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On the convergence of finite difference scheme for a Schrödinger type equation

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Abstract: In the present paper, an initial boundary value problem for the linear Schrödinger equation including the momentum operator is introduced. This problem is discretized by the finite difference method and a difference scheme is presented. Moreover, an estimate for the solution of the proposed scheme is obtained. Finally, with the help of the estimate, it is proved that the proposed scheme is unconditionally stable and convergent.

Keywords: Schrödinger type equation, Finite difference method, Stability, Convergence.

1 Introduction

The general form of a Schrödinger type equation (StE) is as follows:

$$\varepsilon u_t + F_2(\varsigma,t;u)u_{\varsigma\varsigma} + F_1(\varsigma,t;u)u_{\varsigma} + F_0(\varsigma,t;u)u = 0, \tag{1}$$

where $\varepsilon = const.$, $u(\varsigma,t)$ is the wave's complex amplitude; $u_t = \frac{\partial u}{\partial t}$, $u_\varsigma = \frac{\partial u}{\partial \varsigma}$, $u_{\varsigma\varsigma} = \frac{\partial^2 u}{\partial \varsigma^2}$. Equation (1) describes the slow variation of the function $u(\varsigma,t)$ in a medium with quadratic dispersion [1]. The diversified versions of equation (1) and its applications have been studied widely in many fields such as hydrodynamics, water waves, optical fiber setting, photonics, nonlinear transmission lines, Bose-Einstein condensates, plasma physics [2]. In equation (1), the variables ς and t have different meanings according to the context of its application areas. Here, ς and t denote the space and time variables, respectively.

In the present paper, we study an initial boundary value problem (IBVP) for a particular case of equation (1), which is a linear Schrödinger equation including a momentum operator, in the form

$$i\frac{\partial u}{\partial t} + p_0\frac{\partial^2 u}{\partial \varsigma^2} + ip_1\frac{\partial u}{\partial \varsigma} - p(\varsigma)u + q(t)u = \rho(\varsigma,t),\ (\varsigma,t)\in\Omega, \tag{2}$$

$$u(\varsigma,0) = \eta(\varsigma),\ \varsigma\in I, \tag{3}$$

$$u(0,t) = u(l,t) = 0,\ t\in Q, \tag{4}$$

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where $i=\sqrt{-1}$, $I=(0,l)$, $Q=(0,T)$, $\Omega=I\times Q$, $p_0,p_1>0$ are real numbers; $p(\varsigma)$ and $q(t)$ are real valued functions such that

$$0<p(\varsigma)\le\mu_0 \text{ almost everywhere } (a.e.) \text{ in } I,\ \mu_0=const.>0\ , \tag{5}$$

$$q\in L_2(Q),\ |q(t)|\le b_0,\ \left|\frac{dq(t)}{dt}\right|\le b_1 \ a.e. \text{ in } Q, \tag{6}$$

$b_0,b_1>0$ are given numbers; $\eta\in H_0^2(I)$, $\rho\in H^{0,1}(\Omega)$. Here, almost everywhere means that a property is said to hold almost everywhere in any set E if it holds in E except in some subset of E with measure zero. Also, $H_0^2(I)$, $H^{0,1}(\Omega)$ are Sobolev spaces defined as in [3] and $L_2(Q)$ is a Hilbert space with the inner product $\langle u,v\rangle=\int\limits_Q u(t)\bar{v}(t)dt$ for any $u,v\in L_2(Q)$.

In this work, we examine the solution of problem (2)-(4) with the help of the finite difference method. For this, firstly, we constitute a difference scheme for IBVP (2)-(4). Later, we obtain an estimate for the solution of difference scheme. Finally, by using the estimate obtained we show that the scheme is unconditionally stable and is convergent. According to characteristics of the coefficients $F\alpha$, $\alpha=0,1,2$ in (1), we obtain the varied forms of linear and nonlinear Schrödinger equations from equation (1). The solutions by finite difference method of IBVPs for linear Schrödinger equations obtained from equation (1) in case of $F_0(\varsigma,t;u)=F_0(\varsigma,t)$ are previously analyzed in the works [4,5,6,7]. In these papers, the coefficient F_1 is usually zero. But, in [6], there is a nonzero real valued function. Also, when $F_2(\varsigma,t;u)=const.$, $F_1(\varsigma,t;u)=0$, $F_0(\varsigma,t;u)=F_0(\varsigma,t;u)$ in (1), we obtain the nonlinear Schrödinger equations such that the solutions of such equations by the finite difference method are studied in [7,8,9,10,11,12,13,14]. The stability, error and convergence of the method have been demonstrated in most of these papers.

As different from previous studies, in this paper, we work out an IBVP in the form (2)-(4) for linear Schrödinger equation including a momentum operator with coefficients $\varepsilon=i$, $F_2(\varsigma,t;u)=p_0$, $F_1(\varsigma,t;u)=ip_1$, $F_0(\varsigma,t;u)=-p(\varsigma)+q(t)$, which is more comprehensive and current than the problems studied before.

Based on results in [15], we write the next theorem for problem (2)-(4). It is easily proved by the Galerkin's method.

Theorem 1. *Assume that (5) and (6) are satisfied and $\eta\in H_0^2(I)$, $\rho\in H^{0,1}(\Omega)$. Then, there exists a unique solution $u\in B_0\equiv C^0(Q_T,H_0^2(I))\cap C^1(Q_T,L_2(I))$ of problem (2)-(4) for any $t\in Q_T=[0,T]$ and the following estimate holds*

$$||u(.,t)||^2_{H_0^2(I)}+\left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(I)}\le c_0\left(||\eta||^2_{H_0^2(I)}+||\rho||^2_{H^{0,1}(\Omega)}\right), \tag{7}$$

where $c_0>0$ is a constant independent of η,ρ,t.

2 Notations, some useful lemmas and difference scheme

In this section, we present some notations and give some lemmas and theorems used in the paper. Let I be discretized using by grid points $\varsigma_j=jh-\frac{h}{2}$, $j=1,2,...,A-1$, $\varsigma_1-\frac{h}{2}=0$, $\varsigma_{A-1}+\frac{h}{2}=l$, $h=\frac{l}{A-1}$ and let Q_T be divided by $t_k=k\tau$, $k=0,1,...,B$ with $\tau=\frac{T}{B}$, where A,B are any positive integers. Let u_{jk}, $j=0,1,...,A$, $k=0,1,...,B$ be the numerical approximation of $u(\varsigma,t)$ at the point (ς_j,t_k). Also, we define the finite difference operators

$$D_t^- u_{jk} = \frac{u_{jk} - u_{jk-1}}{\tau},\ D_\varsigma^- u_{jk} = \frac{u_{jk} - u_{j-1k}}{h},$$
$$D_\varsigma^+ u_{jk} = \frac{u_{j+1k} - u_{jk}}{h},\ \ D_\varsigma^2 u_{jk} = \frac{D_\varsigma^+ u_{jk} - D_\varsigma^- u_{jk}}{h} = \frac{u_{j+1k} - 2u_{jk} + u_{j-1k}}{h^2},$$

and some discrete inner product and norms as

$$(v,w) = h\sum_{j=1}^{A-1} v_j \overline{w}_j,\ \|v\|_2 = \sqrt{h\sum_{j=1}^{A-1}\left|v_j\right|^2},\ \|v\|_\infty = \max_{1\le j\le A-1}\left|v_j\right|,\left\|D_\varsigma^+ v\right\|_2 = \sqrt{h\sum_{j=1}^{A-1}\left|D_\varsigma^+ v_j\right|^2}$$

for any grid functions v,w, where the symbols $\|.\|_2$ and $\|.\|_\infty$ indicate the discrete norms on spaces $L_2(I)$ and $L_\infty(I)$, respectively and $\overline{w}_j$ means the complex-conjugate of w_j. Also, throughout this paper, we denote the positive constants independent from τ, h and m by c_r, $r = 1,2,...,12$.

With these designations, we write the finite difference scheme of problem (2)-(4) for $j = 1,2,...,A-1, k = 1,2,...,B$ as

$$iD_t^- u_{jk} + p_0 D_\varsigma^2 u_{jk} + ip_1 D_\varsigma^- u_{jk} - p_j u_{jk} + q_k u_{jk} = \rho_{jk}, \tag{8}$$
$$u_{j0} = \eta_j,\ \ j = 0,1,...,A, \tag{9}$$
$$u_{0k} = u_{Ak} = 0,\ \ k = 1,2,...B, \tag{10}$$

where the functions p_j, q_k, ρ_{jk} and η_j for $j = 1,2,...,A-1, k = 1,2,...,B$ are Steklov averages of the functions $p(\varsigma)$, $q(t)$, $\rho(\varsigma,t)$ and $\eta(\varsigma)$ respectively, defined by

$$p_j = \frac{1}{h}\int_{\varsigma_j - h/2}^{\varsigma_j + h/2} p(\varsigma)d\varsigma,$$

$$q_k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} q(t)dt,$$

$$\rho_{jk} = \frac{1}{\tau h}\int_{t_{k-1}}^{t_k}\int_{\varsigma_j - h/2}^{\varsigma_j + h/2} \rho(\varsigma,t)d\varsigma dt,$$

$$\eta_j = \frac{1}{h}\int_{\varsigma_j - h/2}^{\varsigma_j + h/2} \eta(\varsigma)d\varsigma,\ \eta_0 = \eta_A = 0$$

[16]. Also, from conditions (5) and (6), the inequalities

$$0 \le p_j \le \mu_0,\ j = 1,2,...,A-1, \tag{11}$$
$$|q_k| \le b_0,\ \ k = 1,2,...B,\ \left|D_t^- q_k\right| \le b_1,\ \ k = 2,...B, \tag{12}$$

are written.

We need the following lemmas and theorem.

Lemma 1.*(Discrete Gronwall's Inequality [17]): Assume that the nonnegative grid functions* $\{v(s), y(s),\ s = 1,2,...,S,\ S\tau = T\}$ *satisfy the inequality*

$$v(s) \leq y(s) + \tau \sum_{r=1}^{s} B_r v(r),$$

where $B_r\ (r = 1,2,...,S)$ *are nonnegative constant. Then, for any* $0 \leq s \leq S$*, there is*

$$v(s) \leq y(s) \exp\left(s\tau \sum_{r=1}^{s} B_r\right).$$

Lemma 2. *(Summation by Parts Formula): For any two grid functions*

$v, w \in \{v : v = (v_0, v_1, v_2, ..., v_{A-1}, v_A),\ v_0 = v_A = 0\}$, we have

$$h \sum_{j=1}^{A-1} \left(D_{\varsigma}^2 v_j\right) \overline{w}_j = -h \sum_{j=1}^{A} \left(D_{\varsigma}^- v_j\right)\left(D_{\varsigma}^- \overline{w}_j\right).$$

Lemma 3. *($\in$ −Cauchy′s inequality [18]): For any* $\in > 0$ *and arbitrary a and b, the inequality*

$$ab \leq \frac{\in}{2} a^2 + \frac{1}{2\in} b^2$$

is valid.

Theorem 2. *(Fubini's Theorem [19]): Let the function* $f(\varsigma, y)$ *be integrable over* $\Theta_{p+q} = \Theta_p \times \Theta_q$*. Then,* $f(\varsigma, y)$ *is integrable with respect to* $y \in \Theta_p$ *for almost all* $\varsigma \in \Theta_q$*, is integrable with respect to* $\varsigma \in \Theta_q$ *for almost all* $y \in \Theta_p$*, the functions* $\int_{\Theta_p} f(\varsigma, y) dy$ *and* $\int_{\Theta_q} f(\varsigma, y) d\varsigma$ *are integrable with respect to* $\varsigma \in \Theta_q$ *and* $y \in \Theta_p$*, respectively, and*

$$\int_{\Theta_{p+q}} f d\varsigma dy = \int_{\Theta_q} d\varsigma \int_{\Theta_p} f dy = \int_{\Theta_p} dy \int_{\Theta_q} f d\varsigma,$$

where Θ_p *is a* $p-$*dimensional bounded region in variables* $y = (y_1, y_2, ...y_p), \Theta_q$ *is a* $q-$*dimensional bounded region in variables* $\varsigma = (\varsigma_1, \varsigma_2, ...\varsigma_q)$.

3 The stability of scheme (8)-(10)

We firstly get an estimate for the solution of difference scheme (8)-(10). By this estimate, we provide the proof of unconditional stability of the difference scheme.

Theorem 3. *Assume that (5) and (6) are satisfied and* $\eta \in H_0^2(I), \rho \in H^{0,1}(\Omega)$*. Then, the solution* u_{jm} *of difference scheme (8)-(10) for any* $m \in \{1,2,...,B\}$ *satisfies the estimate*

$$h \sum_{j=1}^{A-1} |u_{jm}|^2 + 2h \sum_{k=1}^{m} \sum_{j=1}^{A-1} |u_{jk} - u_{jk-1}|^2 + 2p_1\tau \sum_{k=1}^{m} |u_{A-1k}|^2$$
$$+2p_1\tau \sum_{k=1}^{m} \sum_{j=1}^{A-1} |u_{jk} - u_{j-1k}|^2 \leq c_1 \left(h \sum_{j=1}^{A-1} |\eta_j|^2 + \tau h \sum_{k=1}^{B} \sum_{j=1}^{A-1} |\rho_{jk}|^2\right). \quad (13)$$

Proof. It is clear that for any scheme function ξ_{jk} such that $\xi_{0k}=\xi_{Ak}=0$ for $k=1,2,...B$, scheme (8)-(10) is equivalent to the summation identity

$$ih\sum_{j=1}^{A-1}D_t^- u_{jk}\overline{\xi}_{jk}+p_0h\sum_{j=1}^{A-1}D_\varsigma^2 u_{jk}\overline{\xi}_{jk}+ip_1h\sum_{j=1}^{A-1}D_\varsigma^- u_{jk}\overline{\xi}_{jk}-h\sum_{j=1}^{A-1}p_j u_{jk}\overline{\xi}_{jk}+h\sum_{j=1}^{A-1}q_k u_{jk}\overline{\xi}_{jk}=h\sum_{j=1}^{A-1}\rho_{jk}\overline{\xi}_{jk}, \tag{14}$$

where $\overline{\xi}_{jk}$ is the complex conjugate of ξ_{jk}. Substituting $\tau\overline{u}_{jk}$ for $\overline{\xi}_{jk}$ in (14) and applying the formula of summation by parts, we get

$$ih\tau\sum_{j=1}^{A-1}D_t^- u_{jk}\overline{u}_{jk}-p_0h\tau\sum_{j=1}^{A-1}\left|D_\varsigma^- u_{jk}\right|^2+ip_1h\tau\sum_{j=1}^{A-1}D_\varsigma^- u_{jk}\overline{u}_{jk}-h\tau\sum_{j=1}^{A-1}p_j\left|u_{jk}\right|^2+h\tau\sum_{j=1}^{A-1}q_k\left|u_{jk}\right|^2=h\tau\sum_{j=1}^{A-1}\rho_{jk}\overline{u}_{jk}. \tag{15}$$

By subtracting its complex conjugate from (15) and using the relations

$$\tau\left(D_t^- u_{jk}\overline{u}_{jk}+D_t^-\overline{u}_{jk}u_{jk}\right)=\left|u_{jk}\right|^2-\left|u_{jk-1}\right|^2+\left|u_{jk}-u_{jk-1}\right|^2, \tag{16}$$

$$h\left(D_\varsigma^- u_{jk}\overline{u}_{jk}+D_\varsigma^-\overline{u}_{jk}u_{jk}\right)=\left|u_{jk}\right|^2-\left|u_{j-1k}\right|^2+\left|u_{jk}-u_{j-1k}\right|^2, \tag{17}$$

we have

$$h\sum_{j=1}^{A-1}\left(\left|u_{jk}\right|^2-\left|u_{jk-1}\right|^2+\left|u_{jk}-u_{jk-1}\right|^2\right)+p_1\tau\sum_{j=1}^{A-1}\left(\left|u_{jk}\right|^2-\left|u_{j-1k}\right|^2+\left|u_{jk}-u_{j-1k}\right|^2\right)=2h\tau\sum_{j=1}^{A-1}Im\left(\rho_{jk}\overline{u}_{jk}\right) \tag{18}$$

for $k=1,2,...B$. If we sum all equalities in (18) in k from 1 to $m\le B$ and consider

$$\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left(\left|u_{jk}\right|^2-\left|u_{jk-1}\right|^2\right)=\sum_{j=1}^{A-1}\left(\left|u_{jm}\right|^2-\left|u_{j0}\right|^2\right)=\sum_{j=1}^{A-1}\left|u_{jm}\right|^2-\sum_{j=1}^{A-1}\left|\eta_j\right|^2 \tag{19}$$

$$\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left(\left|u_{jk}\right|^2-\left|u_{j-1k}\right|^2\right)=\sum_{k=1}^{m}\left(\left|u_{A-1k}\right|^2-\left|u_{0k}\right|^2\right)=\sum_{k=1}^{m}\left|u_{A-1k}\right|^2 \tag{20}$$

with (9) and (10), we obtain from (18) the inequality

$$\begin{aligned}&h\sum_{j=1}^{A-1}\left|u_{jm}\right|^2+h\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|u_{jk}-u_{jk-1}\right|^2+p_1\tau\sum_{k=1}^{m}\left|u_{A-1k}\right|^2\\&+p_1\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|u_{jk}-u_{j-1k}\right|^2\le 2h\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|\rho_{jk}\right|\left|u_{jk}\right|+h\sum_{j=1}^{A-1}\left|\eta_j\right|^2.\end{aligned} \tag{21}$$

By $\varepsilon-Cauchy's$ and Young's inequalities it is written that

$$\begin{aligned}2h\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|\rho_{jk}\right|\left|u_{jk}\right|&=2\tau h\sum_{j=1}^{A-1}\left|\rho_{jm}\right|\left|u_{jm}\right|+2h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|\rho_{jk}\right|\left|u_{jk}\right|\\&\le\varepsilon\tau h\sum_{j=1}^{A-1}\left|\rho_{jm}\right|^2+\frac{h\tau}{\varepsilon}\sum_{j=1}^{A-1}\left|u_{jm}\right|^2+h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|\rho_{jk}\right|^2+h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|u_{jk}\right|^2\\&\le 2T\tau h\sum_{j=1}^{A-1}\left|\rho_{jm}\right|^2+\frac{h}{2}\sum_{j=1}^{A-1}\left|u_{jm}\right|^2+h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|\rho_{jk}\right|^2+h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|u_{jk}\right|^2\end{aligned}$$

with $\varepsilon = 2\tau$ and $\tau \leq T$. Thus,

$$h\sum_{j=1}^{A-1}|u_{jm}|^2+2h\sum_{k=1}^{m}\sum_{j=1}^{A-1}|u_{jk}-u_{jk-1}|^2+2p_1\tau\sum_{k=1}^{m}|u_{A-1k}|^2+2p_1\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}|u_{jk}-u_{j-1k}|^2$$
$$\leq 4Th\tau\sum_{k=1}^{B}\sum_{j=1}^{A-1}|\rho_{jk}|^2+2h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}|u_{jk}|^2+2h\sum_{j=1}^{A-1}|\eta_j|^2 \tag{22}$$

for *any* $m \in \{1,2,...,B\}$ is obtained from (21). From the non-negativeness of all terms in the left-hand side of (22), it is written that

$$h\sum_{j=1}^{A-1}|u_{jm}|^2 \leq 4Th\tau\sum_{k=1}^{B}\sum_{j=1}^{A-1}|\rho_{jk}|^2+2h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}|u_{jk}|^2+2h\sum_{j=1}^{A-1}|\eta_j|^2. \tag{23}$$

Applying the discrete Gronwall's inequality to (23), we achieve

$$h\sum_{j=1}^{A-1}|u_{jm}|^2 \leq c_2\left(h\sum_{j=1}^{A-1}|\eta_j|^2+\tau h\sum_{k=1}^{B}\sum_{j=1}^{A-1}|\rho_{jk}|^2\right) \text{ for } any\ m\in\{1,2,...,B\}. \tag{24}$$

If we use inequality (24) in (22), we get for *any* $m \in \{1,2,...,B\}$,

$$h\sum_{j=1}^{A-1}|u_{jm}|^2+2h\sum_{k=1}^{m}\sum_{j=1}^{A-1}|u_{jk}-u_{jk-1}|^2+2p_1\tau\sum_{k=1}^{m}|u_{A-1k}|^2+2p_1\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}|u_{jk}-u_{j-1k}|^2$$
$$\leq c_3\left(h\sum_{j=1}^{A-1}|\eta_j|^2+\tau h\sum_{k=1}^{B}\sum_{j=1}^{A-1}|\rho_{jk}|^2\right) \tag{25}$$

which completes the proof.

Theorem 4. *Suppose that u^1_{jk}, u^2_{jk} are solutions of difference scheme (8)-(10) with initial values η^1_j, η^2_j and right sides ρ^1_{jk}, ρ^2_{jk}, respectively. Assume that the conditions of theorem 3 are fulfilled. Let $\Phi_{jk} = u^1_{jk} - u^2_{jk}$. Then, for any $m \in \{1,2,...,B\}$ and $h,\tau > 0$,*

$$h\sum_{j=1}^{A-1}|\Phi_{jm}|^2 \leq c_4\left(h\sum_{j=1}^{A-1}|\eta^1_j-\eta^2_j|^2+h\tau\sum_{k=1}^{B-1}\sum_{j=1}^{A-1}|\rho^1_{jk}-\rho^2_{jk}|^2\right).$$

Hence, difference scheme (8)-(10) is unconditionally stable.

Proof. It is clear that for $j = 1,2,...,A-1, k = 1,2,...B$, Φ_{jk} is the solution of scheme

$$iD^-_t\Phi_{jk}+p_0D^2_\varsigma\Phi_{jk}+ip_1D^-_\varsigma\Phi_{jk}-p_j\Phi_{jk}+q_k\Phi_{jk}=\rho^1_{jk}-\rho^2_{jk}, \tag{26}$$
$$\Phi_{j0}=\eta^1_j-\eta^2_j,\quad j=0,1,...,A, \tag{27}$$
$$\Phi_{0k}=\Phi_{Ak}=0,\quad k=1,2,...B, \tag{28}$$

which is equivalent to summation identity

$$ih\sum_{j=1}^{A-1}D^-_t\Phi_{jk}\overline{\kappa}_{jk}+p_0h\sum_{j=1}^{A-1}D^2_\varsigma\Phi_{jk}\overline{\kappa}_{jk}+ip_1h\sum_{j=1}^{A-1}D^-_\varsigma\Phi_{jk}\overline{\kappa}_{jk}$$
$$-h\sum_{j=1}^{A-1}p_j\Phi_{jk}\overline{\kappa}_{jk}+h\sum_{j=1}^{A-1}q_k\Phi_{jk}\overline{\kappa}_{jk}=h\sum_{j=1}^{A-1}\left(\rho^1_{jk}-\rho^2_{jk}\right)\overline{\kappa}_{jk} \tag{29}$$

for any scheme function κ_{jk} such that $\kappa_{0k} = \kappa_{Ak} = 0$ for $k = 1,2,...B$. If we substitute $\tau\overline{\Phi}_{jk}$ for $\overline{\kappa}_{jk}$ in (29) and apply the formula of summation by parts, we get

$$
\begin{aligned}
& ih\tau\sum_{j=1}^{A-1} D_t^- \Phi_{jk}\overline{\Phi}_{jk} - p_0 h\tau\sum_{j=1}^{A-1}\left|D_\varsigma^- \Phi_{jk}\right|^2 + ip_1 h\tau\sum_{j=1}^{A-1} D_\varsigma^- \Phi_{jk}\overline{\Phi}_{jk} - h\tau\sum_{j=1}^{A-1} p_j\left|\Phi_{jk}\right|^2 \\
& + h\tau\sum_{j=1}^{A-1} q_k\left|\Phi_{jk}\right|^2 = h\tau\sum_{j=1}^{A-1}\left(\rho_{jk}^1 - \rho_{jk}^2\right)\overline{\Phi}_{jk}.
\end{aligned}
$$

If we continue the process similarly to the proof of Theorem 3, we obtain for *any* $m \in \{1,2,...,B\}$,

$$
\begin{aligned}
& h\sum_{j=1}^{A-1}\left|\Phi_{jm}\right|^2 + 2h\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|\Phi_{jk} - \Phi_{jk-1}\right|^2 + 2p_1\tau\sum_{k=1}^{m}\left|\Phi_{A-1k}\right|^2 + 2p_1\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|\Phi_{jk} - \Phi_{j-1k}\right|^2 \\
& \leq 4Th\tau\sum_{k=1}^{B}\sum_{j=1}^{A-1}\left|\rho_{jk}^1 - \rho_{jk}^2\right|^2 + 2h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|\Phi_{jk}\right|^2 + h\sum_{j=1}^{A-1}\left|\eta_j^1 - \eta_j^2\right|^2. \qquad (30)
\end{aligned}
$$

Since all terms in the left-hand side of (30) are non-negative, the inequality

$$
h\sum_{j=1}^{A-1}\left|\Phi_{jm}\right|^2 \leq 4Th\tau\sum_{k=1}^{B}\sum_{j=1}^{A-1}\left|\rho_{jk}^1 - \rho_{jk}^2\right|^2 + 2h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|\Phi_{jk}\right|^2 + h\sum_{j=1}^{A-1}\left|\eta_j^1 - \eta_j^2\right|^2 \qquad (31)
$$

is written. If we apply discrete Gronwall's Inequality to (31), we obtain

$$
h\sum_{j=1}^{A-1}\left|\Phi_{jm}\right|^2 \leq c_5\left(h\tau\sum_{k=1}^{B}\sum_{j=1}^{A-1}\left|\rho_{jk}^1 - \rho_{jk}^2\right|^2 + h\sum_{j=1}^{A-1}\left|\eta_j^1 - \eta_j^2\right|^2\right) \text{ for } \textit{any } m \in \{1,2,...,B\}
$$

which completes the proof.

4 The convergence of scheme (8)-(10)

In this section, we prove that the solution u_{jk} of scheme (8)-(10) is convergent to the exact solution U_{jk} of $u(\varsigma,t)$, which U_{jk} is defined by

$$
U_{jk} = \frac{1}{\tau h}\int_{t_{k-1}}^{t_k}\int_{\varsigma_j - \frac{h}{2}}^{\varsigma_j + \frac{h}{2}} u(\varsigma,t)\,d\varsigma dt,\ j = 1,2,...,A-1,\ k = 1,2,...B, \qquad (32)
$$

$$
U_{j0} = \eta_j,\ j = 0,1,...,A,\ U_{0k} = U_{Ak} = 0,\ k = 1,2,...B. \qquad (33)
$$

For this, let's show the error of scheme (8)-(10) by $e_{jk} = u_{jk} - U_{jk}$ at (ς_j, t_k). It is clear that e_{jk} satisfies the following system:

$$
iD_t^- e_{jk} + p_0 D_\varsigma^2 e_{jk} + ip_1 D_\varsigma^- e_{jk} - p_j e_{jk} + q_k e_{jk} = I_{jk},\ j = 1,2,...,A-1,\ k = 1,2,...B, \qquad (34)
$$

$$
e_{j0} = 0,\ j = 0,1,...,A,\ \ e_{0k} = e_{Ak} = 0,\ k = 1,2,...B, \qquad (35)
$$

where

$$
\begin{aligned}
I_{jk} = & \frac{1}{\tau h}\int_{t_{k-1}}^{t_k}\int_{\varsigma_j - \frac{h}{2}}^{\varsigma_j + \frac{h}{2}}\left(i\frac{\partial u}{\partial t} + p_0\frac{\partial^2 u}{\partial \varsigma^2} + ip_1\frac{\partial u}{\partial \varsigma} - p(\varsigma)u + q(t)u\right)d\varsigma dt \qquad (36) \\
& - iD_t^- U_{jk} - p_0 D_\varsigma^2 U_{jk} - ip_1 D_\varsigma^- U_{jk} + p_j U_{jk} - q_k U_{jk}.
\end{aligned}
$$

Theorem 5. *Presume that the conditions of Theorem (1) are fulfilled. Then, the error e_{jk} of scheme (8)-(10) satisfies:*

$$h\sum_{j=1}^{A-1}\left|e_{jm}\right|^2 \le c_{12}\left(w_\tau^0+w_h^0+w_h^1+\tau^2+h^2\right) \text{ for any } m\in\{1,2,...,B\},$$

where $w_\tau^0 \longrightarrow 0$, $w_h^0 \longrightarrow 0$, $w_h^1 \longrightarrow 0$ as $\tau,h \longrightarrow 0$. Hence, the solution u_{jk} of scheme (8)-(10) converges to the solution U_{jk} of problem (2)-(4).

Proof. System (34)-(35) is equivalent to the summation identity

$$\begin{aligned}&h\sum_{j=1}^{A-1} iD_t^- e_{jk}\overline{\vartheta}_{jk}+p_0h\sum_{j=1}^{A-1} D_\varsigma^2 e_{jk}\overline{\vartheta}_{jk}+ip_1h\sum_{j=1}^{A-1} D_\varsigma^- e_{jk}\overline{\vartheta}_{jk}\\&-h\sum_{j=1}^{A-1} p_j e_{jk}\overline{\vartheta}_{jk}+h\sum_{j=1}^{A-1} q_k e_{jk}\overline{\vartheta}_{jk}=h\sum_{j=1}^{A-1} I_{jk}\overline{\vartheta}_{jk},\ k=1,2,...B\end{aligned} \tag{37}$$

for any grid function ϑ_{jk} such that $\vartheta_{0k}=\vartheta_{Ak}=0$ for $k=1,2,...B$, where $\overline{\vartheta}_{jk}$ is the complex conjugate of ϑ_{jk}. With the help of the formula of summation by parts for $\overline{\vartheta}_{jk}=\tau\overline{e}_{jk}$ in (37), we get

$$\begin{aligned}&h\tau\sum_{j=1}^{A-1} i\left(D_t^- e_{jk}\right)\overline{e}_{jk}-p_0h\tau\sum_{j=1}^{A-1}\left|D_\varsigma^- e_{jk}\right|^2+ip_1h\tau\sum_{j=1}^{A-1}\left(D_\varsigma^- e_{jk}\right)\overline{e}_{jk}\\&-h\tau\sum_{j=1}^{A-1} p_j\left|e_{jk}\right|^2+h\tau\sum_{j=1}^{A-1} q_k\left|e_{jk}\right|^2=h\tau\sum_{j=1}^{A-1} I_{jk}\overline{e}_{jk}.\end{aligned} \tag{38}$$

If we subtract its conjugate from (38) and use relations (16), (17), we obtain for $k=1,2,...B$

$$h\sum_{j=1}^{A-1}\left[\left|e_{jk}\right|^2-\left|e_{jk-1}\right|^2+\left|e_{jk}-e_{jk-1}\right|^2\right]+p_1\tau\sum_{j=1}^{A-1}\left[\left|e_{jk}\right|^2-\left|e_{j-1k}\right|^2+\left|e_{jk}-e_{j-1k}\right|^2\right]=2h\tau\sum_{j=1}^{A-1} Im\left(I_{jk}\overline{e}_{jk}\right). \tag{39}$$

Let's sum all equalities in (39) in k from 1 to $m\le B$ and use equalities (19) and (20) for e_{jk} with (35). Thus, we have

$$h\sum_{j=1}^{A-1}\left|e_{jm}\right|^2+h\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|e_{jk}-e_{jk-1}\right|^2+p_1\tau\sum_{k=1}^{m}\left|e_{A-1k}\right|^2+p_1\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|e_{jk}-e_{j-1k}\right|^2\le 2h\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|I_{jk}\right|\left|e_{jk}\right|$$

which is equivalent to

$$\begin{aligned}&h\sum_{j=1}^{A-1}\left|e_{jm}\right|^2+h\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|e_{jk}-e_{jk-1}\right|^2+p_1\tau\sum_{k=1}^{m}\left|e_{A-1k}\right|^2+p_1\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|e_{jk}-e_{j-1k}\right|^2\\&\le 2h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|I_{jk}\right|\left|e_{jk}\right|+2h\tau\sum_{j=1}^{A-1}\left|I_{jm}\right|\left|e_{jm}\right|.\end{aligned}$$

By $\varepsilon-Cauchy's$ inequality from inequality above, it is written that

$$\begin{aligned}&h\sum_{j=1}^{A-1}\left|e_{jm}\right|^2+h\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|e_{jk}-e_{jk-1}\right|^2+p_1\tau\sum_{k=1}^{m}\left|e_{A-1k}\right|^2+p_1\tau\sum_{k=1}^{m}\sum_{j=1}^{A-1}\left|e_{jk}-e_{j-1k}\right|^2\\&\le 2h\tau\frac{1}{2\varepsilon}\sum_{j=1}^{A-1}\left|e_{jm}\right|^2+2h\tau\frac{\varepsilon}{2}\sum_{j=1}^{A-1}\left|I_{jm}\right|^2+h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|I_{jk}\right|^2+h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|e_{jk}\right|^2\end{aligned} \tag{40}$$

which implies that

$$h\sum_{j=1}^{A-1}\left|e_{jm}\right|^2\le(4T+2)h\tau\sum_{k=1}^{B}\sum_{j=1}^{A-1}\left|I_{jk}\right|^2+2h\tau\sum_{k=1}^{m-1}\sum_{j=1}^{A-1}\left|e_{jk}\right|^2 \tag{41}$$

noting that $\tau \leq T$ by $\varepsilon = 2\tau$. In (41), by discrete Gronwall's inequality, we obtain

$$h\sum_{j=1}^{A-1}\left|e_{jm}\right|^{2} \leq c_{6}h\tau\sum_{k=1}^{B}\sum_{j=1}^{A-1}\left|I_{jk}\right|^{2} \text{ for any } m\in\{1,2,...B\}. \tag{42}$$

Let's denote the grid function I_{jk} as follows

$$I_{jk} = I_{jk}^{1}+I_{jk}^{2}+I_{jk}^{3}+I_{jk}^{4}+I_{jk}^{5} \text{ for } j=1,2,...,A-1, k=1,2,...B,$$

where

$$I_{jk}^{1} = \frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} i\frac{\partial u}{\partial t}d\varsigma dt - iD_t^{-}U_{jk}, \tag{43}$$

$$I_{jk}^{2} = \frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} p_0\frac{\partial^2 u}{\partial \varsigma^2}d\varsigma dt - p_0 D_\varsigma^2 U_{jk}, \tag{44}$$

$$I_{jk}^{3} = \frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} ip_1\frac{\partial u}{\partial \varsigma}d\varsigma dt - ip_1 D_\varsigma^{-} U_{jk}, \tag{45}$$

$$I_{jk}^{4} = \frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} -p(\varsigma)u(\varsigma,t)d\varsigma dt + p_j U_{jk}, \tag{46}$$

$$I_{jk}^{5} = \frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} q(t)u(\varsigma,t)d\varsigma dt - q_k U_{jk}. \tag{47}$$

From (32) and (43) for $j=1,2,...,A-1$, $k=2,3,...B$, it is written that

$$I_{jk}^{1} = \frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} i\frac{\partial u}{\partial t}d\varsigma dt - iD_t^{-}U_{jk} = \frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} i\frac{\partial u}{\partial t}d\varsigma dt - i\frac{1}{h\tau^2}\left[\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}(u(\varsigma,t)-u(\varsigma,t-\tau))d\varsigma dt\right]$$

$$= \frac{i}{h\tau^2}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{-\tau}^{0}\left(\frac{\partial u(\varsigma,t)}{\partial t}-\frac{\partial u(\varsigma,t+\theta)}{\partial t}\right)d\theta d\varsigma dt$$

which implies that

$$\left|I_{jk}^{1}\right| \leq \frac{1}{h\tau^2}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{-\tau}^{0}\left|\frac{\partial u(\varsigma,t)}{\partial t}-\frac{\partial u(\varsigma,t+\theta)}{\partial t}\right|d\theta d\varsigma dt. \tag{48}$$

In (48), using Fubini's Theorem and Cauchy-Schwarz inequality, we achieve

$$h\tau\sum_{k=2}^{B}\sum_{j=1}^{A-1}\left|I_{jk}^{1}\right|^{2} \leq \frac{1}{\tau}\int\limits_{-\tau}^{0}\left[\int\limits_{\Omega}\left|\frac{\partial u(\varsigma,t)}{\partial t}-\frac{\partial u(\varsigma,t+\theta)}{\partial t}\right|^{2}d\varsigma dt\right]d\theta. \tag{49}$$

As known, any function belonging to $L_2(\Omega)$ is continuous in the norm of $L_2(\Omega)$. So since $\partial u/\partial t \in L_2(\Omega)$, for a given $\varepsilon > 0$, a number $\sigma > 0$ can be found such that

$$\left(\int\limits_{\Omega} \left| \frac{\partial u(\varsigma,t)}{\partial t} - \frac{\partial u(\varsigma,t+\theta)}{\partial t} \right|^2 d\varsigma dt \right)^{1/2} < \varepsilon$$

for all $|\theta| \leq \tau < \sigma$ [19]. Therefore, we write

$$\tau h \sum_{k=2}^{B} \sum_{j=1}^{A-1} \left| I_{jk}^1 \right|^2 \leq w_\tau^0, \tag{50}$$

where

$$w_\tau^0 = \frac{1}{\tau} \int\limits_{-\tau}^{0} \left(\int\limits_{\Omega} \left| \frac{\partial u(\varsigma,t)}{\partial t} - \frac{\partial u(\varsigma,t+\theta)}{\partial t} \right|^2 d\varsigma dt \right) d\theta, \quad w_\tau^0 > 0$$

and w_τ^0 converges to zero since $\theta \to 0$ as $\tau \to 0$. Similarly, from (32) and (43) for $k = 1$, we have

$$\tau h \sum_{j=1}^{A-1} \left| I_{j1}^1 \right|^2 \leq 4 \int\limits_{0}^{\tau} \left\| \frac{\partial u(.,t)}{\partial t} \right\|_{L_2(I)}^2 dt \leq c_7 \tau$$

by (7). Combining the last inequality with (50), we obtain

$$\tau h \sum_{k=1}^{B} \sum_{j=1}^{A-1} \left| I_{jk}^1 \right|^2 \leq c_8 \tau + w_\tau^0. \tag{51}$$

From (32) and (44) for $j = 2,3,...,A-2$, $k = 1,2,...B$, it is written that

$$\begin{aligned} I_{jk}^2 &= \frac{1}{h\tau} \int\limits_{t_{k-1}}^{t_k} \int\limits_{\varsigma_j - \frac{h}{2}}^{\varsigma_j + \frac{h}{2}} p_0 \frac{\partial^2 u(\varsigma,t)}{\partial \varsigma^2} d\varsigma dt - p_0 \left[\frac{U_{j+1k} - 2U_{jk} + U_{j-1k}}{h^2} \right] \\ &= \frac{p_0}{h^3\tau} \int\limits_{t_{k-1}}^{t_k} \int\limits_{\varsigma_j - \frac{h}{2}}^{\varsigma_j + \frac{h}{2}} \int\limits_{\varsigma}^{\varsigma+h} \int\limits_{\zeta-h}^{\zeta} \frac{\partial^2 u(\varsigma,t)}{\partial \varsigma^2} d\phi d\zeta d\varsigma dt - \frac{p_0}{h^3\tau} \int\limits_{t_{k-1}}^{t_k} \int\limits_{\varsigma_j - \frac{h}{2}}^{\varsigma_j + \frac{h}{2}} \int\limits_{\varsigma}^{\varsigma+h} \int\limits_{\zeta-h}^{\zeta} \frac{\partial^2 u(\phi,t)}{\partial \phi^2} d\phi d\zeta d\varsigma dt \\ &= \frac{p_0}{h^3\tau} \int\limits_{t_{k-1}}^{t_k} \int\limits_{\varsigma_j - \frac{h}{2}}^{\varsigma_j + \frac{h}{2}} \int\limits_{0}^{h} \int\limits_{-h}^{0} \left(\frac{\partial^2 u(\varsigma,t)}{\partial \varsigma^2} - \frac{\partial^2 u(\varsigma+\zeta+\phi,t)}{\partial \varsigma^2} \right) d\phi d\zeta d\varsigma dt \end{aligned}$$

which implies that

$$\left| I_{jk}^2 \right| \leq \frac{p_0}{h^3\tau} \int\limits_{t_{k-1}}^{t_k} \int\limits_{\varsigma_j - \frac{h}{2}}^{\varsigma_j + \frac{h}{2}} \int\limits_{0}^{h} \int\limits_{-h}^{0} \left| \frac{\partial^2 u(\varsigma,t)}{\partial \varsigma^2} - \frac{\partial^2 u(\varsigma+\zeta+\phi,t)}{\partial \varsigma^2} \right| d\phi d\zeta d\varsigma dt.$$

In above inequality, by using Fubini's Theorem and Cauchy-Schwarz inequality, we get

$$\tau h\sum_{k=1}^{B}\sum_{j=2}^{A-2}\left|I_{jk}^{2}\right|^{2}\leq\frac{2p_{0}}{h^{2}}\int_{0}^{h}\int_{-h}^{0}\left(\int_{\Omega}\left|\frac{\partial^{2}u(\varsigma,t)}{\partial\varsigma^{2}}-\frac{\partial^{2}u(\varsigma+\zeta,t)}{\partial\varsigma^{2}}\right|^{2}d\varsigma dt\right)d\phi d\zeta$$
$$+\frac{2p_{0}}{h^{2}}\int_{0}^{h}\int_{-h}^{0}\left(\int_{\Omega}\left|\frac{\partial^{2}u(\varsigma+\zeta,t)}{\partial\varsigma^{2}}-\frac{\partial^{2}u(\varsigma+\zeta+\phi,t)}{\partial\varsigma^{2}}\right|^{2}d\varsigma dt\right)d\phi d\zeta.$$

Since $\partial^2 u/\partial\varsigma^2\in L_2(\Omega)$, for a given $\varepsilon>0$, a number $\sigma>0$ can be found such that

$$\left(\int_{\Omega}\left|\frac{\partial^{2}u(\varsigma,t)}{\partial\varsigma^{2}}-\frac{\partial^{2}u(\varsigma+\zeta,t)}{\partial\varsigma^{2}}\right|^{2}d\varsigma dt\right)^{1/2}<\frac{\varepsilon}{2}$$

$$\left(\int_{\Omega}\left|\frac{\partial^{2}u(\varsigma+\zeta,t)}{\partial\varsigma^{2}}-\frac{\partial^{2}u(\varsigma+\zeta+\phi,t)}{\partial\varsigma^{2}}\right|^{2}d\varsigma dt\right)^{1/2}<\frac{\varepsilon}{2}$$

for $|\zeta|\leq h<\sigma$ and $|\phi|\leq h<\sigma$ [19]. Since $\zeta\longrightarrow 0$ and $\phi\longrightarrow 0$ as $h\longrightarrow 0$, it is clear that

$$\int_{\Omega}\left|\frac{\partial^{2}u(\varsigma,t)}{\partial\varsigma^{2}}-\frac{\partial^{2}u(\varsigma+\zeta,t)}{\partial\varsigma^{2}}\right|^{2}d\varsigma dt\longrightarrow 0 \text{ and } \int_{\Omega}\left|\frac{\partial^{2}u(\varsigma+\zeta,t)}{\partial\varsigma^{2}}-\frac{\partial^{2}u(\varsigma+\zeta+\phi,t)}{\partial\varsigma^{2}}\right|^{2}d\varsigma dt\longrightarrow 0.$$

Thus, we can write

$$h\tau\sum_{k=1}^{B}\sum_{j=2}^{A-2}\left|I_{jk}^{2}\right|^{2}\leq w_{h}^{0}, \tag{52}$$

where

$$w_{h}^{0}=\frac{2p_{0}^{2}}{h^{2}}\int_{0}^{h}\int_{-h}^{0}\left(\int_{\Omega}\left|\frac{\partial^{2}u(\varsigma,t)}{\partial\varsigma^{2}}-\frac{\partial^{2}u(\varsigma+\zeta,t)}{\partial\varsigma^{2}}\right|^{2}d\varsigma dt\right)d\phi d\zeta$$
$$+\frac{2p_{0}^{2}}{h^{2}}\int_{0}^{h}\int_{-h}^{0}\left(\int_{\Omega}\left|\frac{\partial^{2}u(\varsigma+\zeta,t)}{\partial\varsigma^{2}}-\frac{\partial^{2}u(\varsigma+\zeta+\phi,t)}{\partial\varsigma^{2}}\right|^{2}d\varsigma dt\right)d\phi d\zeta.$$

So we say w_h^0 converges to zero as $h\to 0$. From (32) and (44) for $j=1$ and $j=A-1$, we get

$$\left|I_{1k}^{2}\right|\leq\frac{3p_{0}}{h\tau}\int_{t_{k-1}}^{t_{k}}\int_{\varsigma_{1}-\frac{h}{2}}^{\varsigma_{1}+\frac{h}{2}}\left|\frac{\partial^{2}u(\varsigma,t)}{\partial\varsigma^{2}}\right|d\varsigma dt \text{ for } k=1,2,...B,$$

$$\left|I_{A-1k}^{2}\right|\leq\frac{3p_{0}}{h\tau}\int_{t_{k-1}}^{t_{k}}\int_{\varsigma_{A-1}-\frac{h}{2}}^{\varsigma_{A-1}+\frac{h}{2}}\left|\frac{\partial^{2}u(\varsigma,t)}{\partial\varsigma^{2}}\right|d\varsigma dt \text{ for } k=1,2,...B$$

which is equivalent to

$$h\tau\sum_{k=1}^{B}\left|I_{1k}^{2}\right|^{2}+h\tau\sum_{k=1}^{B}\left|I_{A-1k}^{2}\right|^{2}\leq 9p_{0}^{2}\left(\int_{0}^{h}\left\|\frac{\partial^{2}u(\varsigma,.)}{\partial\varsigma^{2}}\right\|_{L_{2}(0,T)}^{2}d\varsigma+\int_{l-h}^{l}\left\|\frac{\partial^{2}u(\varsigma,.)}{\partial\varsigma^{2}}\right\|_{L_{2}(0,T)}^{2}d\varsigma\right). \tag{53}$$

From (53), we can easily say that the term $h\tau\sum\limits_{k=1}^{B}\left|I_{1k}^{2}\right|^{2}+h\tau\sum\limits_{k=1}^{B}\left|I_{A-1k}^{2}\right|^{2}$ converges to zero as $h\longrightarrow 0$. Combining (53) with (52), we obtain

$$\tau h\sum_{k=1}^{B}\sum_{j=1}^{A-1}\left|I_{jk}^{2}\right|^{2}\le w_{h}^{0}. \tag{54}$$

By formulas (32) and (45) for $j=2,3,...,A-2$, $k=1,2,...B$, we write

$$I_{jk}^{3}=\frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} ip_1\frac{\partial u}{\partial\varsigma}d\varsigma dt-ip_1\frac{\left(U_{jk}-U_{j-1k}\right)}{h}=\frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}} ip_1\frac{\partial u}{\partial\varsigma}d\varsigma dt-\frac{ip_1}{h^2\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left(u(\varsigma,t)-u(\varsigma-h,t)\right)d\varsigma dt$$

which implies that

$$\left|I_{jk}^{3}\right|\le\frac{p_1}{\tau h^2}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{-h}^{0}\left|\frac{\partial u(\varsigma,t)}{\partial\varsigma}-\frac{\partial u(\varsigma+\zeta,t)}{\partial\varsigma}\right|d\zeta d\varsigma dt. \tag{55}$$

From (55), by means of Fubini's theorem and Cauchy-Schwarz inequality, we get

$$h\tau\sum_{k=1}^{B}\sum_{j=2}^{A-2}\left|I_{jk}^{3}\right|^{2}\le\frac{p_1^2}{h}\int\limits_{-h}^{0}\left(\int\limits_{\Omega}\left|\frac{\partial u(\varsigma,t)}{\partial\varsigma}-\frac{\partial u(\varsigma+\zeta,t)}{\partial\varsigma}\right|^{2}d\varsigma dt\right)d\zeta.$$

Since $\partial u/\partial\varsigma\in L_2(\Omega)$, for a given $\varepsilon>0$, a number $\sigma>0$ can be found such that

$$\left(\int\limits_{\Omega}\left|\frac{\partial u(\varsigma,t)}{\partial\varsigma}-\frac{\partial u(\varsigma+\zeta,t)}{\partial\varsigma}\right|^{2}d\varsigma dt\right)^{1/2}<\varepsilon$$

for $|\zeta|\le h<\sigma$. Since $\zeta\longrightarrow 0$ as $h\longrightarrow 0$, we can write

$$h\tau\sum_{k=1}^{B}\sum_{j=2}^{A-2}\left|I_{jk}^{3}\right|^{2}\le w_{h}^{1}, \tag{56}$$

where

$$w_h^1=\frac{p_1^2}{h}\int\limits_{-h}^{0}\left(\int\limits_{\Omega}\left|\frac{\partial u(\varsigma,t)}{\partial\varsigma}-\frac{\partial u(\varsigma+\zeta,t)}{\partial\varsigma}\right|^{2}d\varsigma dt\right)d\zeta$$

and w_h^1 converges to zero as $h\to 0$. Similarly, from (32) and (45) for $j=1$ and $j=A-1$, we get

$$\tau h\sum_{k=1}^{B}\left|I_{1k}^{3}\right|^{2}\le 4p_1^2\int\limits_{0}^{h}\left(\int\limits_{0}^{T}\left|\frac{\partial u(\varsigma,t)}{\partial\varsigma}\right|^{2}dt\right)d\varsigma=4p_1^2\int\limits_{0}^{h}\left\|\frac{\partial u(\varsigma,.)}{\partial\varsigma}\right\|_{L_2(0,T)}^{2}d\varsigma,$$

$$\tau h\sum_{k=1}^{B}\left|I_{A-1k}^{3}\right|^{2}\le 4p_1^2\int\limits_{l-h}^{l}\left(\int\limits_{0}^{T}\left|\frac{\partial u(\varsigma,t)}{\partial\varsigma}\right|^{2}dt\right)d\varsigma=4p_1^2\int\limits_{l-h}^{l}\left\|\frac{\partial u(\varsigma,.)}{\partial\varsigma}\right\|_{L_2(0,T)}^{2}d\varsigma$$

which implies that

$$\tau h\sum_{k=1}^{B}\left|I_{1k}^{3}\right|^{2}+\tau h\sum_{k=1}^{B}\left|I_{A-1k}^{3}\right|^{2}\le 4p_1^2\left(\int\limits_{0}^{h}\left\|\frac{\partial u(\varsigma,.)}{\partial\varsigma}\right\|_{L_2(0,T)}^{2}d\varsigma+\int\limits_{l-h}^{l}\left\|\frac{\partial u(\varsigma,.)}{\partial\varsigma}\right\|_{L_2(0,T)}^{2}d\varsigma\right). \tag{57}$$

In (57), since $h\tau\sum\limits_{k=1}^{B}\left|I_{1k}^{3}\right|^{2}+h\tau\sum\limits_{k=1}^{B}\left|I_{A-1k}^{3}\right|^{2}\longrightarrow 0$ as $h\longrightarrow 0$, combining (56) with (57), we can write

$$\tau h\sum_{k=1}^{B}\sum_{j=1}^{A-1}\left|I_{jk}^{3}\right|^{2}\leq w_{h}^{1}. \tag{58}$$

From (32) and (46) for $j=1,2,...,A-1$, $k=1,2,...B$, it is written that

$$I_{jk}^{4}=\frac{1}{\tau h}U_{jk}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left(p_j-p(\varsigma)\right)d\varsigma dt+\frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}p(\varsigma)\left(U_{jk}-u(\varsigma,t)\right)d\varsigma dt=\frac{1}{\tau h}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}p(\varsigma)\left(U_{jk}-u(\varsigma,t)\right)d\varsigma dt$$

which implies that

$$\left|I_{jk}^{4}\right|\leq\frac{\mu_0}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left|U_{jk}-u(\varsigma,t)\right|d\varsigma dt \tag{59}$$

by condition (5). Since

$$U_{jk}-u(\varsigma,t)=\frac{1}{\tau h}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{t}^{\theta}\frac{\partial u(\rho,\eta)}{\partial\eta}d\eta d\rho d\theta+\frac{1}{\tau h}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{\varsigma}^{\rho}\frac{\partial u(\gamma,t)}{\partial\gamma}d\gamma d\rho d\theta,$$

it is obtained that

$$\begin{aligned}\left|I_{jk}^{4}\right|&\leq\frac{\mu_0}{\tau h}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left[\frac{1}{\tau h}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{t}^{\theta}\left|\frac{\partial u(\zeta,\phi)}{\partial\phi}\right|d\phi d\zeta d\theta+\frac{1}{h\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{\varsigma}^{\rho}\left|\frac{\partial u(\gamma,t)}{\partial\gamma}\right|d\gamma d\zeta d\theta\right]d\varsigma dt\\&\leq\frac{\mu_0}{(\tau h)^2}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{t_{k-1}}^{t_k}\left|\frac{\partial u(\zeta,\phi)}{\partial\phi}\right|d\phi d\zeta d\theta d\varsigma dt+\frac{\mu_0}{(\tau h)^2}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left|\frac{\partial u(\gamma,t)}{\partial\gamma}\right|d\gamma d\zeta d\theta d\varsigma dt\\&=\frac{\mu_0}{h}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left|\frac{\partial u(\varsigma,t)}{\partial t}\right|d\varsigma dt+\frac{\mu_0}{\tau}\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left|\frac{\partial u(\varsigma,t)}{\partial\varsigma}\right|d\varsigma dt.\end{aligned} \tag{60}$$

In (60), by Cauchy-Schwarz and Young's inequalities, we get

$$\left|I_{jk}^{4}\right|^{2}\leq\frac{2\mu_0^2\tau}{h}\left(\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left|\frac{\partial u(\varsigma,t)}{\partial t}\right|^{2}d\varsigma dt\right)+\frac{2\mu_0^2h}{\tau}\left(\int\limits_{t_{k-1}}^{t_k}\int\limits_{\varsigma_j-\frac{h}{2}}^{\varsigma_j+\frac{h}{2}}\left|\frac{\partial u(\varsigma,t)}{\partial\varsigma}\right|^{2}d\varsigma dt\right)$$

which implies that

$$\tau h\sum_{k=1}^{B}\sum_{j=1}^{A-1}\left|I_{jk}^{4}\right|^{2}\leq 2\mu_0^2\tau^2\left\|\frac{\partial u}{\partial t}\right\|_{L_2(\Omega)}^{2}+2\mu_0^2h^2\left\|\frac{\partial u}{\partial\varsigma}\right\|_{L_2(\Omega)}^{2}\leq c_9\left(\tau^2+h^2\right) \tag{61}$$

by the estimate (7).

Similarly to the computations obtaining inequality (61), from (32) and (47) for $j=1,2,...,A-1$, $k=1,2,...,B$, we obtain

$$\tau h \sum_{k=1}^{B} \sum_{j=1}^{A-1} \left| I_{jk}^{5} \right|^{2} \leq 2b_0^2 \tau^2 \left\| \frac{\partial u}{\partial t} \right\|_{L_2(\Omega)} + 2b_0^2 h^2 \left\| \frac{\partial u}{\partial \varsigma} \right\|_{L_2(\Omega)} \leq c_{10} \left(\tau^2 + h^2 \right) \tag{62}$$

by the estimate (7).

Thus, from (51), (54), (58), (61) and (62), we have

$$\tau h \sum_{k=1}^{B} \sum_{j=1}^{A-1} \left| I_{jk} \right|^{2} \leq w_\tau^0 + w_h^0 + w_h^1 + c_{11} \left(\tau^2 + h^2 \right). \tag{63}$$

Inserting (63) into (42), we achieve

$$h \sum_{j=1}^{A-1} \left| e_{jm} \right|^{2} \leq c_{12} \left(w_\tau^0 + w_h^0 + w_h^1 + \tau^2 + h^2 \right) \text{ for any } \{ m \in 1, 2, \ldots, B \}.$$

This completes the proof.

5 Conclusion

In the present paper, a finite difference scheme for Schrödinger type equation including the momentum operator has been constructed. Unconditional stability and convergence of the proposed scheme have been proved. Here, it is worth mentioning that the considered equation in discretized problem contains a momentum operator. Such problems focussing on the solution of Schrödinger type equations including momentum operators by finite difference method have been very slightly studied in literature. Hence, our paper is more comprehensive and current than previous works.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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