New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2022.490

Dynamical analysis and solutions of nonlinear difference equations of twenty-fourth order

Lama Sh. Aljoufi[1], *M. B. Almatrafi*[2]

[1]Department of Mathematics, College of Science, Jouf University, P.O. Box: 2014, Sakaka, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Taibah University, Saudi Arabia

Received: 11 May 2022, Accepted: 8 June 2022
Published online: 5 December 2022.

Abstract: This paper discusses the behaviors and solutions of some rational recursive relations of twenty-fourth order using the iteration technique and the modulus operator. The stability of the equilibrium points are comprehensively analyzed. We also present other properties such as periodicity, oscillation and bounded solutions. Some numerical examples are obviously given to ensure the validity of the theoretical work. These examples are plotted using MATLAB. The proposed techniques can be utilized to be applied on other nonlinear equations.

Keywords: equilibrium, asymptotic stability, periodicity, exact and numerical solutions.

1 Introduction

The evolution of a certain natural phenomena is often described over a course of time using differential equations. However, many real life problems can be sometimes modeled using discrete time steps which lead to difference equations. Hence, recursive equations have an effective and powerful role in mathematics. They are successfully utilized to investigate some applications in engineering, physics, biology, economic and others. For instance, recursive equations have been well used in modeling some natural phenomena such as the size of a population, the Fibonacci sequence, the drug in the blood system, the transmission of information, the pricing of a certain commodity, the propagation of annual plants, and others [1]. In addition, some scholars have used difference equations to find the numerical solutions of some differential equations. More specifically, discretizing a given differential equation gives a difference equation. For example, Runge–Kutta scheme is obtained from discretizing a first order differential equation. The development of technology has motivated the use of recurrence equations as approximations to partial differential equations. It is worth mentioning that fractional order difference equations are often utilized to investigate some real life phenomena emerging in nonlinear sciences.

Most properties of recursive expressions have been widely discussed by some researchers. For example, researchers have explored the stability, periodicity, boundedness and solutions of some recursive equations. We here present some published works. Alayachi et al. [2] analyzed the local and global attractivity, periodicity and the solutions of a sixth order difference equation. Some numerical examples have been also presented in [2]. In [3], Sanbo and Elsayed presented the periodicity, stability and some solutions of a fifth order recursive equation. Almatrafi and Alzubaidi [4] discussed the dynamical behaviors of an eighth order difference relation and showed some 2D figures for the obtained results. Moreover, Ahmed et al. [5], found new solutions and investigated the dynamical analysis for some nonlinear

* Corresponding author e-mail: mmutrafi@taibahu.edu.sa

difference relations of fifteenth order. The authors in [6] obtained novel structures for the solutions of a rational recursive relation. The local and global stability, boundedness, periodicity and solutions of a second order difference equation were investigated in [7]. The authors proved that the local asymptotic stability of the equilibrium point implies global asymptotic stability. In [8], Kara and Yazlik expressed the solutions of a $(k+l)$-order recursive equation and investigated the asymptotic stability of the constructed results of the problem when $k=3$, and $l=k$. Finally, Elsayed [9] analyzed the qualitative behaviors of a nonlinear recursive equation. More discussions about nonlinear recursive problems can be seen in refs. [10,11,12,13,14,15,16,17,18,19,20].

The motivation of writing this article arises from the investigation of fifteenth order difference equations given in [5]. We consider more sophisticated rational difference equations of twenty-fourth order. Therefore, this work aims to analyze some dynamical properties such as equilibrium points, local and global behaviors, boundedness, and analytic solutions of the nonlinear recursive equations

$$\chi_{r+1} = \frac{\chi_{r-23}}{\pm 1 \pm \prod_{i=0}^{5} \chi_{r-(4i+3)}}, \quad r = 0,1,2,....$$

Here, the initial values $\chi_{-23}, \chi_{-22}, ..., \chi_0$ are arbitrary non-zero real numbers. In this work, we also illustrate some 2D figures with the help of MATLAB to validate the obtained results.

Definition 1. *We consider mod* $(\kappa,4) = \kappa - 4\lfloor \frac{\kappa}{4} \rfloor$, *where* $\lfloor \Lambda \rfloor$ *is the greatest integer less than or equal to the real number* Λ.

2 Dynamical analysis of $\chi_{r+1} = \frac{\chi_{r-23}}{1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}$

In this section, we implement specific strategies to investigate the dynamical behaviors of the difference equation

$$\chi_{r+1} = \frac{\chi_{r-23}}{1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}, \quad r = 0,1,2,..., \tag{1}$$

with the initial conditions $\chi_{-\iota}$, $\iota = 0,1,2,...,23$. More specifically, the attractivity and boundedness of the solutions of Eq. (1) are investigated.

Theorem 1. *Assume that* $\{\chi_r\}_{r=-23}^{\infty}$ *is a solution to Eq. (1). Then, for* $r = 0,1,2,...$, *we have*

$$\chi_{24r-\kappa} = \varepsilon_\kappa \prod_{i=0}^{r-1} \left(\frac{1+(6i+\eta_\kappa - 1)\mu_\kappa}{1+(6i+\eta_\kappa)\mu_\kappa} \right), \tag{2}$$

where $\mu_\kappa = \prod_{j=0}^{5} \varepsilon_{mod(\kappa,4)+4j}$, $\eta_\kappa = 6 - \left[\frac{\kappa}{4}\right]$, $\chi_{-k} = \varepsilon_\kappa$, $r\mu_\kappa \neq -1$, $r \in \{1,2,3,...\}$, *and* $\kappa = 0,1,2,...,23$.

Proof The solutions are true at $r=0$. Let $r>0$ and suppose that the results are true at $r-1$, as follows:

$$\chi_{24r-24-\kappa} = \varepsilon_\kappa \prod_{i=0}^{r-2} \left(\frac{1+(6i+\eta_\kappa - 1)\mu_\kappa}{1+(6i+\eta_\kappa)\mu_\kappa} \right). \tag{3}$$

Using Eq. (1) and Eq. (3) gives

$$\begin{aligned}
\chi_{24r-23} &= \frac{\chi_{24r-47}}{1+\chi_{24r-27}\chi_{24r-31}\chi_{24r-35}\chi_{24r-39}\chi_{24r-43}\chi_{24r-47}} \\
&= \frac{\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{23}-1)\mu_{23}}{1+(6i+\eta_{23})\mu_{23}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{4j+3}\prod_{i=0}^{r-2}\left(\frac{1+\left(6i+\eta_{4j+3}-1\right)\mu_{4j+3}}{1+\left(6i+\eta_{4j+3}\right)\mu_{4j+3}}\right)\right)} \\
&= \frac{\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{1+(6i+1)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}\right)}{1+\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{1+(6i+6)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}\right)} \\
&= \varepsilon_{23}\prod_{i=0}^{r-1}\frac{1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{1+(6i+1)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}.
\end{aligned}$$

Furthermore, utilizing Eq. (1) and Eq. (3), we have

$$\begin{aligned}
\chi_{24r-22} &= \frac{\chi_{24r-46}}{1+\chi_{24r-26}\chi_{24r-30}\chi_{24r-34}\chi_{24r-38}\chi_{24r-42}\chi_{24r-46}} \\
&= \frac{\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{22}-1)\mu_{22}}{1+(6i+\eta_{22})\mu_{22}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{4j+2}\prod_{i=0}^{r-2}\left(\frac{1+\left(6i+\eta_{4j+2}-1\right)\mu_{4j+2}}{1+\left(6i+\eta_{4j+2}\right)\mu_{4j+2}}\right)\right)} \\
&= \frac{\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{1+(6i+1)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right)}{1+\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{1+(6i+6)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right)} \\
&= \varepsilon_{22}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{1+(6i+1)\,\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right).
\end{aligned}$$

Also, from Eq. (1) and Eq. (3), we have

$$\begin{aligned}
\chi_{24r-21} &= \frac{\chi_{24r-45}}{1+\chi_{24r-25}\chi_{24r-29}\chi_{24r-33}\chi_{24r-37}\chi_{24r-41}\chi_{24r-45}} \\
&= \frac{\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{21}-1)\mu_{21}}{1+(6i+\eta_{21})\mu_{21}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{4j+1}\prod_{i=0}^{r-2}\left(\frac{1+\left(6i+\eta_{4j+1}-1\right)\mu_{4j+1}}{1+\left(6i+\eta_{4j+1}\right)\mu_{4j+1}}\right)\right)} \\
&= \frac{\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{1+(6i+1)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right)}{1+\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{1+(6i+6)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right)} \\
&= \varepsilon_{21}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{1+(6i+1)\,\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right).
\end{aligned}$$

Finally, using Eq. (1) and Eq. (3) gives

$$\begin{aligned}\chi_{24r-20} &= \frac{\chi_{24r-44}}{1+\chi_{24r-24}\chi_{24r-28}\chi_{24r-32}\chi_{24r-36}\chi_{24r-40}\chi_{24r-44}} \\ &= \frac{\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{1+(6i+\eta_{20}-1)\mu_{20}}{1+(6i+\eta_{20})\mu_{20}}\right)}{1+\prod_{j=0}^{5}\left(\varepsilon_{4j}\prod_{i=0}^{r-2}\left(\frac{1+\left(6i+\eta_{4j}-1\right)\mu_{4j}}{1+\left(6i+\eta_{4j}\right)\mu_{4j}}\right)\right)} \\ &= \frac{\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{1+(6i+1)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right)}{1+\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{1+(6i+6)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right)} \\ &= \varepsilon_{20}\prod_{i=0}^{r-1}\left(\frac{1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{1+(6i+1)\,\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right).\end{aligned}$$

We similarly can obtain other relations of Eq. (2).

Theorem 2. *Suppose that $\chi_{-23},\chi_{-22},\ldots,\chi_0 \in [0,\infty)$. Then, every solution of Eq. (1) is bounded.*

Proof: Let $\{\chi_r\}_{r=-23}^{\infty}$ be a solution to Eq. (1). Then, from Eq. (1), one obtains

$$0 \le \chi_{r+1} = \frac{\chi_{r-23}}{1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}} \le \chi_{r-23}, \quad \forall\, r \ge 0.$$

Thus, the sequence $\{\chi_{24r-i}\}_{r=0}^{\infty}$, $i = 0,1,\ldots,23$ is decreasing and is bounded from above by $\tau = \max\{\chi_{-23},\chi_{-22},\ldots,\chi_0\}$.

Theorem 3. *Equation (1) has a unique fixed point which is $\overline{\chi} = 0$.*

Proof: Using Eq. (1) gives

$$\overline{\chi} = \frac{\overline{\chi}}{1+\overline{\chi}^6},$$

which leads to

$$\overline{\chi}+\overline{\chi}^7 = \overline{\chi}.$$

Hence, $\overline{\chi}^7 = 0$. This gives that $\overline{\chi} = 0$.

Theorem 4. *Let $\chi_{-23},\chi_{-22},\ldots,\chi_0 \in [0,\infty)$. Then, the fixed point $\overline{\chi} = 0$ of Eq. (1) is locally stable.*

Proof: Assume that $\varepsilon > 0$, and let $\{\chi_\Omega\}_{\Omega=-23}^{\infty}$ be a solution to Eq. (1) with

$$\sum_{j=0}^{23}\left|\chi_{-j}\right| < \varepsilon.$$

Now, it is sufficient to show that $|\chi_1| < \varepsilon$. Note that

$$0 < \chi_1 = \frac{\chi_{-23}}{1+\chi_{-3}\chi_{-7}\chi_{-11}\chi_{-15}\chi_{-19}\chi_{-23}} \le \chi_{-23} < \varepsilon.$$

The proof is done.

Theorem 5. *Let $\chi_{-23},\chi_{-22},\ldots,\chi_0 \in [0,\infty)$. Then, the fixed point $\overline{\chi} = 0$ of Eq. (1) is globally asymptotically stable.*

Proof: In Theorem 4, we showed that the fixed point $\overline{\chi} = 0$ is locally stable. Let $\{\chi_r\}_{r=-23}^{\infty}$ be a positive solution to Eq. (1). Then, it is needed to prove that $\lim_{r\to\infty}\chi_r = \overline{\chi} = 0$. Note that Theorem 2 gives $\chi_{r+1} < \chi_{r-23}$, $\forall\, r \ge 0$. The sequences

$\{\chi_{24r-i}\}_{r=0}^{\infty}$, $i = 0,1,...,23$ are decreasing and bounded which means that the sequences $\{\chi_{24r-i}\}_{r=0}^{\infty}, i = 0,1,...,23$ approach to a limit $Z_i \geq 0$. Hence,

$$Z_{23} = \frac{Z_{23}}{1+Z_3Z_7Z_{11}Z_{15}Z_{19}Z_{23}}, Z_{22} = \frac{Z_{22}}{1+Z_2Z_6Z_{10}Z_{14}Z_{18}Z_{22}}, ..., Z_0 = \frac{Z_0}{1+Z_0Z_4Z_8Z_{12}Z_{16}Z_{20}}.$$

This leads to $Z_0 = Z_1 = ... = Z_{23} = 0$.

3 Dynamical analysis of $\chi_{r+1} = \frac{\chi_{r-23}}{1-\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}$

This part investigates the qualitative behavior of the recursive equation

$$\chi_{r+1} = \frac{\chi_{r-23}}{1-\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}, \quad r = 0,1,2,... \tag{4}$$

where the initial conditions $\chi_{-\iota}$, $\iota = 0,1,2,...,23$, are real numbers.

Theorem 6. *Assume that $\{\chi_r\}_{r=-23}^{\infty}$ is a solution to Eq. (4). Then, for $r = 0,1,2,...$*

$$\chi_{24r-\kappa} = \varepsilon_\kappa \prod_{i=0}^{r-1} \left(\frac{-1+(6i+\eta_\kappa-1)\mu_\kappa}{-1+(6i+\eta_\kappa)\mu_\kappa} \right). \tag{5}$$

Here, $\mu_\kappa = \prod_{j=0}^{5} \varepsilon_{mod(\kappa,4)+4j}$, $\eta_\kappa = 6 - \left[\frac{\kappa}{4}\right]$, $\chi_{-\kappa} = \varepsilon_{k\kappa}$, $r\mu_\kappa \neq 1$, $r \in \{1,2,3,...\}$, and $\kappa = 0,1,2,...,23$.

Proof: Equation (5) is true for $r = 0$. We let $r > 0$ and suppose that the solutions are true at $r-1$. Hence,

$$\chi_{24r-24-\kappa} = \varepsilon_\kappa \prod_{i=0}^{r-2} \left(\frac{-1+(6i+\eta_\kappa-1)\mu_\kappa}{-1+(6i+\eta_\kappa)\mu_\kappa} \right). \tag{6}$$

Utilizing Eq. (4) and Eq. (6) yields

$$\begin{aligned}
\chi_{24r-23} &= \frac{\chi_{24r-47}}{1-\chi_{24r-27}\chi_{24r-31}\chi_{24r-35}\chi_{24r-39}\chi_{24r-43}\chi_{24r-47}} \\
&= \frac{\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{23}-1)\mu_{23}}{-1+(6i+\eta_{23})\mu_{23}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{4j+3}\prod_{i=0}^{r-2}\left(\frac{-1+\left(6i+\eta_{4j+3}-1\right)\mu_{4j+3}}{-1+\left(6i+\eta_{4j+3}\right)\mu_{4j+3}}\right)\right)} \\
&= \frac{\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{-1+(6i+1)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}\right)}{1-\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{-1+(6i+6)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}\right)} \\
&= \varepsilon_{23}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}{-1+(6i+1)\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23}}\right).
\end{aligned}$$

Furthermore, from Eq. (4) and Eq. (6), we have

$$\begin{aligned}
\chi_{24r-22} &= \frac{\chi_{24r-46}}{1-\chi_{24r-26}\chi_{24r-30}\chi_{24r-34}\chi_{24r-38}\chi_{24r-42}\chi_{24r-46}} \\
&= \frac{\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{22}-1)\mu_{22}}{-1+(6i+\eta_{22})\mu_{22}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{4j+2}\prod_{i=0}^{r-2}\left(\frac{-1+\left(6i+\eta_{4j+2}-1\right)\mu_{4j+2}}{-1+\left(6i+\eta_{4j+2}\right)\mu_{4j+2}}\right)\right)} \\
&= \frac{\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{-1+(6i+1)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right)}{1-\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{-1+(6i+6)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right)} \\
&= \varepsilon_{22}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}{-1+(6i+1)\,\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22}}\right).
\end{aligned}$$

Moreover, from Eq. (4) and Eq. (6), we have

$$\begin{aligned}
\chi_{24r-21} &= \frac{\chi_{24r-45}}{1-\chi_{24r-25}\chi_{24r-29}\chi_{24r-33}\chi_{24r-37}\chi_{24r-41}\chi_{24r-45}} \\
&= \frac{\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{21}-1)\mu_{21}}{-1+(6i+\eta_{21})\mu_{21}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{4j+1}\prod_{i=0}^{r-2}\left(\frac{-1+\left(6i+\eta_{4j+1}-1\right)\mu_{4j+1}}{-1+\left(6i+\eta_{4j+1}\right)\mu_{4j+1}}\right)\right)} \\
&= \frac{\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{-1+(6i+1)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right)}{1-\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{-1+(6i+6)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right)} \\
&= \varepsilon_{21}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}{-1+(6i+1)\,\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21}}\right).
\end{aligned}$$

Finally, Eq. (4) and Eq. (6) lead to

$$\begin{aligned}
\chi_{24r-20} &= \frac{\chi_{24r-44}}{1-\chi_{24r-24}\chi_{24r-28}\chi_{24r-32}\chi_{24r-36}\chi_{24r-40}\chi_{24r-44}} \\
&= \frac{\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{-1+(6i+\eta_{20}-1)\mu_{20}}{-1+(6i+\eta_{20})\mu_{20}}\right)}{1-\prod_{j=0}^{5}\left(\varepsilon_{4j}\prod_{i=0}^{r-2}\left(\frac{-1+\left(6i+\eta_{4j}-1\right)\mu_{4j}}{-1+\left(6i+\eta_{4j}\right)\mu_{4j}}\right)\right)} \\
&= \frac{\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{-1+(6i+1)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right)}{1-\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}\prod_{i=0}^{r-2}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{-1+(6i+6)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right)} \\
&= \varepsilon_{20}\prod_{i=0}^{r-1}\left(\frac{-1+(6i)\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}{-1+(6i+1)\,\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20}}\right).
\end{aligned}$$

Other relations can be similarly done.

Theorem 7. *Equation (4) has a unique equilibrium point $\overline{\chi}=0$, which is non-hyperbolic.*

Proof: Equation (4) leads to

$$\overline{\chi}=\frac{\overline{\chi}}{1-\overline{\chi}^6},$$

from which we have

$$\overline{\chi}-\overline{\chi}^7=\overline{\chi}.$$

Hence, $\overline{\chi}^7 = 0$. As a result, $\overline{\chi} = 0$. Next, we define a function

$$h\,(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{x_1}{1 - x_1x_2x_3x_4x_5x_6},$$

on I^6 where I is a subset of $\mathbb{R}$ such that $0 \in I$ and $f(I^6) \subseteq I$. It is obvious that h is continuously differentiable on I^6. Thus,

$$\begin{aligned}
h_{x_1}(x_1,x_2,x_3,x_4,x_5,x_6) &= \frac{1}{(1-x_1x_2x_3x_4x_5x_6)^2}, \quad h_{x_2}(x_1,x_2,x_3,x_4,x_5,x_6) = \frac{x_1^2x_3x_4x_5x_6}{(1-x_1x_2x_3x_4x_5x_6)^2},\\
h_{x_3}(x_1,x_2,x_3,x_4,x_5,x_6) &= \frac{x_1^2x_2x_4x_5x_6}{(1-x_1x_2x_3x_4x_5x_6)^2}, \quad h_{x_4}(x_1,x_2,x_3,x_4,x_5,x_6) = \frac{x^2x_2x_3x_5x_6}{(1-x_1x_2x_3x_4x_5x_6)^2},\\
h_{x_5}(x_1,x_2,x_3,x_4,x_5,x_6) &= \frac{x_1^2x_2x_3x_4x_6}{(1-x_1x_2x_3x_4x_5x_6)^2}, \quad h_{x_6}(x_1,x_2,x_3,x_4,x_5,x_6) = \frac{x_1^2x_2x_3x_4x_5}{(1-x_1x_2x_3x_4x_5x_6)^2}.
\end{aligned}$$

Therefore,

$h_{x_1}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi}) = 1,\ h_{x_2}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi}) = h_{x_3}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi}) = h_{x_4}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi}) = h_{x_5}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi}) = h_{x_6}(\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi},\overline{\chi}) = 0.$

We now obtain the linearized equation of Eq. (4) about $\overline{\chi} = 0$, which is given by

$$\chi_{\Omega+1} = \chi_{\Omega-23}, \tag{7}$$

whose characteristic equation is

$$\lambda^{24} - 1 = 0.$$

This means that

$$|\lambda_i| = 1, \quad i = 1, 2, \ldots, 24.$$

As a result, $\overline{\chi}$ is a non hyperbolic equilibrium point.

4 Dynamical analysis of $\chi_{r+1} = \frac{\chi_{r-23}}{-1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}$

This section is assigned to show the solutions of the difference problem

$$\chi_{r+1} = \frac{\chi_{r-23}}{-1+\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}, \quad r = 0, 1, 2, \ldots, \tag{8}$$

where $\chi_{-\iota}$, $\iota = 0, 1, 2, \ldots, 23$ are real numbers. We also present the periodicity and the oscillation of the solutions.

Theorem 8. *Assume that $\{\chi_r\}_{r=-23}^{\infty}$ is a solution to Eq. (8). Then, for $r = 0, 1, 2, \ldots$, we have*

$$\chi_{24r-\kappa} = \frac{\varepsilon_\kappa}{(-1+\mu_\kappa)^{r\alpha_\kappa}}. \tag{9}$$

Here, $\mu_\kappa = \prod_{j=0}^{5} \varepsilon_{mod(\kappa,4)+4j}$, $\alpha_\kappa = (-1)^{\left[\frac{\kappa}{4}\right]+1}$, $\chi_{-\kappa} = \varepsilon_\kappa$, $\mu_\kappa \neq 1$, and $\kappa = 0, 1, 2, \ldots, 23$.

Proof: Equation (9) is true for $r = 0$. Let $r > 0$ and assume that the results are true at $r - 1$, as follows:

$$\chi_{24r-24-\kappa} = \frac{\varepsilon_\kappa}{(-1+\mu_\kappa)^{(r-1)\alpha_\kappa}}. \tag{10}$$

From Eq. (8) and Eq. (10), we have

$$
\begin{aligned}
\chi_{24r-23} &= \frac{\chi_{24r-47}}{-1+\chi_{24r-27}\chi_{24r-31}\chi_{24r-35}\chi_{24r-39}\chi_{24r-43}\chi_{24r-47}} \\
&= \frac{\frac{\varepsilon_{23}}{(-1+\mu_{23})^{r-1}}}{-1+\varepsilon_3(-1+\mu_3)^{r-1}\frac{\varepsilon_7}{(-1+\mu_7)^{r-1}}\varepsilon_{11}(-1+\mu_{11})^{r-1}\frac{\varepsilon_{15}}{(-1+\mu_{15})^{r-1}}\varepsilon_{19}(-1+\mu_{19})^{r-1}\frac{\varepsilon_{23}}{(-1+\mu_{23})^{r-1}}} \\
&= \frac{\varepsilon_{23}}{(-1+\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})^{r-1}(-1+\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})} \\
&= \frac{\varepsilon_{23}}{(-1+\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})^{r}}.
\end{aligned}
$$

We also obtain from Eq. (8) and Eq. (10) that

$$
\begin{aligned}
\chi_{24r-22} &= \frac{\chi_{24r-46}}{-1+\chi_{24r-26}\chi_{24r-30}\chi_{24r-34}\chi_{24r-38}\chi_{24r-42}\chi_{24r-46}} \\
&= \frac{\frac{\varepsilon_{22}}{(-1+\mu_{22})^{r-1}}}{-1+\varepsilon_2(-1+\mu_2)^{r-1}\frac{\varepsilon_6}{(-1+\mu_6)^{r-1}}\varepsilon_{10}(-1+\mu_{10})^{r-1}\frac{\varepsilon_{14}}{(-1+\mu_{14})^{r-1}}\varepsilon_{18}(-1+\mu_{18})^{r-1}\frac{\varepsilon_{22}}{(-1+\mu_{22})^{r-1}}} \\
&= \frac{\varepsilon_{22}}{(-1+\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})^{r-1}(-1+\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})} \\
&= \frac{\varepsilon_{22}}{(-1+\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})^{r}}.
\end{aligned}
$$

Furthermore, Eq. (8) and Eq. (10) give

$$
\begin{aligned}
\chi_{24r-21} &= \frac{\chi_{24r-45}}{-1+\chi_{24r-25}\chi_{24r-29}\chi_{24r-33}\chi_{24r-37}\chi_{24r-41}\chi_{24r-45}} \\
&= \frac{\frac{\varepsilon_{21}}{(-1+\mu_{21})^{r-1}}}{-1+\varepsilon_1(-1+\mu_1)^{r-1}\frac{\varepsilon_5}{(-1+\mu_5)^{r-1}}\varepsilon_9(-1+\mu_9)^{r-1}\frac{\varepsilon_{13}}{(-1+\mu_{13})^{r-1}}\varepsilon_{17}(-1+\mu_{17})^{r-1}\frac{\varepsilon_{21}}{(-1+\mu_{21})^{r-1}}} \\
&= \frac{\varepsilon_{21}}{(-1+\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})^{r-1}(-1+\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})} \\
&= \frac{\varepsilon_{21}}{(-1+\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})^{r}}.
\end{aligned}
$$

Finally, we use Eq. (8) and Eq. (10) to have

$$
\begin{aligned}
\chi_{24r-20} &= \frac{\chi_{24r-44}}{-1+\chi_{24r-24}\chi_{24r-28}\chi_{24r-32}\chi_{24r-36}\chi_{24r-40}\chi_{24r-44}} \\
&= \frac{\frac{\varepsilon_{20}}{(-1+\mu_{20})^{r-1}}}{-1+\varepsilon_0(-1+\mu_0)^{r-1}\frac{\varepsilon_4}{(-1+\mu_4)^{r-1}}\varepsilon_8(-1+\mu_8)^{r-1}\frac{\varepsilon_{12}}{(-1+\mu_{12})^{r-1}}\varepsilon_{16}(-1+\mu_{16})^{r-1}\frac{\varepsilon_{20}}{(-1+\mu_{20})^{r-1}}} \\
&= \frac{\varepsilon_{20}}{(-1+\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})^{r-1}(-1+\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})} \\
&= \frac{\varepsilon_{20}}{(-1+\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})^{r}}.
\end{aligned}
$$

One can similarly proved other relations.

Theorem 9. *Equation (8) has three equilibrium points* 0 *and* $\pm\sqrt[6]{2}$, *which are non-hyperbolic.*

Proof: It can be similarly done as the proof of Theorem 7.

Theorem 10. *Equation (8) is periodic of period* 24 *if and only if* $\mu_\kappa = 2$, *for* $\kappa = 0,1,...,23$, *which take the following form:*

$$\chi_{24r-\kappa} = \varepsilon_\kappa,\ \kappa = 0,1,...,23,\ and\ r = 0,1,2,...$$

Proof: The proof is done by using Theorem 8.

Theorem 11. *Let* $\varepsilon_0, \varepsilon_1, ..., \varepsilon_{23} \in (0,1)$. *Then, the solution* $\{\chi_\Omega\}_{\Omega=-23}^{\infty}$ *oscillates about the equilibrium point* $\overline{\chi} = 0$, *with positive semicycles of length* 24, *and negative semicycles of length* 24.

Proof: Using Theorem 8, we find $\chi_1, \chi_2, ..., \chi_{24} < 0$ and $\chi_{25}, \chi_{26}, ..., \chi_{48} > 0$. Thus, the result follows by induction.

5 Dynamical analysis of $\chi_{r+1} = \frac{\chi_{r-23}}{-1-\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}}$

This section presents the periodicity, oscillation, and the solutions of the recursive problem

$$\chi_{r+1} = \frac{\chi_{r-23}}{-1-\chi_{r-3}\chi_{r-7}\chi_{r-11}\chi_{r-15}\chi_{r-19}\chi_{r-23}},\quad r = 0,1,2,..., \tag{11}$$

where $\chi_{-\iota},\ \iota = 0,1,2,...,23$, are real numbers.

Theorem 12. *Let* $\{\chi_r\}_{r=-23}^{\infty}$ *be a solution to Eq. (11). Then, for* $r = 0,1,2,...$, *we have*

$$\chi_{24r-\kappa} = \frac{\varepsilon_\kappa}{(-1-\mu_\kappa)^{r\alpha_\kappa}}. \tag{12}$$

Here, $\mu_\kappa = \prod_{j=0}^{5} \varepsilon_{mod(\kappa,4)+4j}$, $\alpha_\kappa = (-1)^{\left[\frac{\kappa}{4}\right]+1}$ *and* $\chi_{-\kappa} = \varepsilon_\kappa$, *with* $\mu_\kappa \neq -1,\ \kappa = 0,1,2,...,23$.

Proof: The results are true at $r = 0$. We let $r > 0$ and suppose that the solutions are true at $r-1$ as follows:

$$\chi_{24r-24-\kappa} = \frac{\varepsilon_\kappa}{(-1-\mu_\kappa)^{(r-1)\alpha_\kappa}}. \tag{13}$$

Next, from Eq. (11) and Eq. (13), we obtain

$$\begin{aligned}
\chi_{24r-23} &= \frac{\chi_{24r-47}}{-1-\chi_{24r-27}\chi_{24r-31}\chi_{24r-35}\chi_{24r-39}\chi_{24r-43}\chi_{24r-47}} \\
&= \frac{\frac{\varepsilon_{23}}{(-1-\mu_{23})^{r-1}}}{-1-\varepsilon_3(-1-\mu_3)^{r-1}\frac{\varepsilon_7}{(-1-\mu_7)^{r-1}}\varepsilon_{11}(-1-\mu_{11})^{r-1}\frac{\varepsilon_{15}}{(-1-\mu_{15})^{r-1}}\varepsilon_{19}(-1-\mu_{19})^{r-1}\frac{\varepsilon_{23}}{(-1-\mu_{23})^{r-1}}} \\
&= \frac{\varepsilon_{23}}{(-1-\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})^{r-1}(-1-\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})} \\
&= \frac{\varepsilon_{23}}{(-1-\varepsilon_3\varepsilon_7\varepsilon_{11}\varepsilon_{15}\varepsilon_{19}\varepsilon_{23})^{r}}.
\end{aligned}$$

Moreover, utilizing Eq. (11) and Eq. (13) leads to

$$\begin{aligned}
\chi_{24r-22} &= \frac{\chi_{24r-46}}{-1-\chi_{24r-26}\chi_{24r-30}\chi_{24r-34}\chi_{24r-38}\chi_{24r-42}\chi_{24r-46}} \\
&= \frac{\frac{\varepsilon_{22}}{(-1-\mu_{22})^{r-1}}}{-1-\varepsilon_2(-1-\mu_2)^{r-1}\frac{\varepsilon_6}{(-1-\mu_6)^{r-1}}\varepsilon_{10}(-1-\mu_{10})^{r-1}\frac{\varepsilon_{14}}{(-1-\mu_{14})^{r-1}}\varepsilon_{18}(-1-\mu_{18})^{r-1}\frac{\varepsilon_{22}}{(-1-\mu_{22})^{r-1}}} \\
&= \frac{\varepsilon_{22}}{(-1-\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})^{r-1}(-1-\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})} \\
&= \frac{\varepsilon_{22}}{(-1-\varepsilon_2\varepsilon_6\varepsilon_{10}\varepsilon_{14}\varepsilon_{18}\varepsilon_{22})^{r}}.
\end{aligned}$$

We also use Eq. (11) and Eq. (13) to have

$$\begin{aligned}
\chi_{24r-21} &= \frac{\chi_{24r-45}}{-1-\chi_{24r-25}\chi_{24r-29}\chi_{24r-33}\chi_{24r-37}\chi_{24r-41}\chi_{24r-45}} \\
&= \frac{\frac{\varepsilon_{21}}{(-1-\mu_{21})^{r-1}}}{-1-\varepsilon_1(-1-\mu_1)^{r-1}\frac{\varepsilon_5}{(-1-\mu_5)^{r-1}}\varepsilon_9(-1-\mu_9)^{r-1}\frac{\varepsilon_{13}}{(-1-\mu_{13})^{r-1}}\varepsilon_{17}(-1-\mu_{17})^{r-1}\frac{\varepsilon_{21}}{(-1-\mu_{21})^{r-1}}} \\
&= \frac{\varepsilon_{21}}{(-1-\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})^{r-1}(-1-\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})} \\
&= \frac{\varepsilon_{21}}{(-1-\varepsilon_1\varepsilon_5\varepsilon_9\varepsilon_{13}\varepsilon_{17}\varepsilon_{21})^{r}}.
\end{aligned}$$

Finally, Eq. (11) and Eq. (13) give

$$\begin{aligned}
\chi_{24r-20} &= \frac{\chi_{24r-44}}{-1-\chi_{24r-24}\chi_{24r-28}\chi_{24r-32}\chi_{24r-36}\chi_{24r-40}\chi_{24r-44}} \\
&= \frac{\frac{\varepsilon_{20}}{(-1-\mu_{20})^{r-1}}}{-1-\varepsilon_0(-1-\mu_0)^{r-1}\frac{\varepsilon_4}{(-1-\mu_4)^{r-1}}\varepsilon_8(-1-\mu_8)^{r-1}\frac{\varepsilon_{12}}{(-1-\mu_{12})^{r-1}}\varepsilon_{16}(-1-\mu_{16})^{r-1}\frac{\varepsilon_{20}}{(-1-\mu_{20})^{r-1}}} \\
&= \frac{\varepsilon_{20}}{(-1-\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})^{r-1}(-1-\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})} \\
&= \frac{\varepsilon_{20}}{(-1-\varepsilon_0\varepsilon_4\varepsilon_8\varepsilon_{12}\varepsilon_{16}\varepsilon_{20})^{r}}.
\end{aligned}$$

Similarly, we can show other formulas.

Theorem 13. *Equation (11) has a unique equilibrium point $\overline{\chi}=0$, which is non-hyperbolic.*

Proof: It can be similarly done as the proof of Theorem 7.

Theorem 14. *Equation (11) is periodic of period 24 if and only if $\mu_\kappa=-2$, $\kappa=0,1,\ldots,23$, which have the form*

$$\chi_{24r-k}=\varepsilon_\kappa \qquad \kappa=0,1,\ldots,23 \;\; and \;\; r=0,1,2,\ldots$$

Proof: It can be easily done by using Theorem 12.

Theorem 15. *Let $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{23}\in(0,\infty)$. Then, the solution $\{\chi_r\}_{r=-23}^{\infty}$ oscillates about the equilibrium point $\overline{\chi}=0$, with positive semicycles of length 24, and negative semicycles of length 24.*

Proof: Using Theorem 12, we obtain that $\chi_1,\chi_2,\ldots,\chi_{24}<0$, and $\chi_{25},\chi_{26},\ldots,\chi_{48}>0$. Using induction achieves the proof.

6 Numerical investigation

This part is added to guarantee that the constructed results are correct. We present some 2D example under specific initial conditions.

Example 1. This example presents the behavior of Eq. (1) under the initial conditions $\chi_{-23}=0.1$, $\chi_{-22}=0.5$, $\chi_{-21}=5.3$, $\chi_{-20}=2$, $\chi_{-19}=0.54$, $\chi_{-18}=1.6$, $\chi_{-17}=0.07$, $\chi_{-16}=0.9$, $\chi_{-15}=0.10$, $\chi_{-14}=3$, $\chi_{-13}=5$, $\chi_{-12}=3$, $\chi_{-11}=4$, $\chi_{-10}=5$, $\chi_{-9}=6.8$, $\chi_{-8}=7.8$, $\chi_{-7}=2.8$, $\chi_{-6}=9$, $\chi_{-5}=2.9$, $\chi_{-4}=1.8$, $\chi_{-3}=6.8$, $\chi_{-2}=9.3$, $\chi_{-1}=1.9$ and $\chi_{0}=9.8$, as shown in Figure 1 (left).

Example 2. Figure 1 (right) illustrates the dynamical behavior of Eq. (4) when $\chi_{-23}=1$, $\chi_{-22}=5$, $\chi_{-21}=5.3$, $\chi_{-20}=2$, $\chi_{-19}=0.54$, $\chi_{-18}=6$, $\chi_{-17}=0.07$, $\chi_{-16}=9$, $\chi_{-15}=0.10$, $\chi_{-14}=0.3$, $\chi_{-13}=5$, $\chi_{-12}=0.3$, $\chi_{-11}=0.4$, $\chi_{-10}=5$, $\chi_{-9}=6.8$, $\chi_{-8}=7$, $\chi_{-7}=2$, $\chi_{-6}=9.5$, $\chi_{-5}=2.9$, $\chi_{-4}=1$, $\chi_{-3}=6.8$, $\chi_{-2}=9$, $\chi_{-1}=1$ and $\chi_{0}=9$.

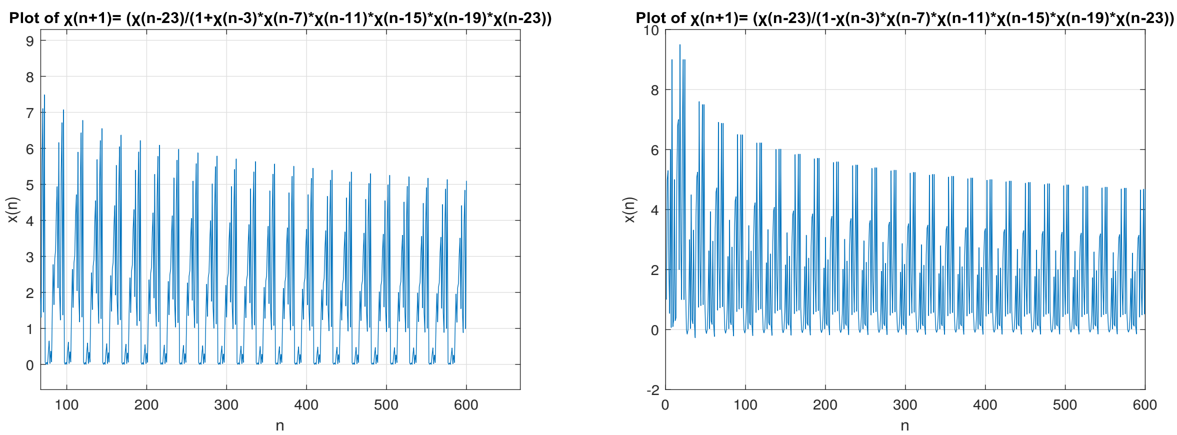

Fig. 1: The left plot presents the behavior of Eq. (1) while the right picture depicts the behavior of Eq. (4).

Example 3. The graph of the Eq. (8) is plotted in Figure 2 (left) under the random values $\chi_{-23}=0.1$, $\chi_{-22}=0.5$, $\chi_{-21}=0.15$, $\chi_{-20}=0.2$, $\chi_{-19}=0.3$, $\chi_{-18}=0.2$, $\chi_{-17}=0.9$, $\chi_{-16}=0.11$, $\chi_{-15}=0.6$, $\chi_{-14}=0.51$, $\chi_{-13}=0.14$, $\chi_{-12}=0.5$, $\chi_{-11}=0.2$, $\chi_{-10}=0.8$, $\chi_{-9}=0.5$, $\chi_{-8}=0.1$, $\chi_{-7}=0.3$, $\chi_{-6}=0.7$, $\chi_{-5}=0.1$, $\chi_{-4}=0.88$, $\chi_{-3}=0.12$, $\chi_{-2}=0.89$, $\chi_{-1}=0.33$ and $\chi_{0}=0.2$.

Example 4. The dynamical behavior of Eq. (11) is shown in Figure 2 (right) when $\chi_{-23}=0.15$, $\chi_{-22}=0.13$, $\chi_{-21}=0.55$, $\chi_{-20}=0.22$, $\chi_{-19}=0.33$, $\chi_{-18}=0.62$, $\chi_{-17}=0.29$, $\chi_{-16}=0.91$, $\chi_{-15}=0.56$, $\chi_{-14}=0.11$, $\chi_{-13}=0.44$, $\chi_{-12}=0.25$, $\chi_{-11}=0.02$, $\chi_{-10}=0.08$, $\chi_{-9}=0.85$, $\chi_{-8}=0.01$, $\chi_{-7}=0.03$, $\chi_{-6}=0.17$, $\chi_{-5}=0.01$, $\chi_{-4}=0.8$, $\chi_{-3}=0.02$, $\chi_{-2}=0.09$, $\chi_{-1}=0.03$ and $\chi_{0}=0.62$.

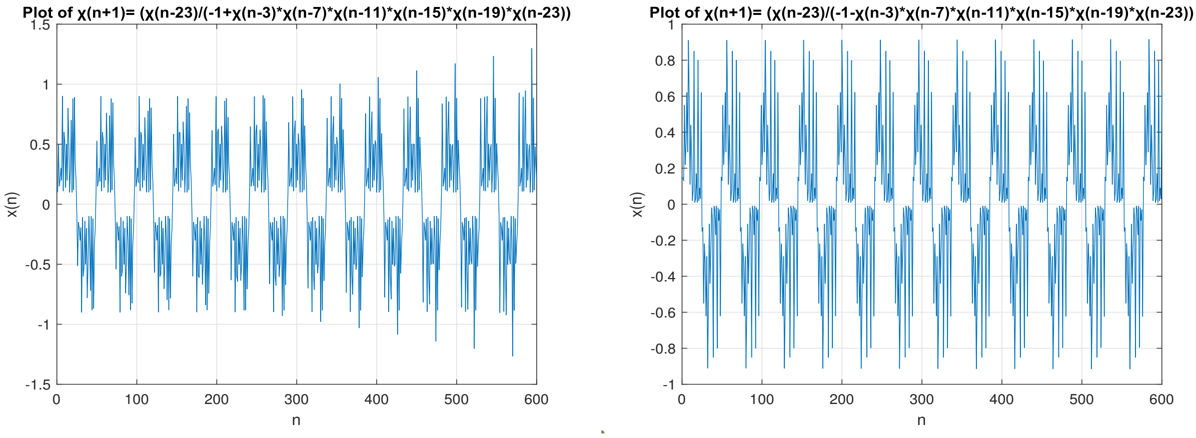

Fig. 2: The left plot illustrates the periodicity of Eq. (8) while the right picture depicts the periodicity of Eq. (11).

7 Conclusion

To sum up, this paper has investigated four main rational difference equations of twenty-fourth order. We have introduced the solutions of the considered equations using modulus operator. In Theorem 1, we have presented and proved the solutions of Eq. (1), while Theorem 2 has shown the boundedness of the solutions of Eq. (1). It has been proved that the fixed point of Eq. (1) is globally stable. Theorem 10 has presented that Eq. (8) is periodic of period 24 if and only if $\mu_\kappa = 2$. Furthermore, in Theorem 12, we have explored the solutions of Eq. (11) which are periodic of period 24 if and only if $\mu_\kappa = -2$. We have also plotted the periodicity of Eq. (8) and Eq. (11) in Figures 2 (left) and 2 (right), respectively. Finally, the used approaches can be simply applied for other nonlinear equations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] S. Elaydi, An Introduction to Difference Equations, 3rd Ed., Springer, USA, 2005.

[2] H. S. Alayachi, M. S. M. Noorani, A. Q. Khan and M. B. Almatrafi, Analytic Solutions and Stability of Sixth Order Difference Equations, Mathematical Problems in Engineering, Volume 2020, Article ID 1230979, 12 page.

[3] A. Sanbo and Elsayed M. Elsayed, Some properties of the solutions of the difference equation $x_{n+1} = ax_n + (bx_nx_{n-4})/(cx_{n-3} + dx_{n-4})$, Open Journal of Discrete Applied Mathematics, 2(2) (2019), 31–47.

[4] M. B. Almatrafi, M. M. Alzubaidi, Analysis of the Qualitative Behaviour of an Eighth-Order Fractional Difference Equation, Open Journal of Discrete Applied Mathematics, 2 (1) (2019), 41-47.

[5] A. M. Ahmed, Samir Al Mohammady, and Lama Sh. Aljouf, Expressions and dynamical behavior of solutions of a class of rational difference equations of fifteenth-order, J. Math. Computer Sci., 25 (2022), 10-22.

[6] A. Alshareef, F. Alzahrani and A. Q. Khan, Dynamics and Solutions' Expressions of a Higher-Order Nonlinear Fractional Recursive Sequence, Mathematical Problems in Engineering, Volume 2021, Article ID 1902473, 12 pages.

[7] S. Kalabusic, M. Nurkanovic, and Z. Nurkanovic, Global Dynamics of Certain Mix Monotone Difference Equation, Mathematics, 6(1), 10 (2018), doi:10.3390/math6010010

[8] Merve Kara, and Yasin Yazlik, Solvability of a (k + l)-order nonlinear difference equation, Tbilisi Mathematical Journal, 14(2) (2021), pp. 271–297.

[9] E. M. Elsayed, Dynamics and behavior of a higher order rational difference equation, J. Nonlinear Sci. Appl., 9 (2016), 1463–1474.

[10] Mohammed B. Almatrafi and Marwa M. Alzubaidi, Qualitative analysis for two fractional difference equations, Nonlinear Engineering, 2020, 9, 265–272.

[11] Lama Sh. Aljoufi, A. M. Ahmed, and Samir Al Mohammady, Global behavior of a third-order rational difference equation, Journal of Mathematics and Computer Science, 25(3) (2022), 296-302.

[12] M. B. Almatrafi, Solutions Structures for Some Systems of Fractional Difference Equations, Open Journal of Mathematical Analysis, 3(1) (2019), 51-61.

[13] M. B. Almatrafi, E.M. Elsayed and Faris Alzahrani, Qualitative Behavior of Two Rational Difference Equations, Fundamental Journal of Mathematics and Applications, 1 (2) (2018), 194-204.

[14] M. B. Almatrafi, E.M. Elsayed, Solutions And Formulae For Some Systems Of Difference Equations, MathLAB Journal , 1 (3) (2018), 356-369.

[15] M. B. Almatrafi, E.M. Elsayed and Faris Alzahrani, Qualitative Behavior of a Quadratic Second-Order Rational Difference Equation, International Journal of Advances in Mathematics, 2019 (1) (2019), 1-14.

[16] T. Khyat and M. R. S. Kulenović, The Invariant Curve Caused by Neimark-Sacker Bifurcation of a Perturbed Beverton-Holt Difference Equation, International Journal of Difference Equations, 12 (2) (2017), 267-280.

[17] Y. Kostrov and Z. Kudlak, On a Second-Order Rational Difference Equation with a Quadratic Term, International Journal of Difference Equations, 11 (2) (2016), 179-202.

[18] K. Liu, P. Li, F. Han, and W. Zhong, Global Dynamics of Nonlinear Difference Equation $x_{n+1} = A + x_n/x_{n-1}x_{n-2}$, Journal of Computational Analysis and Applications, 24 (6) (2018), 1125-1132.

[19] E. M. Elsayed, Dynamics of Recursive Sequence of Order Two, Kyungpook Math. J., 50 (2010), 483-497.

[20] A. Khaliq and Sk.S. Hassan, Dynamics of a Rational Difference Equation $x_{n+1} = ax_n + (\alpha + \beta x_{n-k})/(A + Bx_{n-k})$, International Journal of Advances in Mathematics, 2018 (1) (2018), 159-179.