New Trends in Mathematical Sciences
http://www.ntmsci.com

# Integrals associated with generalized k- Mittag-Leffer function

*Chenaram Choudhary and Palu Choudhary*

Department of Mathematics and Statistics, Jai Narain Vyas University, Jodhpur.

Received: 21 February 2015, Revised: 24 March 2015, Accepted: 19 June 2015
Published online: 4 July 2015

**Abstract:** The aim of the paper is to investigate integrals of Generalized k- Mittag-Leffler function [6], multiplied with Jacobi polynomials, Legendre polynomials, Legendre function, Bessel Maitland function, Hypergeometric function and Generalized hypergeometric function.

**Keywords:** Generalized k- Mittag-Leffler function, Jacobi polynomials, Legendre polynomials, Legendre function, Bessel Maitland function, Hypergeometric function and Generalized hypergeometric function.

## 1 Introduction

The k- Gamma function [5] defined as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, k>0, x\in C\backslash kz^-, \tag{1}$$

where $(x)_{n,k}$ is the k- Pochhammer symbol and is given by

$$(x)_{n,k} = x(x+k)(x+2k)........(x+(n-1)k), \tag{2}$$

$k\in 0, x\in C\backslash kz^-, n\in N^+$. The integral form of the generalized k- Gamma function [5] is given by

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1} dt, \tag{3}$$

where $k\in R, x\in C\backslash kz^-, Re(x)>0$,
from which it follows easily that

$$\Gamma_k(\gamma) = k^{\frac{x}{k}-1}\Gamma\left(\frac{\gamma}{k}\right), \tag{4}$$

and

$$(\gamma)_{nq,k} = (k)^{nq}\left(\frac{\gamma}{k}\right)_{nq} \tag{5}$$

*Let* $\alpha,\beta,\gamma\in C, k\in R, \{Re(\alpha), Re(\beta), Re(\gamma)>0\}$ *and* $q\in(0,1)\cup N$, then the generalized

* Corresponding author e-mail: paluchoudhary@gmail.com

k-Mittag-Leffler function denoted by $GE_{k,\alpha,\beta}^{\gamma,q}(z)$ and defined [6], as

$$GE_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} z^n}{\Gamma_k(\alpha n+\beta) n!} \tag{6}$$

where $(\gamma)_{nq,k}$ is the k- Pochhammer symbol given by equation (2) and $\Gamma_k(x)$ is the k-Gamma function given by equation (3) .

The Generalized Pochhammer symbol (*cf* [2], page 22),

$$(\gamma)_{nq} = \frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)} = q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q}\right)_n \; if\, q \in N. \tag{7}$$

## 2 Integrals with jacobi polynomial

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right) \tag{8}$$

when $\alpha = \beta = 0$ then the polynomial in (8) becomes the Legendre polynomial ( [2], p. 157).

$P_n^{(\alpha,\beta)}(x)$From (8) it follows that is a polynomial of degree n and that

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!} \tag{9}$$

**Theorem 1.** If $\alpha > -1, \beta > -1;\ \eta, \mu, \gamma \in C, Re(\eta) > 0, Re(\mu) > 0, Re(\gamma) > 0, k \in R\, and\, q \in (0,1) \cup N.$ Let is the Jacobi polynomial defined in (8) and Generalized k- Mittag-Leffler function by (6) then we have

$$\int_{-1}^{+1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta} P_n^{\alpha,\beta}(x)\, GE_{k,\eta,\mu}^{\gamma,q}[z(1+x)^h]dx$$

$$= \frac{(-1)^n 2^{n+\delta+1}}{n!} \Gamma(\alpha+n+1) \sum_{r=0}^{\infty} \frac{\Gamma(\delta+hr+1)\Gamma(\delta+hr+\beta+1)}{\Gamma(\delta+hr+\beta+n+1)\Gamma(\delta+hr+\alpha+n+2)}$$

$$\times\, GE_{k,\eta,\mu}^{\gamma,q}(z2^h)\, {}_3F_2\left[\begin{matrix} -\delta, \delta+hr+\beta+1, \delta+hr+1; 1 \\ \delta+hr+\beta+n+1, \delta+hr+\alpha+n+2 \end{matrix}\right] \tag{10}$$

**Proof.** In dealing with the Jacobi polynomial, we have

$$I_1 \equiv \int_{-1}^{+1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta} P_n^{\alpha,\beta}(x)\, GE_{k,\eta,\mu}^{\gamma,q}[z(1+x)^h]dx,$$

making use of (6), we get

$$I_1 \equiv \int_{-1}^{+1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta} P_n^{\alpha,\beta}(x) \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k}[z(1+x)^h]^r}{\Gamma_k(\eta r+\mu)\, r!} dx.$$

Interchanging order of integration and summation, we can write above expression as

$$I_1=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^r}{\Gamma_k(\eta r+\mu)\,r!}\int_{-1}^{+1}x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta+hr}P_n^{\alpha,\beta}(x)\,dx. \tag{11}$$

But we have the formula ([7] p. 52)

$$\begin{aligned}\int_{-1}^{+1}x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta}P_n^{\alpha,\beta}(x)\,dx=&\tfrac{(-1)^n2^{\alpha+\delta+1}}{n!}\tfrac{\Gamma(\delta+1)\Gamma(\alpha+n+1)\Gamma(\delta+\beta+1)}{\Gamma(\delta+\beta+n+1)\Gamma(\delta+\alpha+n+2)}\\ \times{}_3F_2&\left[\begin{matrix}-\lambda,\delta+\beta+1,\delta+1;1\\ \delta+\beta+n+1,\delta+\alpha+n+2\end{matrix}\right],\end{aligned} \tag{12}$$

$\alpha>-1,\beta>-1,Re(\lambda)>-1.$ $\alpha>-1,\beta>-1,Re(\lambda)>-1.$ provided

Now, by using (11) and (12) we have

$$\begin{aligned}I_1=\tfrac{(-1)^n2^{\alpha+\delta+hr+1}}{n!}\Gamma(\alpha+n+1)&\tfrac{\Gamma(\delta+hr+1)\Gamma(\delta+hr+\beta+1)}{\Gamma(\delta+hr+\beta+n+1)\Gamma(\delta+hr+\alpha+n+2)}\\ \times GE_{k,\eta,\mu}^{\gamma,q}(z2^h){}_3F_2&\left[\begin{matrix}-\lambda,\delta+hr+\beta+1,\delta+hr+1;1\\ \delta+hr+\beta+n+1,\delta+hr+\alpha+n+2\end{matrix}\right].\end{aligned}$$

This proves result (10).

**Theorem 2.** If $(\beta)>-1,Re(\gamma)>0,Re(\eta)>0,Re(\mu)>0,\eta,\mu,\gamma\in C,k\in R\,,q\in(0,1)\cup N.$

Then we have

$$\begin{aligned}&\int_{-1}^{+1}(1-x)^{\delta}(1+x)^{\beta}P_n^{\alpha,\beta}(x)\,P_m^{\rho,\sigma}(x)GE_{k,\eta,\mu}^{\gamma,q}[z(1-x)^h]dx\\ &=\tfrac{2^{\delta+\beta+1}\Gamma(\alpha+n+1)\Gamma(1+\rho+m)}{n!m!}\sum_{r=0}^{\infty}\tfrac{(-n)_r(-m)_r(1+\rho+\sigma+m)_r(1+\alpha+\beta+n)_r}{\Gamma(1+\rho+r)\Gamma(\alpha+r+1)(r!)^2}\\ &\quad\times GE_{k,\eta,\mu}^{\gamma,q}(z2^h)\,B(1+\delta+hr+2r,\beta+1).\end{aligned} \tag{13}$$

**Proof.** By Jacobi polynomial, we have

$$I_2\equiv\int_{-1}^{+1}(1-x)^{\delta}(1+x)^{\beta}P_n^{\alpha,\beta}(x)\,P_m^{\rho,\sigma}(x)GE_{k,\eta,\mu}^{\gamma,q}[z(1-x)^h]dx$$

$$=\int_{-1}^{+1}(1-x)^{\delta}(1+x)^{\beta}P_n^{\alpha,\beta}(x)P_m^{\rho,\sigma}(x)\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}[z(1-x)^h]^r}{\Gamma_k(\eta r+\mu)\,r!}dx.$$

Interchanging order of integration and summation, we can write above expression as

$$I_2=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^r}{\Gamma_k(\eta r+\mu)\,r!}\int_{-1}^{+1}(1-x)^{\delta+hr}(1+x)^{\beta}P_n^{\alpha,\beta}(x)\,P_m^{\rho,\sigma}(x)\,dx.$$

Now, using (8) in above expression we get

$$I_2=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^r}{\Gamma_k(\eta r+\mu)\,r!}\frac{(1+\rho)_m}{m!}\sum_{r=0}^{\infty}\frac{(-m)_r(1+\rho+\sigma+m)_r}{(1+\rho)_r2^r r!}\int_{-1}^{+1}(1-x)^{\delta+hr}(1+x)^{\beta}P_n^{\alpha,\beta}(x)\,dx. \tag{14}$$

Again using (8) in (14) , we have

$$I_2=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^r}{\Gamma_k(\eta r+\mu)\,r!}\frac{\Gamma(1+\rho+m)\Gamma(1+\alpha+n)}{m!\,n!}\sum_{r=0}^{\infty}\frac{(-n)_r(-m)_r(1+\rho+\sigma+m)_r(1+\alpha+\beta+n)_r}{\Gamma(1+\rho+r)\Gamma(1+\alpha+r)2^{2r}(r!)^2}\times\int_{-1}^{+1}(1-x)^{\delta+hr+2r}(1+x)^{\beta}\,dx. \tag{15}$$

But by the formula

$$\int_{-1}^{+1}(1-x)^{n+\alpha}(1+x)^{n+\beta}dx=2^{2n+\alpha+\beta+1}B(1+\alpha+n,1+\beta+n). \tag{16}$$

Then (15) becomes,

$$I_2=2^{\delta+\beta+1}\frac{\Gamma(1+\rho+m)\Gamma(1+\alpha+n)}{m!\,n!}\sum_{r=0}^{\infty}\frac{(-n)_r(-m)_r(1+\rho+\sigma+m)_r(1+\alpha+\beta+n)_r}{\Gamma(1+\rho+r)\Gamma(1+\alpha+r)2^{2r}(r!)^2}\times GE_{k,\eta,\mu}^{\gamma,q}(z2^h)\,B(1+\delta+hr+2r,1+\beta)\,.$$

This proves result (13)

**Theorem 3.** If $Re(\alpha)>1, Re(\beta)>-1, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, \eta,\mu,\gamma\in C, k\in R\ and\ q\in(0,1)\cup N$. Then the following relation holds true

$$\int_{-1}^{+1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{\alpha,\beta}(x)\,GE_{k,\eta,\mu}^{\gamma,q}[z(1-x)^h(1+x)^t]dx$$

$$=\frac{2^{\rho+\sigma+1}(1+\alpha)_n}{n!}\sum_{r=0}^{\infty}\frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r\,r!}\,GE_{k,\eta,\mu}^{\gamma,q}(z2^{h+t})\,B(1+\rho+hr+r,1+\sigma+tr). \tag{17}$$

**Proof.** By invoking Jacobi polynomial and generalized k- Mittag-Leffler function, we have

$$I_3\equiv\int_{-1}^{+1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{\alpha,\beta}(x)\,GE_{k,\eta,\mu}^{\gamma,q}[z(1-x)^h(1+x)^t]dx$$

$$=\int_{-1}^{+1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{\alpha,\beta}(x)\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}[z(1-x)^h(1+x)^t]^r}{\Gamma_k(\eta r+\mu)\,r!}dx,$$

interchanging order of integration and summation, we get

$$I_3=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^r}{\Gamma_k(\eta r+\mu)\,r!}\int_{-1}^{+1}(1-x)^{\rho+hr}(1+x)^{\sigma+tr}P_n^{\alpha,\beta}(x)\,dx. \tag{18}$$

Now, by using (8) in (18), we obtain

$$I_3=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^r}{\Gamma_k(\eta r+\mu)\,r!}\frac{(1+\alpha)_n}{n!}\sum_{r=0}^{\infty}\frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r2^r r!}\int_{-1}^{+1}(1-x)^{\rho+hr+r}(1+x)^{\sigma+tr}\,dx. \tag{19}$$

Making use of (16) in (19), we have

$$I_3=2^{\rho+\sigma+1}\frac{(1+\alpha)_n}{n!}\sum_{r=0}^{\infty}\frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r2^r r!}GE_{k,\eta,\mu}^{\gamma,q}(z2^{h+t})B(1+\rho+hr+r,1+\sigma+tr).$$

This proves Theorem 3.

**Theorem 4.** If $Re(\alpha)>1, Re(\beta)>-1, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, \eta,\mu,\gamma\in C, k\in R\ and\ q\in(0,1)\cup N$. Then the following relation holds true

$$I_4 \equiv \int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{\alpha,\beta}(x)\, GE_{k,\eta,\mu}^{\gamma,q}[z(1-x)^h(1+x)^{-t}]dx$$

$$= \frac{2^{\rho+\sigma+1}(1+\alpha)_n}{n!} \sum_{r=0}^{\infty} \frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r\, r!}\, GE_{k,\eta,\mu}^{\gamma,q}(z2^{h-t})B(1+\rho+hr+r, 1+\sigma-tr). \quad (20)$$

**Proof.** This Theorem can be proved on similar lines as Theorem 3.

**Theorem 5.** If $Re(\alpha) > 1, Re(\beta) > -1, Re(\gamma) > 0, Re(\eta) > 0, Re(\mu) > 0, \eta, \mu, \gamma \in C, k \in R\ and\ q \in (0,1) \cup N$. Then the following relation holds true

$$\int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{\alpha,\beta}(x)\, GE_{k,\eta,\mu}^{\gamma,q}[z(1+x)^{-h}]dx$$

$$= \frac{2^{\rho+\sigma+1}(1+\alpha)_n}{n!} \sum_{r=0}^{\infty} \frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r\, r!}\, GE_{k,\eta,\mu}^{\gamma,q}(z2^{-h})B(1+\rho+r, 1+\sigma-hr). \quad (21)$$

**Proof.** We have

$$I_5 = \int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{\alpha,\beta}(x)\, GE_{k,\eta,\mu}^{\gamma,q}[z(1+x)^{-h}]dx.$$

By using (6), we can write

$$I_5 = \int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\sigma} P_n^{\alpha,\beta}(x) \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k}[z(1+x)^{-h}]^r}{\Gamma_k(\eta r+\mu)\, r!} dx.$$

Interchanging order of summation and integration, we have

$$= \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k} z^r}{\Gamma_k(\eta r+\mu)\, r!} \int_{-1}^{+1} (1-x)^{\rho} (1+x)^{\sigma-hr} P_n^{\alpha,\beta}(x)\, dx. \quad (22)$$

Now using (8) in (22), we get

$$I_5 = \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k} z^r}{\Gamma_k(\eta r+\mu)\, r!} \frac{(1+\alpha)_n}{n!} \sum_{r=0}^{\infty} \frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r 2^r r!} \int_{-1}^{+1} (1-x)^{\rho+r}(1+x)^{\sigma-hr}\, dx, \quad (23)$$

using (16) in (23), we have

$$I_5 = 2^{\rho+\sigma+1} \frac{(1+\alpha)_n}{n!} \sum_{r=0}^{\infty} \frac{(-n)_r(1+\alpha+\beta+n)_r}{(1+\alpha)_r 2^r r!}\, GE_{k,\eta,\mu}^{\gamma,q}(z2^{-h})B(1+\rho+hr+r, 1+\sigma+tr).$$

This proves Theorem 5.

## 3 Special cases

**(i)** If we replace $\delta$ by $\lambda$-1 and put $\alpha = \beta = \rho = \sigma = 0$ then the integral $I_2$ transforms into the following integral involving Legendre polynomial [2],

$$I_6 \equiv \int_{-1}^{+1} (1-x)^{\lambda-1} P_n(x)\, GE_{k,\eta,\mu}^{\gamma,q}[z(1-x)^h]dx$$

$$= 2^{\lambda} \sum_{r=0}^{\infty} \frac{(-n)_r(-m)_r(1+m)_r(1+n)_r}{(r!)^2\,(r!)^2}\; GE^{\gamma,q}_{k,\eta,\mu}(z2^h)\,B(\lambda+hr+2r,1). \tag{24}$$

**(ii)** If $\alpha=\beta=0$ $\rho$ is replaced by $\rho$ -1 and $\sigma$ by $\sigma$ -1, then $I_3$ transforms into the following integral involving Legendre polynomial [2],

$$I_7 \equiv \int_{-1}^{+1} (1-x)^{\rho-1}(1+x)^{\sigma-1} P_n(x)\, GE^{\gamma,q}_{k,\eta,\mu}[z(1-x)^h(1+x)^t]dx$$

$$= 2^{\rho+\sigma-1} \sum_{r=0}^{\infty} \frac{(-n)_r(1+n)_r}{(r!)^2}\; GE^{\gamma,q}_{k,\eta,\mu}(z2^{h+t})\,B(\rho+hr+r,\sigma+tr). \tag{25}$$

**(iii)** If $\alpha$=$\beta$=0 , $\rho$ is replaced by $\rho$-1 and $\sigma$ by $\sigma$-1, , then $I_4$ transforms into the following integral involving Legendre polynomial [2],

$$I_8 \equiv \int_{-1}^{+1} (1-x)^{\rho-1}(1+x)^{\sigma-1} P_n(x)\, GE^{\gamma,q}_{k,\eta,\mu}[z(1-x)^h(1+x)^{-t}]dx$$

$$= 2^{\rho+\sigma-1} \sum_{r=0}^{\infty} \frac{(-n)_r(1+n)_r}{(r!)^2}\; GE^{\gamma,q}_{k,\eta,\mu}(z2^{h-t})\,B(\rho+hr+r,\sigma-tr). \tag{26}$$

## 4 Integral involving Bessel Maitland function

The special case of the Wright function ([10], vol. 3, section 18.1) and ([3],) in the form

$$\phi(B,b;z) =_0 \psi_1\,[-;(B,b);z] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Bk+b)} \frac{z^k}{k!}, \tag{27}$$

with $z,b \in C, B \in R$ when $B=\delta$, $b=v+1$ and $z$ is replaced by the function $\phi(\delta,v+1;z)$ is defined by $J^{\delta}_{v}$ , which is known as the Bessel Maitland function (*cf* [ 8], p. 352),

$$J^{\delta}_{v}(z) \equiv \phi(\delta,v+1;-z) = \sum_{r=0}^{\infty} \frac{1}{\Gamma(\delta r+v+1)} \frac{(-z)^r}{r!}. \tag{28}$$

**Theorem 6.** If $\eta,\mu,\gamma \in C, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, k\in R, q\in(0,1)\cup N, \alpha-\delta\alpha>-1, \alpha>0, Re(\rho+1)>0$. Then we have the following integral involving Bessel Maitland function,

$$\int_0^{\infty} x^{\rho} J^{\delta}_{v}(x) GE^{\gamma,q}_{k,\eta,\mu}(zx^{\alpha})\,dx = \sum_{r=0}^{\infty} \frac{\Gamma(\rho+\alpha r+1)}{\Gamma(1+v-\delta-\delta(\rho+\alpha r))} GE^{\gamma,q}_{k,\eta,\mu}(z). \tag{29}$$

**Proof.** We have,

$$I_9 \equiv \int_0^{\infty} x^{\rho} J^{\delta}_{v}(x) GE^{\gamma,q}_{k,\eta,\mu}(zx^{\alpha})\,dx,$$

$$= \int_0^{\infty} x^{\rho} J^{\delta}_{v}(x) \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k}(zx^{\alpha})^r}{\Gamma_k(\eta r+\mu)r!}\,dx,$$

interchanging order of summation and integration, we get

$$I_9 = \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k} z^r}{\Gamma_k(\eta r+\mu)r!} \int_0^{\infty} x^{\rho+\alpha r} J^{\delta}_{v}(x)\,dx. \tag{30}$$

By well known formula, ( *cf* [7],p.55)

$$\int_0^\infty x^\rho J_v^\delta(x)dx = \frac{\Gamma(\rho+1)}{\Gamma(1+v-\delta-\delta\rho)}, \tag{31}$$

where $Re(\rho) > -1, 0 < \delta < 1$. We can write (30) as

$$I_9 = \sum_{r=0}^{\infty} \frac{\Gamma(\rho+\alpha r+1)}{\Gamma(1+v-\delta-\delta(\rho+\alpha r))} GE_{k,\eta,\mu}^{\gamma,q}(z).$$

This proves Theorem 6.

## 5 Integrals with Legendre functions

The Legendre functions are solution of Legendre's differentials equation ([9], section 3.1, vol. 1)

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + [v(v+1) - \mu^2(1-z^2)^{-1}]w = 0, \tag{32}$$

where $z, v\mu$ unrestricted. If we substitute $w = (z^2-1)^{\frac{1}{2}\mu}v$ in (32) becomes

$$(1-z^2)\frac{d^2v}{dz^2} - 2(\mu+1)z\frac{dv}{dz} + (v-\mu)(v+\mu+1)v = 0, \tag{33}$$

and if $\xi = \frac{1}{2} - \frac{1}{2}z$ as the independent variable, this differential equation becomes,

$$\xi(1-\xi)\frac{d^2v}{d\xi^2} + (1-2\xi)(\mu+1)\frac{dv}{d\xi} + (v-\mu)(v+\mu+1)v = 0. \tag{34}$$

This is the Gauss hypergeometric type equation with $a = \mu - v$, $b = v+\mu+1$ and $c = \mu+1$ .
Hence it follows that the function

$$w = P_v^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} F\left[-v, v+1; 1-\mu; \frac{1}{2} - \frac{1}{2}z\right], |1-z| < 2, \tag{35}$$

is a solution of (32).

**Theorem 7.** If $\eta, \mu, \gamma \in C, Re(\gamma) > 0, Re(\eta) > 0, Re(\mu) > 0, k \in R$, $q \in (0,1) \cup N and\, \delta$ is non negative integer. Then the integral involving Legendre function of first kind written as,

$$\begin{aligned}\textstyle\int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}} P_v^\delta(x) GE_{k,\eta,\mu}^{\gamma,q}(zx^\alpha)dx &= \tfrac{(-1)^\delta \pi^{\frac{1}{2}} 2^{-\sigma-\delta}\Gamma(1+\delta+v)}{\Gamma(1-\delta+v)} \\ &\times \textstyle\sum_{r=0}^\infty \tfrac{\Gamma(\sigma+\alpha r)}{\Gamma\left[\frac{1}{2}+\frac{(\sigma+\alpha r)}{2}+\frac{\delta}{2}-\frac{v}{2}\right]\Gamma\left[1+\frac{(\sigma+\alpha r)}{2}+\frac{\delta}{2}+\frac{v}{2}\right]} GE_{k,\eta,\mu}^{\gamma,q}(z2^{-\alpha}).\end{aligned} \tag{36}$$

**Proof.** The integral involving Legendre function of first kind is

$$I_{10} \equiv \int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}} P_v^\delta(x) GE_{k,\eta,\mu}^{\gamma,q}(zx^\alpha)dx$$

$$= \int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}} P_v^\delta(x) \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k}(zx^\alpha)^r}{\Gamma_k(\eta r+\mu)\, r!} dx,$$

interchanging order of integration and summation, we can write

$$I_{10}=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^{r}}{\Gamma_{k}(\eta r+\mu)r!}\int_{0}^{1}x^{\alpha r+\sigma-1}(1-x^{2})^{\frac{\delta}{2}}P_{v}^{\delta}(x)dx.$$

Above integral (36)can be solved by using the formula ( [9], section 3.12, vol. 1).

$$\int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}}P_v^{\delta}(x)dx=\frac{(-1)^{\delta}\pi^{\frac{1}{2}}2^{-\sigma-\delta}\Gamma(\sigma)\Gamma(1+\delta+v)}{\Gamma\left[\frac{1}{2}+\frac{\sigma}{2}+\frac{\delta}{2}-\frac{v}{2}\right]\Gamma\left[1+\frac{\sigma}{2}+\frac{\delta}{2}+\frac{v}{2}\right]\Gamma(1-\delta+v)}, \tag{37}$$

where $Re(\sigma)>0,\ \delta=1,2,3,.....\ Re(\sigma)>0,\ \delta=1,2,3,.....$

Now, with the help of (32), and (36) can be written as

$$I_{10}=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^{r}}{\Gamma_{k}(\eta r+\mu)r!}\frac{(-1)^{\delta}\pi^{\frac{1}{2}}2^{-\sigma-\alpha r-\delta}\Gamma(\sigma+\alpha r)\Gamma(1+\delta+v)}{\Gamma\left[\frac{1}{2}+\frac{(\sigma+\alpha r)}{2}+\frac{\delta}{2}-\frac{v}{2}\right]\Gamma\left[1+\frac{(\sigma+\alpha r)}{2}+\frac{\delta}{2}+\frac{v}{2}\right]\Gamma(1-\delta+v)},$$

where $\eta,\mu,\gamma\in C, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, k\in R,\ q\in(0,1)\cup N, Re(\sigma)>0, Re(\delta)>1$. Therefore,

$$\begin{aligned}\int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}}P_v^{\delta}(x)GE_{k,\eta,\mu}^{\gamma,q}(zx^{\alpha})dx=&\frac{(-1)^{\delta}\pi^{\frac{1}{2}}2^{-\sigma-\delta}\Gamma(1+\delta+v)}{\Gamma(1-\delta+v)}\\ \times\sum_{r=0}^{\infty}&\frac{\Gamma(\sigma+\alpha r)}{\Gamma\left[\frac{1}{2}+\frac{(\sigma+\alpha r)}{2}+\frac{\delta}{2}-\frac{v}{2}\right]\Gamma\left[1+\frac{(\sigma+\alpha r)}{2}+\frac{\delta}{2}+\frac{v}{2}\right]}GE_{k,\eta,\mu}^{\gamma,q}(z2^{-\alpha}).\end{aligned}$$

This proves result (36).

**Theorem 8.** Let $\eta,\mu,\gamma\in C, Re(\gamma)>0, Re(\eta)>0, Re(\sigma)>0, Re(\mu)>0, k\in R,\ q\in(0,1)\cup N and\ Re(\delta)>1$ Then the integral involving Legendre function of first kind is

$$\begin{aligned}\int_0^1 x^{\sigma-1}(1-x^2)^{-\frac{\delta}{2}}P_v^{\delta}(x)GE_{k,\eta,\mu}^{\gamma,q}(zx^{\alpha})dx=&\pi^{\frac{1}{2}}2^{\delta-\sigma}\\ \times\sum_{r=0}^{\infty}&\frac{\Gamma(\sigma+\alpha r)}{\Gamma\left[\frac{1}{2}+\frac{(\sigma+\alpha r)}{2}-\frac{\delta}{2}-\frac{v}{2}\right]\Gamma\left[1+\frac{(\sigma+\alpha r)}{2}-\frac{\delta}{2}-\frac{v}{2}\right]}GE_{k,\eta,\mu}^{\gamma,q}(z2^{-\alpha}).\end{aligned} \tag{38}$$

**Proof.** The integral associated with Legendre function of first kind is,

$$\begin{aligned}I_{11}&\equiv\int_0^1 x^{\sigma-1}(1-x^2)^{-\frac{\delta}{2}}P_v^{\delta}(x)GE_{k,\eta,\mu}^{\gamma,q}(zx^{\alpha})dx\\ &=\int_0^1 x^{\sigma-1}(1-x^2)^{-\frac{\delta}{2}}P_v^{\delta}(x)\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}(zx^{\alpha})^{r}}{\Gamma_k(\eta r+\mu)r!}dx,\end{aligned}$$

interchanging order of integration and summation, we can write,

$$I_{11}=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^{r}}{\Gamma_{k}(\eta r+\mu)r!}\int_{0}^{1}x^{\alpha r+\sigma-1}(1-x^{2})^{-\frac{\delta}{2}}P_{v}^{\delta}(x)dx. \tag{39}$$

Now, we have the formula ( [9], section 3.12, vol. 1)

$$\int_0^1 x^{\sigma-1}(1-x^2)^{-\frac{\delta}{2}}P_v^{\delta}(x)dx=\frac{\pi^{\frac{1}{2}}2^{\delta-\sigma}\Gamma(\sigma)}{\Gamma\left[\frac{1}{2}+\frac{\sigma}{2}-\frac{\delta}{2}-\frac{v}{2}\right]\Gamma\left[1+\frac{\sigma}{2}-\frac{\delta}{2}-\frac{v}{2}\right]}, \tag{40}$$

where $Re(\sigma) > 0$, $\delta = 1, 2, 3, \ldots$. Finally, by using (40) in (39), we get

$$I_{11} = \pi^{\frac{1}{2}} 2^{\delta-\sigma} \sum_{r=0}^{\infty} \frac{\Gamma(\sigma+\alpha r)}{\Gamma\left[\frac{1}{2} + \frac{(\sigma+\alpha r)}{2} - \frac{\delta}{2} - \frac{v}{2}\right] \Gamma\left[1 + \frac{(\sigma+\alpha r)}{2} - \frac{\delta}{2} - \frac{v}{2}\right]} GE_{k,\eta,\mu}^{\gamma,q}(z2^{-\alpha}).$$

## 6 Integrals with Hermite polynomials

$H_n(x)$ Hermite polynomials ( [2], p. 187) may be defined as the following relation,

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!} \tag{41}$$

valid for all finite and . Since

$$\exp(2xt - t^2) = \exp(2xt)\exp(-t^2)$$

$$= \left(\sum_{n=0}^{\infty} \frac{(2x)^n t^n}{n!}\right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n/2} \frac{(-1)^k (2x)^{n-2k} t^n}{k!(n-2k)!}.$$

By (41), we can write

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!}, \tag{42}$$

$n$ $H_n(x)$is a polynomial of degree precisely in $x$ and

$$H_n(x) = 2^n x^n + \pi_{n-2}(x), \tag{43}$$

$\pi_{n-2}(x)\pi_{n-2}(x)$where is a polynomial of degree (n-2) in $x$.

**Theorem 9.** Let $\eta, \mu, \gamma \in C, Re(\gamma) > 0, Re(\eta) > 0, Re(\mu) > 0, k \in R$, $q \in (0,1) \cup N, Re(\sigma) > 0\, and\, Re(\delta) > 1$. Then the relation holds true,

$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2v}(x) GE_{k,\eta,\mu}^{\gamma,q}(zx^{-2h}) dx$$

$$= \pi^{\frac{1}{2}} 2^{2(v-\rho)} \sum_{r=0}^{\infty} \frac{\Gamma(2\rho - 2hr + 1)}{\Gamma(\rho - hr - v + 1)} GE_{k,\eta,\mu}^{\gamma,q}(z2^{2h}). \tag{44}$$

**Proof.** The integral associated with Hermite polynomial can be written as,

$$I_{12} \equiv \int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2v}(x) GE_{k,\eta,\mu}^{\gamma,q}(zx^{-2h}) dx$$

$$= \int_{-\infty}^{\infty} x^{2\rho} e^{-x^2} H_{2v}(x) \sum_{r=0}^{\infty} \frac{(\gamma) rq, k (zx^{-2h})^r}{\Gamma_k(\eta r + \mu) r!} dx,$$

interchanging the order of integration and summation, we get

$$=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}z^{r}}{\Gamma_{k}(\eta r+\mu)r!}\int_{-\infty}^{\infty}x^{2\rho-2h}e^{-x^{2}}H_{2v}(x)dx. \tag{45}$$

Now, by the formula ( [7], p. 59)

$$\int_{-\infty}^{\infty}x^{2\rho}e^{-x^{2}}H_{2v}(x)dx = \pi^{1/2}\frac{2^{2(v-\rho)}\Gamma(2\rho+1)}{\Gamma(\rho-v+1)}.$$

Hence, (45) can be written as

$$I_{12}=\pi^{\frac{1}{2}}2^{2(v-\rho)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho-2hr+1)}{\Gamma(\rho-hr-v+1)}GE_{k,\eta,\mu}^{\gamma,q}(z2^{2h}).$$

This proves result (46).

**Theorem 10.** Let $\eta,\mu,\gamma\in C, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, k\in R$, $q\in(0,1)\cup N, Re(\sigma)>0, Re(\delta)>1$. Then the relation holds true,

$$I_{13}\equiv\int_{-\infty}^{\infty}x^{2\rho}e^{-x^{2}}H_{2v}(x)GE_{k,\eta,\mu}^{\gamma,q}(zx^{2h})dx=\pi^{\frac{1}{2}}2^{2(v-\rho)}\sum_{r=0}^{\infty}\frac{\Gamma(2\rho+2hr+1)}{\Gamma(\rho+hr-v+1)}GE_{k,\eta,\mu}^{\gamma,q}(z2^{-2h}). \tag{46}$$

**Proof.** This Theorem can be proved on similar lines as Theorem 9.

## 7 Integrals with hypergeometric function

The function

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}}\frac{z^{n}}{n!}, \tag{47}$$

$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$,is known as Hypergeometric function ( [2], p. 45) for c neither zero nor a negative integer, in (47) the notation is the factorial function.

**Theorem 11.** Let $\eta,\mu,\gamma\in C, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, k\in R$ *and* $q\in(0,1)\cup N$.Then the integral involving hypergeometric function can be written as

$$\int_{1}^{\infty}x^{-\rho}(x-1)^{\sigma-1}_{2}F_{1}\begin{bmatrix}v+\sigma-\rho,\lambda+\sigma-\rho;(1-x)\\ \sigma;\end{bmatrix}GE_{k,\eta,\mu}^{\gamma,q}(zx)\,dx$$

$$=GE_{k,\eta,\mu}^{\gamma,q}(zx)\,{}_{2}F_{1}\begin{bmatrix}v+\sigma-\rho,\lambda+\sigma-\rho;-1\\ \sigma;\end{bmatrix}B(r+\sigma,\rho-2r-\sigma). \tag{48}$$

**Proof.** The integral involving hypergeometric function is,

$$I_{14}\equiv\int_{1}^{\infty}x^{-\rho}(x-1)^{\sigma-1}_{2}F_{1}\begin{bmatrix}v+\sigma-\rho,\lambda+\sigma-\rho;(1-x)\\ \sigma;\end{bmatrix}GE_{k,\eta,\mu}^{\gamma,q}(zx)dx,$$

$$=\sum_{r=0}^{\infty}\frac{(\gamma)_{rq,k}}{\Gamma_{k}(\eta r+\mu)r!}\int_{1}^{\infty}x^{-\rho+r}(x-1)^{\sigma-1}_{2}F_{1}\begin{bmatrix}v+\sigma-\rho,\lambda+\sigma-\rho;(1-x)\\ \sigma;\end{bmatrix}dx.$$

Let $x = t+1$,then

$$I_{14} = \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k}}{\Gamma_k(\eta r+\mu) r!} \int_1^{\infty} t^{\sigma-1}(t+1)^{r-\rho}_2 F_1 \left[\begin{matrix} v+\sigma-\rho, \lambda+\sigma-\rho; -t \\ \sigma; \end{matrix}\right] dt,$$

$$= \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k}}{\Gamma_k(\eta r+\mu) r!} \sum_{r=0}^{\infty} \frac{(-1)^r (v+\sigma-\rho)_r (\lambda+\sigma-\rho)_r}{(\sigma)_r r!} \int_1^{\infty} t^{r+\sigma-1}(t+1)^{r-\rho} dt,$$

$$= GE^{\gamma,q}_{k,\eta,\mu}(z)_2 F_1 \left[\begin{matrix} v+\sigma-\rho, \lambda+\sigma-\rho; -1 \\ \sigma; \end{matrix}\right] B(\sigma+r, \rho-2r-\sigma).$$

This proves Theorem 11.

## 8 Integrals involving generalized hypergeometric function

A generalized hypergeometric function ([2],p.73) is defined by

$$_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, ...., \alpha_p; z \\ \beta_1, \beta_2, ...., \beta_q; \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!}, \tag{49}$$

where no denominator parameter $\beta_j$ is allowed to be zero or negative integer.

**Theorem 12.** Let $\eta, \mu, \gamma \in C, Re(\gamma) > 0, Re(\eta) > 0, Re(\mu) > 0, k \in R$ and $q \in (0,1) \cup N$.
Then the integral involving generalized hypergeometric function can be written as

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}_p F_q \left[(g_p);(h_q); ax^{\alpha}(t-x)^{\beta}\right] GE^{\gamma,q}_{k,\eta,\mu}[zx^u(t-x)^v]\, dx \tag{50}$$

$$= t^{\sigma+\rho-1} \sum_{r=0}^{\infty} f(r) t^{(\alpha+\beta)r} GE^{\gamma,q}_{k,\eta,\mu}(zt^{u+v})\, B(\rho+ur+\alpha r, \sigma+vr+\beta r),$$

where $Re(\alpha) \geq 0, Re(v) \geq o$, both are not zero simultaneously.

**Proof.** The integral involving generalized hypergeometric function is

$$I_{15} \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1}_p F_q \left[(g_p);(h_q); ax^{\alpha}(t-x)^{\beta}\right] GE^{\gamma,q}_{k,\eta,\mu}[zx^u(t-x)^v]\, dx,$$

$$= \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k} z^r}{\Gamma_k(\eta r+\mu) r!} t^{vr+\sigma-1} \int_0^t x^{ur+\rho-1} \left(1-\frac{x}{t}\right)^{vr+\sigma-1} {}_pF_q \left[(g_p);(h_q); ax^{\alpha}(t-x)^{\beta}\right] dx.$$

Let $x = st$, then

$$I_{15} = \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k} z^r t^{(v+u)r}}{\Gamma_k(\eta r+\mu) r!} t^{\sigma+\sigma-1} \int_0^1 s^{ur+\rho-1} (1-s)^{vr+\sigma-1} {}_pF_q \left[(g_p);(h_q); as^{\alpha} t^{\alpha+\beta} (1-s)^{\beta}\right] ds.$$

$$= t^{\sigma+\sigma-1} \sum_{r=0}^{\infty} \frac{(\gamma)_{rq,k} (zt^{v+u})^r}{\Gamma_k(\eta r+\mu) r!} \int_0^1 s^{ur+\alpha r+\rho-1} (1-s)^{vr+\beta r+\sigma-1} \sum_{r=0}^{\infty} \frac{(g_p)_r}{(h_q)_r} \frac{t^{(\alpha+\beta)r} a^r}{r!} ds,$$

where

$$f(r)=\sum_{r=0}^{\infty}\frac{(g_p)_r}{(h_q)_r}\frac{a_r}{r!}=\frac{(g_1)_r.....(g_p)_r}{(h_1)_r.....(h_q)_r}\frac{a^r}{r!} \tag{51}$$

and $\alpha$, $\beta$ are non negative integer such that $\alpha+\beta\geq 1$.

**Theorem 13.** Let $\eta,\mu,\gamma\in C, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, k\in R$ *and* $q\in(0,1)\cup N$.
Then the integral involving generalized hypergeometric function can be written as

$$I_{16}\equiv\int_0^t x^{\rho-1}(t-x)^{\sigma-1}_p F_q\left[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}\right] GE^{\gamma,q}_{k,\eta,\mu}[zx^{-u}(t-x)^{-v}]\,dx$$

$$=t^{\sigma+\rho-1}\sum_{r=0}^{\infty}f(r)t^{(\alpha+\beta)r}GE^{\gamma,q}_{k,\eta,\mu}(zt^{-u-v})B(\rho-ur+\alpha r,\sigma-vr+\beta r). \tag{52}$$

where f(r) is defined by (51).

**Proof.** This theorem can be proved on similar lines as Theorem 12.

**Theorem 14.** Let $\eta,\mu,\gamma\in C, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, k\in R$ *and* $q\in(0,1)\cup N$.
Then the integral involving generalized hypergeometric function can be written as

$$I_{17}\equiv\int_0^t x^{\rho-1}(t-x)^{\sigma-1}_p F_q\left[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}\right] GE^{\gamma,q}_{k,\eta,\mu}[zx^{u}(t-x)^{-v}]\,dx$$

$$=t^{\sigma+\rho-1}\sum_{r=0}^{\infty}f(r)t^{(\alpha+\beta)r}GE^{\gamma,q}_{k,\eta,\mu}(zt^{u-v})B(\rho+ur+\alpha r,\sigma-vr+\beta r). \tag{53}$$

where f(r) is defined by (51).

**Proof.** This Theorem can be proved on similar lines as Theorem 12.

**Theorem 15.** Let $\eta,\mu,\gamma\in C, Re(\gamma)>0, Re(\eta)>0, Re(\mu)>0, k\in R$ *and* $q\in(0,1)\cup N$.
Then the integral involving generalized hypergeometric function can be written as

$$I_{18}\equiv\int_0^t x^{\rho-1}(t-x)^{\sigma-1}_p F_q\left[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}\right] GE^{\gamma,q}_{k,\eta,\mu}[zx^{-u}(t-x)^{v}]\,dx$$

$$=t^{\sigma+\rho-1}\sum_{r=0}^{\infty}f(r)t^{(\alpha+\beta)r}GE^{\gamma,q}_{k,\eta,\mu}(zt^{-u+v})B(\rho-ur+\alpha r,\sigma+vr+\beta r). \tag{54}$$

where f(r) is defined by (51).

**Proof.** This theorem can be proved on similar lines as Theorem 12.

## 9 Conclusion

In this article we have obtained various integrals involving Generalized k- Mittag-Leffler function. If we set q=1, then theorems established in this paper reduces for k- Mittag-Leffler function [5]. Further if we set k=1, then the results for the function earlier given by [1] are also obtained.

## 10 Acknowledgment

The Second author would like to thank NATIONAL BOARD OF HIGHER MATHEMATICS, Dept. of Atomic Energy, Govt.of India, Mumbai for the financial assistance under PDF sanction no. 2/40(47)/2013/R&D-II/275.

## References

[1] D.K. Singh and R. Rawat, Integral involving generalized Mittag-Leffler function, J. Frac. Cal. App., 42, 2013.

[2] E.D. Rainville, *Special Functions*, New York, Macmillan, 1960.

[3] E. M. Wright, The asymptotic expansion of the generalized Bessel functions, Proceedings London Mathematical Society, 38, 1934.

[4] E. M.Wright, The asymptotic expansion of the generalized hypergeometric functions, Journal London Mathematical Society,10, 1935.

[5] G.A. Dorrego and R.A. Cerutti, The k- Mittag-Leffler function, Int. J.Contemp. Math. Sci., 7, 2012.

[6] K.S. Gehlot, The Generalized k- Mittag-Leffler function, Int. J. contemp. Math. Sci.,7:45, 2012.

[7] V. P. Saxena , *The I-Function,* Anamaya publisher, New Delhi, 2008.

[8] V. S. Kiryakova, *Generalized Fractional Calculus and Applications*. Pitman Research Notes in Mathematics, 301, John Wiley and Sons, New York, 1994.

[9] W. Erdelyi et al, *Higher Transcendental Functions*, 1, McGraw Hill, 1953.

[10] W. Erdelyi, Magnus, F. Oberhettinger, and F. G Tricomi, *Higher Transcendental Functions*, 3, McGraw- Hill, New York, NY USA, 1955.