asymptotically stable.

Remark 1. *In the present study, although we only consider two additive delayed nonlinear systems, our theoretical results can be easily extended to multiple additive delayed nonlinear systems as follows*

$$\dot{x}(t) = Ax(t) + Bx(t - \sum_{i=}^{n} u_i(t)) + Cf(x(t)) + Df(x(t - \sum_{i=}^{n} u_i(t))),$$

and assumptions on the time delays $u_i(t)$ can be described by

$$0 \leq u_i(t) \leq u_i < \infty, \; \dot{u}_i(t) \leq \delta_i < \infty \,.$$

4 Illustrative examples

In this section, we present two illustrative examples to show the advantage of Theorem 1 and Theorem 2 and the effectiveness of obtained results.

Example 1. *As a special case of system (1), we deal the following the nonlinear system with time delay:*

$$\dot{x}(t) = Ax(t) + Bx(t-u(t)) + Cf(x(t)) + Df(x(t-u(t))), \tag{25}$$

where $x(t) = [x_1(t),\, x_2(t)]^T$,

$$A = \begin{bmatrix} -2.3 & -3.6 \\ 4.1 & -6 \end{bmatrix},\ B = \begin{bmatrix} -0.1 & 0.1 \\ -0.2 & 0.3 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix},$$

and

$$0 \le u(t) = 0.1 + 0.2\sin^2(t) \le 0.3 = u,\ \dot{u}(t) \le 0.4. \tag{26}$$

Considering the LMI given with (5) for $\alpha = 0.2,\ \beta = 0.02, \varepsilon_1 = 18.5$ *and* $\varepsilon_2 = 64.2$, *let us choose*

$$K = \begin{bmatrix} 12 & 0 \\ 0 & 18 \end{bmatrix},\ L = \begin{bmatrix} 4.6 & -0.04 \\ -0.04 & 2.5 \end{bmatrix},\ M = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix},\ N = \begin{bmatrix} 2 & 0.2 \\ 0.2 & 2 \end{bmatrix}.$$

Thus, by using MATLAB-Simulink and considering the above assumptions, it can be easily obtained that all eigenvalues of LMI defined by (5) are $\lambda_{\max}(\Xi) \le -0.1562$. *Consequently, the system (25) with (26) is asymptotically stable according to Theorem (1).*

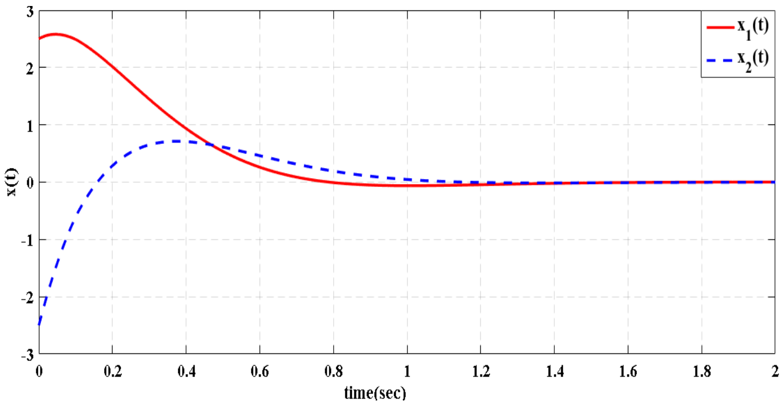

Fig. 1: Trajectories of solutions $x(t)$ of the system (25) with (26).

Example 2. *As a special case of the system (14), we deal the following nonlinear system with two additive time delays:*

$$\dot{x}(t) = Ax(t) + Bx(t-u_1(t)-u_2(t)) + Cf(x(t)) + Df(x(t-u_1(t)-u_2(t))), \tag{27}$$

where $x(t) = [x_1(t),\, x_2(t)]^T$,

$$A = \begin{bmatrix} -3.4 & -3.2 \\ 1 & -5.8 \end{bmatrix},\ B = \begin{bmatrix} -0.1 & 0.1 \\ -0.8 & 0.1 \end{bmatrix},\ C = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.5 \end{bmatrix},\ D = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix},$$

and

$$0 \le u_1(t) = 0.1\sin^2(t) \le 0.1 = u_1,\ \dot{u}_1(t) \le 0.2, \tag{28}$$

$$0 \le u_2(t) = 0.3\sin^2(t) \le 0.3 = u_2,\ \dot{u}_2(t) \le 0.6. \tag{29}$$

Considering the LMI given with (16) for $\alpha = 0.3,\ \beta = 0.04,\ \varepsilon_1 = 48.5$ *and* $\varepsilon_2 = 12.2$, *let us choose*

$$K = \begin{bmatrix} 10 & 0 \\ 0 & 12 \end{bmatrix}, L_1 = \begin{bmatrix} 5.6 & -0.04 \\ -0.04 & 2.5 \end{bmatrix}, L_2 = \begin{bmatrix} 1.6 & 0.14 \\ 0.14 & 0.5 \end{bmatrix}, L_3 = \begin{bmatrix} 10.2 & -0.04 \\ -0.04 & 1.5 \end{bmatrix},$$

$$L_4 = \begin{bmatrix} 5.6 & -0.04 \\ -0.04 & 1 \end{bmatrix}, M_1 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, M_3 = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}, M_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, by using MATLAB-Simulink and considering the above assumptions, it can be easily obtained that all eigenvalues of LMI defined by (16) are $\lambda_{\max}(\Omega) \le -0.1351$. *Consequently, the system (27) with (28) and (29) is asymptotically stable according to Theorem (2).*

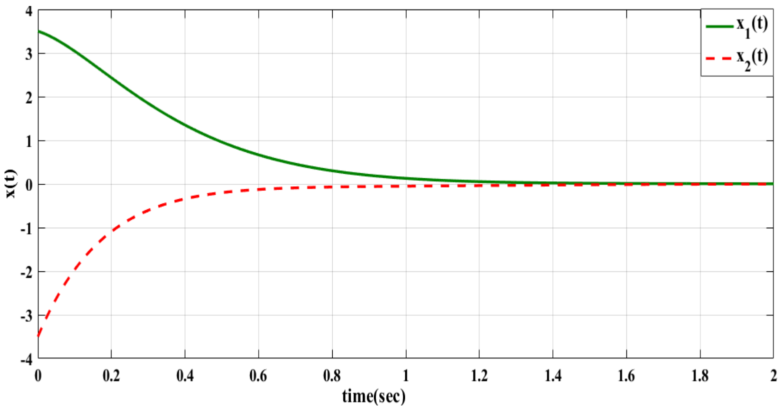

Fig. 2: Trajectories of solutions $x(t)$ of the system (27) with (28) and (29).

When the theoretical results of Examples 1 and 2 are analyzed, it is seen that the zero solutions of the considered nonlinear systems are stable under different initial conditions. These are confirmed by the corresponding simulation results of Figure 1 and Figure 2.

5 Conclusion

In this manuscript, delay dependent asymptotically stability has been investigated under some sufficient conditions for nonlinear systems with time delay and additive time delays. On the basis of an appropriate LKF, delay dependent stability criteria have been established to be stable in terms of LMIs. Considered to the stability criteria in the literature, the obtained our criteria are simple and practical. Two simple examples with their numerical simulations are presented (Figure 1 and Figure 2) to demonstrate the applicability of proposed method by using the MATLAB-Simulink. In addition, the results of this manuscript can be used to solve delayed classical and fractional optimal control problems with novel methodologies such as those discussed in [30,31,32]. Consequently, the achieved theoretical results in this manuscript extend and generalize the current ones in the literature.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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