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Finite-time behaviour of solutions to nonlinear parabolic equation

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Abstract: In this paper we consider nonlinear parabolic equation with power type nonlinearity which such types of problems occur in many mathematical models of applied science. We show that there are solutions under some conditions on initial data which blow up in finite time with positive initial energy.

Keywords: Finite-time behaviour, nonlinear parabolic equation, generalized concavity.

1 Introduction

Many physical problems can be modelled mathematically in the form of evolution equations. We can not obtain a well-defined solution for these equations without adding suitable additional conditions (initial and boundary conditions).

In the last quarter century, partial differential equations became one of the most active areas of mathematics research since it helped us to find answers to many phenomena of the nonlinear world.

One of the most remarkable type of these singularities is what we call the Blow-up phenomena. Blow-up simply is a form of the spontaneous singularities appear when one or more of the depending variables go to infinity as time goes to a certain finite time.

We consider the following second order nonlinear parabolic equation :

$$u_t - \Delta u + k\left(\int_{\Omega} u^2 dx\right) u - g\left(u, \nabla \mathrm{u}\right) = |u|^p u \tag{1}$$

$$u(x,t) = 0,\ x \in \partial\Omega, t > 0 \tag{2}$$

$$u(x,0) = u_0(x),\ \ x \in \Omega \tag{3}$$

where $\Omega \subseteq \mathbb{R}^n,\ n \geq 1$ is bounded domain with a sufficiently smooth boundary $\partial\Omega$. Also the constant k is positive number and $p \geq 2$. Assume that $u_0(x)$ is given function satisfying

$$u_0 \in H_0^1(\Omega) \cap L_{p+2}(\Omega) \tag{4}$$

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and $g(u,\nabla u)$ is continuous function which have the relation

$$|g(u,\nabla u)| \le c_1 (|u| + |\nabla u|) \tag{5}$$

with some positive $c_1 > 0$.

Existence of solutions to these type of equations are studied in [1] and [8]. Erdem, in [2], studied blow-up solutions to quasilinear parabolic equations

$$u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left(d + |\nabla u|^{m-2}\right), \frac{\partial u}{\partial x_i}\right) + g(u,\nabla u) = f(u)$$

where d is positive constant and f and g are continuous functions which satisfy the following conditions;

$$(u,(f(u)) \ge 2(1+\alpha) G(u) \text{ , } \alpha > 0$$

$$G(u) = \int_0^u f(s)\, ds, \; |g(u,v)| \le c_1 (|u| + |v|), \; c_1 > 0$$

Zhou, in [4], considered the following quasilinear parabolic equation

$$a(x,t) u_t - \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right) = f(u)$$

where $a(x,t) u \ge 0$ is a generalized Lewis function. He obtain blow-up result in finite time if the initial data possesses suitable positive energy.

Shahrouzi, [5], investigated a fourth order nonlinear wave equation with dissipative boundary condition. He showed that there was solutions under some conditions on initial data which blowed up in finite time with positive initial energy.

Kocaman,Yaman, [6], studied the fourth order quasilinear parabolic equation
$u_t - \Delta[(k_0 + k_1 |\Delta u|^{m-2} \Delta u)] - g(x,t,u,\Delta u) = |u|^{p-2} u$
They show that there are solutions in which blow up in finite time with positive initial energy.
Meyvaci [7], investigated the problem
$u_t - \Delta u_t - \Delta u - u^m u_{x_1} + g(t,x,u,\nabla u) = |u|^{m_1} u, x \in \Omega, t > 0$
where m , m_1 are positive numbers and $g(t,x,u,\nabla u)$ is a bounded function. She established sufficient conditions on initial data, m , m_1 and the function $g(t,x,u,\nabla u)$ for the nonblow-up case. Moreover, she obtained the lower and upper bounds for the blow-up time if blow-up happens.
In this work, we consider finite time blow up result for solutions to nonlinear parabolic equation (1)-(3) with positive initial energy.
In this paper, we use the following notations;
$\|u\| = \|u\|_{L_2(\Omega)}$, $\|u\|_{\text{p}} = \|u\|_{L_p(\Omega)}$

are usual lebesgue spaces,

$$(u,v) = \int_{\Omega} uv dx$$

is the inner product.

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \tag{6}$$

is the weighted arithmetic-geometric inequality for $a,b > 0$.

$$\lambda_1 \|u\|^2 \le \|\nabla u\|^2 \tag{7}$$

is the Friedrichs inequality with constant λ_1 which is the first eigenvalue of eigenvalue problem
$-\Delta\varphi = \lambda\varphi,\ \varphi|_{\partial\Omega} = 0$
Let us note the following lemma known as generalized concavity lemma or "Ladyzhenskaya-Kalantarov lemma". It is good tool to obtain the blow up results for dynamical problems.

Lemma 1. *Suppose that a positive, twice differentiable function $\Phi(t)$ satisfies for $t > 0$ the following inequality*
$\Phi(t)\Phi''(t) - (1+\delta)(\Phi'(t))^2 \ge -2M_1\Phi(t)\Phi'(t) - M_2(\Phi'(t))^2$
where $\delta > 0,\ M_1,\ M_2 \ge 0$. If
$\Phi(0) > 0,\ \Phi'(0) > -\gamma_2\delta^{-1}\Phi(0)$
and $M_1 + M_2 > 0$, then $\Phi(t)$ tends to infinity as $t \to t_1 \le t_2$
$t_2 \le \frac{1}{2\sqrt{M_1{}^2+\delta M_2}} \ln \frac{\gamma_1\Phi(0)+\delta\Phi'(0)}{\gamma_2\Phi(0)+\delta\Phi'(0)}$
where $\gamma_1 = -M_1 + \sqrt{M_1^2 + \gamma M_2}$, $\gamma_2 = -M_1 + \sqrt{M_1^2 + \gamma M_2}$.

Proof. See [3]

2 Blow-up result

Theorem 1. *Suppose that the condition (4)-(5) is satisfied. Let $u(x,t)$ be the solution of the problem (1)-(3). Assume the following conditions are valid:*

$$\delta = \sqrt{1+\beta} - 1,\ \ \beta\varepsilon(0,1), \lambda \ge \frac{c_1^2}{2\varepsilon_0 p}\left(1+\lambda_1^{-1}\right), \varepsilon_0 = \frac{p-2\beta}{2(2+p)} \tag{8}$$

$$F_u(0) = -\frac{1}{4}\left(2\lambda + k\|u_0\|^2\right)\|u_0\|^2 - \frac{1}{2}\|\nabla u_0\|^2 + \frac{1}{p+2}\|u_0\|_{p+2}^{p+2} > 0 \tag{9}$$

Then there exists a finite time t_1 such that
$\|u\|^2 \to +\infty$ *as* $t \to t_1^-$.

Proof. For $\lambda > 0$, we make the transformation $u(x,t) = e^{\lambda t}v(x,t)$ in (1) and we obtain the equation

$$v_t + \lambda v - \Delta v + ke^{2\lambda t}\left(\int\limits_\Omega v^2 dx\right)v - \tilde{g}(v,\nabla v) = e^{\lambda pt}|v|^p v \tag{10}$$

with the boundary condition and the initial condition

$$v(x,t) = 0,\ \ x \in \partial\Omega, t > 0, v(x,0) = u_0,\ \ x \in \Omega \tag{11}$$

respectively where $\tilde{g}(v,\nabla v) = e^{-\lambda t}g\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right)$.

Multiply the equation (10) by v_t in $L_2(\Omega)$ to get the relation

$$\|v_t\|^2 + \frac{\lambda}{2}\frac{d}{dt}\|v\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \frac{k}{2}e^{2\lambda t}\left(\int\limits_\Omega v^2 dx\right)\frac{d}{dt}\left(\int\limits_\Omega v^2 dx\right) = (\tilde{g}(v,\nabla v), v_t) + e^{\lambda pt}(|v|^p v, v_t) \tag{12}$$

Using the inequalities (6) and (7) to the first term on the right side of (12) with condition (5), we have

$$|\tilde{g}(v,\nabla v)| = e^{-\lambda t}\left|g\left(e^{\lambda t}v, e^{\lambda t}\nabla v\right)\right| \le e^{-\lambda t}c_1\left(\left|e^{\lambda t}v\right| + \left|e^{\lambda t}\nabla v\right|\right) = c_1\left(|v| + |\nabla v|\right)$$

and

$$\begin{aligned} |(\tilde{g}(v,\nabla v), v_t)| &\le c_1 \int_\Omega |v|\,|v_t|\,dx + c_1 \int_\Omega |\nabla v|\,|v_t|\,dx \\ &\le \frac{c_1^2}{4\varepsilon_0}\|v\|^2 + \varepsilon_0\|v_t\|^2 + \frac{c_1^2}{4\varepsilon_0}\|\nabla v\|^2 + \varepsilon_0\|v_t\|^2 \le \frac{c_1^2}{4\varepsilon_0}\left(\lambda_1^{-1}+1\right)\|\nabla v\|^2 + 2\varepsilon_0\|v_t\|^2 \end{aligned} \tag{13}$$

Substituting the equation (13) in the equation (12) we get the relation

$$-\frac{d}{dt}F_v(t) \le -\|v_t\|^2 + \frac{\lambda k}{2}e^{2\lambda t}\left(\int_\Omega v^2 dx\right)^2 - \frac{\lambda p}{p+2}e^{\lambda pt}\|v\|_{p+2}^{p+2} + \frac{c_1^2}{4\varepsilon_0}\left(\lambda_1^{-1}+1\right)\|\nabla v\|^2 + 2\varepsilon_0\|v_t\|^2 \tag{14}$$

where

$$F_v(t) = -\tfrac{\lambda}{2}\|v\|^2 - \tfrac{1}{2}\|\nabla v\|^2 - \tfrac{k}{4}e^{2\lambda t}\left(\int_\Omega v^2 dx\right)^2 + \tfrac{1}{p+2}e^{\lambda pt}\left(\int_\Omega v^{p+2}dx\right).$$

The differential inequality (14) follows

$$\frac{d}{dt}F_v(t) \ge (1-2\varepsilon_0)\|v_t\|^2 - \frac{c_1^2}{4\varepsilon_0}\left(1+\lambda_1^{-1}\right)\|\nabla v\|^2 - \frac{\lambda k}{2}e^{2\lambda t}\left(\int_\Omega v^2 dx\right)^2 + \frac{\lambda p}{p+2}e^{\lambda pt}\|v\|_{p+2}^{p+2} \tag{15}$$

Regarding the function $F_v(t)$, (15) results in

$$\frac{d}{dt}F_v(t) \ge \lambda pF_v(t) + \left(\frac{\lambda p}{2} - \frac{c_1^2}{4\varepsilon_0}\left(1+\lambda_1^{-1}\right)\right)\|\nabla v\|^2 + (1-2\varepsilon_0)\|v_t\|^2 + \frac{\lambda k}{4}(p-2)e^{2\lambda t}\left(\int_\Omega v^2 dx\right)^2 \tag{16}$$

Using the conditions of theorem; $\lambda \ge \frac{c_1^2}{2\varepsilon_0 p}\left(1+\lambda_1^{-1}\right)$ and $p \ge 2$ then it follows from (16)

$$\frac{d}{dt}F_v(t) \ge (1-2\varepsilon_0)\|v_t\|^2 + \lambda pF_v(t) \tag{17}$$

Choose $\varepsilon_0 = \frac{p-2\beta}{2(2+p)}$, $\beta \varepsilon (0,1)$ to obtain the following inequality

$$\frac{d}{dt}F_v(t) \ge \frac{2(1+\beta)}{p+2}\|v_t\|^2 + \lambda pF_v(t) \tag{18}$$

Solving the differential inequality (18) we have

$$F_v(t) \ge F_v(0)e^{\lambda pt} + \frac{2(1+\beta)}{p+2}\int_0^t \|v_s\|^2 ds \tag{19}$$

It follows that $F_v(t) \ge e^{\lambda pt}F_v(0) \ge F_u(0)$ by assumption (9). So we obtain a lower bound for $F_v(t)$

$$F_v(t) \ge F_u(0) + \frac{2(1+\beta)}{p+2}\int_0^t \|v_s\|^2 ds \tag{20}$$

Multiplying the equation (10) by v in $L_2(\Omega)$ to obtain the relation

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+\lambda\|v\|^2+\|\nabla v\|^2+ke^{2\lambda t}\left(\int\limits_{\Omega}v^2dx\right)^2=(\tilde{g}(v,\nabla v),v)+e^{\lambda pt}\|v\|_{p+2}^{p+2} \tag{21}$$

Using the inequalities (6) to the first term on the right side of (21) under condition (5), we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2\geq-\lambda\|v\|^2-\|\nabla v\|^2-ke^{2\lambda t}\left(\int\limits_{\Omega}v^2dx\right)^2+e^{\lambda pt}\|v\|_{p+2}^{p+2}-c_1\|v\|^2-\frac{\varepsilon_1}{2}\|\nabla v\|^2-\frac{c_1^2}{2\varepsilon_1}\|v\|^2 \tag{22}$$

The inequality (22) follows

$$\frac{d}{dt}\|v\|^2\geq2(p+2)F_v(t)+(\lambda p-2c_2)\|v\|^2+(p-\varepsilon_1)\|\nabla v\|^2+k\left(\frac{p-2}{2}\right)e^{2\lambda t}\left(\int\limits_{\Omega}v^2dx\right)^2 \tag{23}$$

where $c_2=c_1\left(1+\frac{c_1}{2\varepsilon_1}\right)$, $\varepsilon_1\in(0,1)$.

Omit some positive terms on the right side of the inequality (23) to get the following inequality

$$\frac{d}{dt}\|v\|^2\geq2(p+2)F_v(t)-2c_2\|v\|^2 \tag{24}$$

Let us substite the estimate (20) in (24) we have

$$\frac{d}{dt}\|v\|^2\geq4(1+\beta)\int\limits_0^t\|v_s\|^2ds+2(p+2)F_u(0)-2c_2\|v\|^2 \tag{25}$$

Now introduce the function

$$\Phi(t)=\int\limits_0^t\|v\|^2d\tau+C \tag{26}$$

where C is a positive constant will be chosen later. First and second derivatives of (26) as follows

$$\begin{aligned}\Phi'(t)&=\|v\|^2=\|u_0\|^2+2\int\limits_0^t(v,v_s)ds,\\ \Phi''(t)&=\frac{d}{dt}\|v\|^2\end{aligned} \tag{27}$$

Now let us apply the Cauchy-Schwarz inequality and the weighted arithmetic-geometric inequality to get an upper bound for $\Phi'(t)$;

$$\begin{aligned}
&\Phi'(t) \leq \|u_0\|^2 + 2\sqrt{\left(\int_0^t \|v\|^2 ds\right)(\int_0^t \|v_s\|^2 ds)}, \\
&[\Phi'(t)]^2 \leq \left[\|u_0\|^2 + 2\sqrt{\left(\int_0^t \|v\|^2 ds\right)(\int_0^t \|v_s\|^2 ds}\right]^2 \\
&\leq \left[4(1+\varepsilon_2)\left(\int_0^t \|v\|^2 ds\right)\left(\int_0^t \|v_s\|^2 ds\right) + \left(1+\frac{1}{\varepsilon_2}\right)\|u_0\|^4\right]
\end{aligned} \tag{28}$$

Using the relations (26)-(28) we can estimate the term $\Phi(t)\Phi''(t) - (1+\delta)(\Phi'(t))^2$;

$$\begin{aligned}
&\Phi(t)\Phi''(t) - (1+\delta)(\Phi'(t))^2 \geq 4(1+\beta)\left(\int_0^t \|v_s\|^2 ds\right)\left(\int_0^t \|v\|^2 ds\right) \\
&-4(1+\delta)(1+\varepsilon_2)\left(\int_0^t \|v_s\|^2 ds\right)\left(\int_0^t \|v\|^2 ds\right) \\
&-2c_2\Phi(t)\Phi'(t) + 2(p+2)F_u(0)C_0 - (1+\delta)\left(1+\frac{1}{\varepsilon_2}\right)\|u_0\|^4
\end{aligned} \tag{29}$$

The inequality (29) results in
$\Phi(t)\Phi''(t) - (1+\delta)(\Phi'(t))^2 \geq -2c_2\Phi(t)\Phi'(t)$ where $\varepsilon_2 = \delta$, $C = \frac{(1+\delta)^2\|u_0\|^4}{2\delta(P+2)F_u(0)}$, $(1+\delta)^2 = 1+\beta$.

3 Conclusion

The lemma can be applied if

$$C = \frac{(1+\delta)^2\|u_0\|^4}{2\delta(p+2)F_u(0)} \tag{30}$$

So we have $\Phi(t)\Phi''(t) - (1+\delta)(\Phi'(t))^2 \geq -2c_2\Phi(t)\Phi'(t)$, with $M_1 = c_2$, $M_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = -2c_2$. The conditions of lemma $\Phi(0) > 0$, $\Phi'(0) > -\gamma_2\delta^{-1}\Phi(0)$ are satisfied by the positive constant (30) and assumptions (8) and (9). Thus solutions to the problem for nonlinear equation (1)-(3) blow up as

$t \to t_1 \leq \frac{1}{2c_2} ln\left(\frac{(p+2)\delta^2 F_u(0)}{(p+2)\delta^2 F_u(0) - c_2(1+\delta)^2\|u_0\|^2}\right)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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