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Class preserving actor and commutativity degree of isoclinic Lie crossed modules

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Abstract: In this work, we define the class preserving actor and commutativity degree of Lie crossed modules. Then, we obtain some relations about these notions and isoclinic Lie crossed modules.

Keywords: Isoclinism, Class Preserving Actor, Commutativity Degree, Lie Algebra, Lie Crossed Module.

1 Introduction

Isoclinism is an equivalence relation weaker than isomorphism. So, this notion gives rise to a new classification by using isoclinism families. Isoclinism was defined firstly by Hall in [12]. Also, in [16], isoclinic Lie algebras were defined. Numerous researchers, [15,20,21,22], have worked on this notion.

Crossed modules (of groups) were considered as an algebraic model for homotopy 2-types in [24]. So, this notion have been classified as 2-dimensional groups [6,7]. For comprehensive information about this concept, one can see [17,19]. Also, computational applications of crossed modules have been given in [1,2,3]. Additionally, isoclinic crossed modules (of groups) was introduced in [18]. For the notion of Lie crossed module, we refer to [8,9,10]. Some other properties of Lie crossed modules can be found in [5]. Also isoclinic Lie crossed module and n-isoclinic Lie crossed modules were given in [13,14]. Recall that, from [23], the actors of idoclinic crossed modules (of groups) are not isomorphic but their class preserving actors are isomorphic. So, in this paper, we give a notion called class preserving actor of a Lie crossed module and investigate the class preserving actors of isoclinic Lie crossed modules.

Furthermore, the commutativity degree of groups was defined in [11]. The probability of the event for which two elements commute is named as commutativity degree. Calculating the commutativity degree of a finite group is the same as calculating the number of conjugacy classes. Commutativity degree of crossed modules (of groups) has been introduced in [4] and also in the same work it has been shown that isoclinic crossed modules have same commutativity degree. In this paper, we define the commutativity degree of Lie algebras and Lie crossed modules. By using these definitions, we obtain that commutativity degrees of isoclinic Lie crossed modules are equal.

2 Preliminaries

In this section, we give some notions about Lie crossed modules. See [8,9,10], for details.

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We fix a field $\mathbb{K}$ and assume all Lie algebras to be over $\mathbb{K}$.

A Lie crossed module consists of a Lie algebra homomorphism

$$\mathfrak{d} : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$$

with a Lie action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, written $(x,y) \longmapsto [x,y]$, satisfying the following conditions:

(1) $\mathfrak{d}([x,y]) = [x, \mathfrak{d}(y)]$,
(2) $[\mathfrak{d}(y), y'] = [y, y']$,

for all $x \in \mathfrak{g}_0$, $y, y' \in \mathfrak{g}_1$. We will denote such a Lie crossed module by $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$.

Example 1. (1) $\mathfrak{g} \xrightarrow{ad} Der(\mathfrak{g})$ is a Lie crossed module for any Lie algebra $\mathfrak{g}$.
(2)If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{g}$ acts on $\mathfrak{h}$ via adjoint representation and $\mathfrak{h} \xhookrightarrow{inc.} \mathfrak{g}$ is a Lie crossed module.
(3) $\mathfrak{h} \xrightarrow{0} \mathfrak{g}$ is a Lie crossed module, where $\mathfrak{h}$ is a $\mathfrak{g}$-module.

A Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$ is called *aspherical* if $\ker(\mathfrak{d}) = 0$ and *simply connected* if $\operatorname{coker}(\mathfrak{d}) = 0$.
A morphism between Lie crossed modules $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$ and $\mathfrak{g}' : \mathfrak{g}'_1 \xrightarrow{\mathfrak{d}'} \mathfrak{g}'_0$ is a pair (α, β) of Lie algebra homomorphisms $\alpha : \mathfrak{g}_1 \longrightarrow \mathfrak{g}'_1$, $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ such that $\beta\mathfrak{d} = \mathfrak{d}'\alpha$ and $\alpha([x,y]) = [\beta(x), \alpha(y)]$, for all $x \in \mathfrak{g}_0$, $y \in \mathfrak{g}_1$. So, we get a category **XLie** whose objects are the Lie crossed modules and morphisms are Lie crossed module morphisms.

A Lie crossed module $\mathfrak{h} : \mathfrak{h}_1 \xrightarrow{\mathfrak{d}_\mathfrak{h}} \mathfrak{h}_0$ is a *Lie subcrossed module* of a Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}_\mathfrak{g}} \mathfrak{g}_0$ if $\mathfrak{h}_1, \mathfrak{h}_0$ are Lie subalgebras of $\mathfrak{g}_1$, $\mathfrak{g}_0$, respectively, $\mathfrak{d}_\mathfrak{h} = \mathfrak{d}_\mathfrak{g}|_{\mathfrak{h}_1}$ and the action of $\mathfrak{h}_0$ on $\mathfrak{h}_1$ is induced from the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$. If $\mathfrak{h}_1$and $\mathfrak{h}_0$ are ideals of $\mathfrak{g}_1$ and $\mathfrak{g}_0$, respectively, $[x, y'] \in \mathfrak{h}_1$ and $[x', y] \in \mathfrak{h}_1$, for all $x \in \mathfrak{g}_0$, $y \in \mathfrak{g}_1$, $x' \in \mathfrak{h}_0$, $y' \in \mathfrak{h}_1$ then $\mathfrak{h}$ is called an *ideal* of $\mathfrak{g}$. Hence, we get the *quotient Lie crossed module* $\mathfrak{g}/\mathfrak{h} : \mathfrak{g}_1/\mathfrak{h}_1 \xrightarrow{\bar{\mathfrak{d}}} \mathfrak{g}_0/\mathfrak{h}_0$.

A Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$ is called *finite dimensional* if $\mathfrak{g}_1$ and $\mathfrak{g}_0$ are finite dimensional Lie algebras.

Given any Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$, the *center* of $\mathfrak{g}$ is the Lie crossed module $Z(\mathfrak{g}) : \mathfrak{g}_1^{\mathfrak{g}_0} \xrightarrow{\mathfrak{d}|} (St_{\mathfrak{g}_0}(\mathfrak{g}_1) \cap Z(\mathfrak{g}_0))$ where

$$\mathfrak{g}_1^{\mathfrak{g}_0} = \{y \in \mathfrak{g}_1 : \ [x,y] = 0, \text{ for all } x \in \mathfrak{g}_0\}$$

and

$$St_{\mathfrak{g}_0}(\mathfrak{g}_1) = \{x \in \mathfrak{g}_0 : \ [x,y] = 0, \text{ for all } y \in \mathfrak{g}_1\}.$$

The *commutator subcrossed module* $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is defined by

$$[\mathfrak{g}, \mathfrak{g}] : D_{\mathfrak{g}_0}(\mathfrak{g}_1) \xrightarrow{\mathfrak{d}} [\mathfrak{g}_0, \mathfrak{g}_0]$$

where $D_{\mathfrak{g}_0}(\mathfrak{g}_1) = \{[x,y] : x \in \mathfrak{g}_0, y \in \mathfrak{g}_1\}$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is the commutator subalgebra of $\mathfrak{g}_0$.

The analogous version for groups of following proposition can be found in [17].

Proposition 1.*Let $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$ be a Lie crossed module. Then we have the followings;*

(i) *If $\mathfrak{g}$ is simply connected, then $\mathfrak{g}_1^{\mathfrak{g}_0} = Z(\mathfrak{g}_1)$ and $D_{\mathfrak{g}_0}(\mathfrak{g}_1) = [\mathfrak{g}_1, \mathfrak{g}_1]$.*

(ii) *If $\mathfrak{g}$ is aspherical, then $Z(\mathfrak{g}_0) = St_{\mathfrak{g}_0}(\mathfrak{g}_1) \cap Z(\mathfrak{g}_0)$.*

Proof.

(i) Let $y \in \mathfrak{g}_1^{\mathfrak{g}_0}$. Since $\mathfrak{g}$ is simply connected, for every $x \in \mathfrak{g}_0$ there exists $y' \in \mathfrak{g}_1$ such that $\partial(y') = x$. Then $[x,y] = [\partial(y'),y] = 0$ and $[y',y] = 0$. So $y \in Z(\mathfrak{g}_1)$ i.e. $\mathfrak{g}_1^{\mathfrak{g}_0} \subseteq Z(\mathfrak{g}_1)$. Conversely, let $y \in Z(\mathfrak{g}_1)$. From the hypothesis, we have $[x,y] = [\partial(y'),y] = [y',y] = 0$ ($\because y \in Z(\mathfrak{g}_1)$). So $y \in \mathfrak{g}_1^{\mathfrak{g}_0}$ i.e. $Z(\mathfrak{g}_1) \subseteq \mathfrak{g}_1^{\mathfrak{g}_0}$. Let $[x,y] \in D_{\mathfrak{g}_0}(\mathfrak{g}_1)$. Again from the hypothesis, we can say that $[x,y] = [\partial(y'),y] = [y',y] \in [\mathfrak{g}_1,\mathfrak{g}_1]$. So we have $D_{\mathfrak{g}_0}(\mathfrak{g}_1) \subseteq [\mathfrak{g}_1,\mathfrak{g}_1]$. Let $[y,y'] \in [\mathfrak{g}_1,\mathfrak{g}_1]$. Then $[x,y] = [\partial(y'),y] \in D_{\mathfrak{g}_0}(\mathfrak{g}_1)$ i.e. $[\mathfrak{g}_1,\mathfrak{g}_1] \subseteq D_{\mathfrak{g}_0}(\mathfrak{g}_1)$.

(ii) Let $x \in Z(\mathfrak{g}_0)$. Then we have $\partial([x,y]) = [x,\partial(y)] = 0 = \partial(0)$. Since $\mathfrak{g}$ is aspherical, $[x,y] = 0$ i.e. $x \in St_{\mathfrak{g}_0}(\mathfrak{g}_1)$. So, we have $Z(\mathfrak{g}_0) \subseteq St_{\mathfrak{g}_0}(\mathfrak{g}_1)$ i.e. $Z(\mathfrak{g}_0) = St_{\mathfrak{g}_0}(\mathfrak{g}_1) \cap Z(\mathfrak{g}_0)$.

3 Isoclinic Lie crossed modules

In this section, we recall the notion of isoclinism among Lie crossed modules from [13, 14].

Definition 1. *[16] Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. $\mathfrak{g}$ and $\mathfrak{h}$ are isoclinic if there exist isomorphisms $\eta : \mathfrak{g}/Z(\mathfrak{g}) \longrightarrow \mathfrak{h}/Z(\mathfrak{h})$ and $\xi : [\mathfrak{g},\mathfrak{g}] \longrightarrow [\mathfrak{h},\mathfrak{h}]$ between central quotients and commutator subalgebras, respectively, such that the following diagram*

$$\begin{array}{ccc} \mathfrak{g}/Z(\mathfrak{g}) \times \mathfrak{g}/Z(\mathfrak{g}) & \xrightarrow{c_{\mathfrak{g}}} & [\mathfrak{g},\mathfrak{g}] \\ {\scriptstyle \eta\times\eta}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathfrak{h}/Z(\mathfrak{h}) \times \mathfrak{h}/Z(\mathfrak{h}) & \xrightarrow[c_{\mathfrak{h}}]{} & [\mathfrak{h},\mathfrak{h}] \end{array}$$

is commutative where $c_{\mathfrak{g}}, c_{\mathfrak{h}}$ are commutator maps of Lie algebras. The pair (η,ξ) is called an isoclinism from $\mathfrak{g}$ to $\mathfrak{h}$, and denoted by $(\eta,\xi) : \mathfrak{g} \sim \mathfrak{h}$.

Remark. As expected, isoclinism is an equivalence relation.

Example 2.
(1) All abelian Lie algebras are isoclinic to each other. The commutator maps and the pairs (η,ξ) consist of trivial homomorphisms.
(2) Every Lie algebra is isoclinic to a stem Lie algebra whose center is contained in its derived subalgebra.

Now we are going to define the notion of isoclinic Lie crossed modules.

Notation In the rest of the paper, for a given Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$, we denote $\mathfrak{g}/Z(\mathfrak{g})$ by $\overline{\mathfrak{g}_1} \xrightarrow{\overline{\partial}} \overline{\mathfrak{g}_0}$ where $\overline{\mathfrak{g}_1} = \mathfrak{g}_1/\mathfrak{g}_1^{\mathfrak{g}_0}$ and $\overline{\mathfrak{g}_0} = \mathfrak{g}_0/(St_{\mathfrak{g}_0}(\mathfrak{g}_1) \cap Z(\mathfrak{g}_0))$, for the sake of brevity. Next we introduce a new notion from [13, 14].

Definition 2. *The Lie crossed modules $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0$ and $\mathfrak{h} : \mathfrak{h}_1 \xrightarrow{\partial_{\mathfrak{h}}} \mathfrak{h}_0$ are isoclinic if there exist isomorphisms*

$$(\eta_1,\eta_0) : (\overline{\mathfrak{g}_1} \xrightarrow{\overline{\partial_{\mathfrak{g}}}} \overline{\mathfrak{g}_0}) \longrightarrow (\overline{\mathfrak{h}_1} \xrightarrow{\overline{\partial_{\mathfrak{h}}}} \overline{\mathfrak{h}_0})$$

and

$$(\xi_1,\xi_0) : (D_{\mathfrak{g}_0}(\mathfrak{g}_1) \xrightarrow{\partial_{\mathfrak{g}}|} [\mathfrak{g}_0,\mathfrak{g}_0]) \longrightarrow (D_{\mathfrak{h}_0}(\mathfrak{h}_1) \xrightarrow{\partial_{\mathfrak{h}}} [\mathfrak{h}_0,\mathfrak{h}_0])$$

such that the diagrams

$$\begin{array}{ccc} \overline{\mathfrak{g}_1}\times\overline{\mathfrak{g}_0} & \xrightarrow{\quad c_1 \quad} & D_{\mathfrak{g}_0}(\mathfrak{g}_1) \\ {\scriptstyle \eta_1\times\eta_0}\big\downarrow & & \big\downarrow{\scriptstyle \xi_1} \\ \overline{\mathfrak{h}_1}\times\overline{\mathfrak{h}_0} & \xrightarrow[\quad c_1{}' \quad]{} & D_{\mathfrak{h}_0}(\mathfrak{h}_1) \end{array} \tag{1}$$

and

$$\begin{array}{ccc} \overline{\mathfrak{g}_0}\times\overline{\mathfrak{g}_0} & \xrightarrow{\quad c_0 \quad} & \mathfrak{g}_0\wedge\mathfrak{g}_0 \\ {\scriptstyle \eta_0\times\eta_0}\big\downarrow & & \big\downarrow{\scriptstyle \xi_0} \\ \overline{\mathfrak{h}_0}\times\overline{\mathfrak{h}_0} & \xrightarrow[\quad c_0{}' \quad]{} & \mathfrak{h}_0\wedge\mathfrak{h}_0 \end{array} \tag{2}$$

are commutative where (c_1,c_0) and (c_1',c_0') are commutator maps, defined in Proposition 14 in [13], of the Lie crossed modules $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

The pair $((\eta_1,\eta_0),(\xi_1,\xi_0))$ is called an *isoclinism* from $\mathfrak{g}$ to $\mathfrak{h}$ and this situation is denoted by $((\eta_1,\eta_0),(\xi_1,\xi_0)):\mathfrak{g}\sim\mathfrak{h}$.

Example 3. (1) All Abelian Lie crossed modules (crossed modules that coincide with their center) are isoclinic. The pairs $((\eta_1,\eta_0),(\xi_1,\xi_0))$ consist of trivial homomorphisms.

(2) Let (η,ξ) be an isoclinism between the Lie algebras $\mathfrak{g}$ to $\mathfrak{h}$. Then $\mathfrak{g}\xrightarrow{id}\mathfrak{g}$ is isoclinic to $\mathfrak{h}\xrightarrow{id}\mathfrak{h}$ where $(\eta_1,\eta_0)=(\eta,\eta)$, $(\xi_1,\xi_0)=(\xi,\xi)$.

(3) Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ with $\mathfrak{h}+Z(\mathfrak{g})=\mathfrak{g}$. Then $\mathfrak{h}\xhookrightarrow{inc.}\mathfrak{g}$ is isoclinic to $\mathfrak{g}\xrightarrow{id}\mathfrak{g}$. Here (η_1,η_0) and (ξ_1,ξ_0) are defined by $(inc.,inc.)$, (id,id), respectively.

*Remark.*If Lie crossed modules $\mathfrak{g}$ and $\mathfrak{h}$ are simply connected or finite dimensional, then the commutativity of diagrams (1) and (2) in Definition 2 are equivalent to the commutativity of following diagram.

$$\begin{array}{ccc} \mathfrak{g}/Z\wedge(\mathfrak{g})\times\mathfrak{g}/Z\wedge(\mathfrak{g}) & \xrightarrow{\qquad} & \mathfrak{g}\wedge\mathfrak{g} \\ {\scriptstyle (\eta_1,\eta_0)\times(\eta_1,\eta_0)}\big\downarrow & & \big\downarrow{\scriptstyle (\xi_1,\xi_0)} \\ \mathfrak{h}/Z\wedge(\mathfrak{h})\times\mathfrak{h}/Z\wedge(\mathfrak{h}) & \xrightarrow{\qquad} & \mathfrak{h}\wedge\mathfrak{h} \end{array}$$

Proposition 2. *Let $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_\mathfrak{g}}\mathfrak{g}_0$ be a Lie crossed module and $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_\mathfrak{g}|}\mathfrak{h}_0$ be its Lie subcrossed module. If $\mathfrak{g}=\mathfrak{h}+Z(\mathfrak{g})$, i.e. $\mathfrak{g}_1=\mathfrak{h}_1+\mathfrak{g}_1^{\mathfrak{g}_0}$ and $\mathfrak{g}_0=\mathfrak{h}_0+(St_{\mathfrak{g}_0}(\mathfrak{g}_1)\cap Z(\mathfrak{g}_0))$, then $\mathfrak{g}$ is isoclinic to $\mathfrak{h}$.*

Proof. See [14] for details.

Remark. If $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_\mathfrak{g}|}\mathfrak{h}_0$ is a finite dimensional Lie crossed module, then the converse of Proposition 2 is true.

Proposition 3. *Let $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_\mathfrak{g}}\mathfrak{g}_0$ and $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_\mathfrak{h}}\mathfrak{h}_0$ be isoclinic crossed modules.*

(i) *If $\mathfrak{g}$ and $\mathfrak{h}$ are aspherical, then $\mathfrak{g}_0$ and $\mathfrak{h}_0$ are isoclinic Lie algebras.*

(ii) *If $\mathfrak{g}$ and $\mathfrak{h}$ are simply connected, then $\mathfrak{g}_1$ and $\mathfrak{h}_1$ are isoclinic Lie algebras.*

Proof. Let $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_\mathfrak{g}}\mathfrak{g}_0$ and $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_\mathfrak{h}}\mathfrak{h}_0$ be isoclinic Lie crossed modules. Then we have the isomorphisms

$$(\eta_1,\eta_0):(\overline{\mathfrak{g}_1}\xrightarrow{\overline{\partial_\mathfrak{g}}}\overline{\mathfrak{g}_0})\longrightarrow(\overline{\mathfrak{h}_1}\xrightarrow{\overline{\partial_\mathfrak{h}}}\overline{\mathfrak{h}_0})$$

$$(\xi_1,\xi_0):(D_{\mathfrak{g}_0}(\mathfrak{g}_1)\xrightarrow{\partial_\mathfrak{g}|}[\mathfrak{g}_0,\mathfrak{g}_0])\longrightarrow(D_{\mathfrak{h}_0}(\mathfrak{h}_1)\xrightarrow{\partial_\mathfrak{h}}[\mathfrak{h}_0,\mathfrak{h}_0])$$

which makes diagrams (1) and (2) commutative.

(i) Since $\mathfrak{g}$ and $\mathfrak{h}$ are aspherical, we have $Z(\mathfrak{g}_0) \subseteq St_{\mathfrak{g}_0}(\mathfrak{g}_1)$, $Z(\mathfrak{h}_0) \subseteq St_{\mathfrak{h}_0}(\mathfrak{h}_1)$. Consequently, η_0 is an isomorphism between $\mathfrak{g}_0/Z(\mathfrak{g}_0)$ and $\mathfrak{h}_0/Z(\mathfrak{h}_0)$. So the pair (η_0, ξ_0) is an isoclinism from $\mathfrak{g}_0$ to $\mathfrak{h}_0$.

(ii) Since $\mathfrak{g}$ and $\mathfrak{h}$ are simply connected, we have $\mathfrak{g}_1^{\mathfrak{g}_0} = Z(\mathfrak{g}_1)$, $\mathfrak{h}_1^{\mathfrak{h}_0} = Z(\mathfrak{h}_1)$, $D_{\mathfrak{g}_0}(\mathfrak{g}_1) = [\mathfrak{g}_1, \mathfrak{g}_1]$ and $D_{\mathfrak{h}_0}(\mathfrak{h}_1) = [\mathfrak{h}_1, \mathfrak{h}_1]$. So we have the isomorphisms $\eta_1 : \mathfrak{g}_1/Z(\mathfrak{g}_1) \longrightarrow \mathfrak{h}_1/Z(\mathfrak{h}_1)$, $\xi_1 : [\mathfrak{g}_1, \mathfrak{g}_1] \longrightarrow [\mathfrak{h}_1, \mathfrak{h}_1]$. The pair (η_1, ξ_1) is an isoclinism from $\mathfrak{g}_1$ to $\mathfrak{h}_1$, as required.

Proposition 4. *Let* $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}_\mathfrak{g}} \mathfrak{g}_0$ *and* $\mathfrak{h} : \mathfrak{h}_1 \xrightarrow{\mathfrak{d}_\mathfrak{h}} \mathfrak{h}_0$ *be isoclinic finite dimensional Lie crossed modules. Then* $\mathfrak{g}_1$ *and* $\mathfrak{g}_0$ *are isoclinic to* $\mathfrak{h}_1$ *and* $\mathfrak{h}_0$*, respectively.*

*Proof.*Let $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}_\mathfrak{g}} \mathfrak{g}_0$ and $\mathfrak{h} : \mathfrak{h}_1 \xrightarrow{\mathfrak{d}_\mathfrak{h}} \mathfrak{h}_0$ be isoclinic finite Lie crossed modules. Then we have the crossed module isomorphisms

$$(\eta_1, \eta_0) : (\overline{\mathfrak{g}_1} \xrightarrow{\overline{\mathfrak{d}_\mathfrak{g}}} \overline{\mathfrak{g}_0}) \longrightarrow (\overline{\mathfrak{h}_1} \xrightarrow{\overline{\mathfrak{d}_\mathfrak{h}}} \overline{\mathfrak{h}_0})$$
$$(\xi_1, \xi_0) : (D_{\mathfrak{g}_0}(\mathfrak{g}_1) \xrightarrow{\mathfrak{d}_\mathfrak{g}|} [\mathfrak{g}_0, \mathfrak{g}_0]) \longrightarrow (D_{\mathfrak{h}_0}(\mathfrak{h}_1) \xrightarrow{\mathfrak{d}_\mathfrak{h}} [\mathfrak{h}_0, \mathfrak{h}_0])$$

which makes diagrams (1) and (2) commutative. Since $\mathfrak{g}_1$ and $\mathfrak{h}_1$ are finite dimensional, the restriction $\xi_1| : [\mathfrak{g}_1, \mathfrak{g}_1] \longrightarrow [\mathfrak{h}_1, \mathfrak{h}_1]$ is also an isomorphism. Similarly, we have the isomorphisms $\eta_1' : \mathfrak{g}_1/Z(\mathfrak{g}_1) \longrightarrow \mathfrak{h}_1/Z(\mathfrak{h}_1), \eta_1'(aZ(\mathfrak{g}_1)) = a'Z(\mathfrak{h}_1)$, $\eta_0' : \mathfrak{g}_0/Z(\mathfrak{g}_0) \longrightarrow \mathfrak{h}_0/Z(\mathfrak{h}_0), \eta_0'(bZ(\mathfrak{g}_0)) = b'Z(\mathfrak{h}_0)$, and ξ_0 which make $\mathfrak{g}_1$ and $\mathfrak{g}_0$ isoclinic to $\mathfrak{h}_1$ and $\mathfrak{h}_0$, respectively.

4 Class preserving actor of isoclinic Lie crossed modules

In this section, first we recall the definition of actor of Lie crossed module and then define the class preserving actor for any Lie crossed module. Using these definitions, we can get a relation such that the class preserving actors of isoclinic Lie crossed modules are isomorphic.

Suppose $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$ is a Lie crossed module. A *derivation* from $\mathfrak{g}_0$ to $\mathfrak{g}_1$ is the $\mathbb{K}$-linear function $\partial : \mathfrak{g}_0 \longrightarrow \mathfrak{g}_1$ such that

$$\partial[x, x'] = [x, \partial(x')] - [x', \partial(x)],$$

for all $x, x' \in \mathfrak{g}_0$. The set of all derivations is denoted by $Der(\mathfrak{g}_0, \mathfrak{g}_1)$. $Der(\mathfrak{g}_0, \mathfrak{g}_1)$ is a $\mathbb{K}$-Lie algebra with the bracket $[\partial_1, \partial_2] = \partial_1(\mathfrak{d}\partial_2) - \partial_2(\mathfrak{d}\partial_1)$, for all ∂_1, $\partial_2 \in Der(\mathfrak{g}_0, \mathfrak{g}_1)$. Furthermore, *a derivation of a Lie crossed module* $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$ is a pair (α, β) where $\alpha \in Der(\mathfrak{g}_1)$ and $\beta \in Der(\mathfrak{g}_0)$ such that:

$$\begin{array}{ll} (i) & \alpha \in Der(\mathfrak{g}_1),\ \beta \in Der(\mathfrak{g}_0) \\ (ii) & \beta\mathfrak{d} = \mathfrak{d}\alpha \\ (iii) & \alpha([x,y]) = [x, \alpha(y)] + [\beta(x), y] \end{array}$$

for all $x \in \mathfrak{g}_0$ and $y \in \mathfrak{g}_1$. The set of derivations of the Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$ is denoted by $Der(\mathfrak{g})$.

For a given Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\mathfrak{d}} \mathfrak{g}_0$, we have the corresponding crossed module

$$\begin{array}{rcl} \Delta : Der(\mathfrak{g}_0, \mathfrak{g}_1) & \longrightarrow & Der(\mathfrak{g}) \\ \partial & \longmapsto & (\partial\mathfrak{d}, \mathfrak{d}\partial) \end{array}$$

with the action of $Der(\mathfrak{g})$ on $Der(\mathfrak{g}_0, \mathfrak{g}_1)$ given by $(\alpha, \beta) \cdot \partial = \alpha\partial - \partial\beta$, for all $(\alpha, \beta) \in Der(\mathfrak{g})$, $\partial \in Der(\mathfrak{g}_0, \mathfrak{g}_1)$. This crossed module is called the *actor* of $\mathfrak{g}$, and denoted by $Act(\mathfrak{g})$.

As indicated in [18], analogously [23], the actors of isoclinic crossed modules (of groups) are not isomorphic but their class preserving actors are isomorphic. Now, we obtain a similar result for Lie crossed modules. For this, first we define the class preserving actor of any Lie crossed module as follows:

Proposition 5. *Given any Lie crossed module $\mathfrak{g}:\mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$, $Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)=\{\delta\in Der(\mathfrak{g}_0,y)\mid$ there exists $y\in\mathfrak{g}_1$ such that $\delta(x)=[-x,y]$, for all $x\in\mathfrak{g}_0\}$ and $Der_{\mathscr{C}}(\mathfrak{g})=\{(\alpha,\beta)\in Der(\mathfrak{g})\mid$ there exists $x\in\mathfrak{g}_0$ such that $\alpha(y)=[x,y]$, $\beta(x')=[x,x']$, for all $y\in\mathfrak{g}_1, x'\in\mathfrak{g}_0\}$. Then, we obtain the following:*
(a) $Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$ is a Lie subalgebra of $Der(\mathfrak{g}_0,\mathfrak{g}_1)$.
(b) $Der_{\mathscr{C}}(\mathfrak{g})$ is a Lie subalgebra of $Der(\mathfrak{g})$.

Proof. (a) Let $\delta,\delta'\in Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$. We first show that $[\delta,\delta']\in Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$. Since $\delta,\delta'\in Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$, there exist $y,y'\in\mathfrak{g}_1$ such that $\delta(x)=[-x,y]$ and $\delta'(x)=[-x,y']$, for all $x\in\mathfrak{g}_0$. Then,

$$\begin{aligned}
[\delta,\delta'](x) &= \delta(\partial\delta')(x)-\delta'(\partial\delta)(x)\\
&= \delta\partial(\delta'(x))-\delta'\partial(\delta(x))\\
&= \delta\partial([-x,y'])-\delta'\partial([-x,y])\\
&= \delta([-x,\partial(y')])-\delta'([-x,\partial(y)])\ (\partial\sim\text{crossed module})\\
&= [(-x),\delta(\partial(y'))]-[\partial(y'),\delta(-x)]-([(-x),\delta'(\partial(y))]-[\partial(y),\delta'(-x)])\\
&= [-x,[-\partial(y'),y]]-[\partial(y'),[x,y]]-[(-x),[-\partial(y),y']]+[\partial(y),[x,y']]\\
&= [x,[\partial(y'),y]-[\partial(y'),y]-[\partial(y),y']+[\partial(y),y']]\\
&= [x,0]\\
&= [-x,0]
\end{aligned}$$

i.e. $[\delta,\delta']\in Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$. On the other hand, for any $\delta,\delta'\in Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$ we can get $\delta-\delta'\in Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$ and $(k\delta)\in Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$.
(b) By a similar way, we have $[(\alpha,\beta),(\alpha',\beta')]$, $(\alpha,\beta)-(\alpha',\beta')$, $k(\alpha,\beta)\in Der_{\mathscr{C}}(\mathfrak{g})$ for all (α,β), $(\alpha',\beta')\in Der_{\mathscr{C}}(\mathfrak{g})$, $k\in\mathbb{K}$, as required.

Proposition 6.*Suppose $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial}\mathfrak{g}_0$ is a Lie crossed module. $Act_{\mathscr{C}}(\mathfrak{g}):Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)\xrightarrow{\Delta_{\mathscr{C}}} Der_{\mathscr{C}}(\mathfrak{g})$ is a crossed module with the action induced from the action of $Der(\mathfrak{g})$ over $Der(\mathfrak{g}_0,\mathfrak{g}_1)$ such that*

$$\begin{aligned}
Der_{\mathscr{C}}(\mathfrak{g})\times Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1) &\longrightarrow Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)\\
((\alpha,\beta),\delta) &\longmapsto (\alpha,\beta)\cdot\delta=\alpha\delta-\delta\beta.
\end{aligned}$$

Proof. It can be shown by a direct calculation.

Definition 3. *The crossed module*

$$Act_{\mathscr{C}}(\mathfrak{g}):Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)\xrightarrow{\Delta_{\mathscr{C}}} Der_{\mathscr{C}}(\mathfrak{g})$$

defined in Proposition 6 is called the class preserving actor of $\mathfrak{g}$ and is denoted by $Act_{\mathscr{C}}(\mathfrak{g})$ for a Lie crossed module $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial}\mathfrak{g}_0$.

Theorem 1. *Let $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_{\mathfrak{g}}}\mathfrak{g}_0$ and $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_{\mathfrak{h}}}\mathfrak{h}_0$ be isoclinic Lie crossed modules. Then, we have $Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)\cong Der_{\mathscr{C}}(\mathfrak{h}_0,\mathfrak{h}_1)$.*

*Proof.*We have the following isomorphism, such that

$$\begin{aligned}
\phi: Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1) &\longrightarrow Der_{\mathscr{C}}(\mathfrak{h}_0,\mathfrak{h}_1)\\
\delta &\longmapsto \lambda.
\end{aligned}$$

Since $\mathfrak{g}$ and $\mathfrak{h}$ are isoclinic, we have the isomorphisms

$$(\eta_1,\eta_0):(\overline{\mathfrak{g}_1}\xrightarrow{\overline{\partial_{\mathfrak{g}}}}\overline{\mathfrak{g}_0})\longrightarrow(\overline{\mathfrak{h}_1}\xrightarrow{\overline{d_{\mathfrak{h}}}}\overline{\mathfrak{h}_0})$$
$$(\xi_1,\xi_0):(D_{\mathfrak{g}_0}(\mathfrak{g}_1)\xrightarrow{\partial_{\mathfrak{g}}|}[\mathfrak{g}_0,\mathfrak{g}_0])\longrightarrow(D_{\mathfrak{h}_0}(\mathfrak{h}_1)\xrightarrow{\partial_{\mathfrak{h}}|}[\mathfrak{h}_0,\mathfrak{h}_0])$$

which make the diagrams (1) and (2) commutative. Let $\delta\in Der_{\mathscr{C}}(\mathfrak{g}_0,\mathfrak{g}_1)$, $a\in\mathfrak{h}_0-(St_{\mathfrak{h}_0}(\mathfrak{h}_1)\cap Z(\mathfrak{h}_0))$ and $\overline{a}\in\overline{\mathfrak{h}_0}$. Define $\overline{x}=\eta_0^{-1}(\overline{a})$. Since $x\in\mathfrak{g}_0$, there exists an element $y\in\mathfrak{g}_1$ such that $\delta_{\mathfrak{g}}(x)=[-x,y]$. Let $\eta_1(\overline{y})=\overline{y'}$. Defining

$$\lambda(a)=\begin{cases}[-a,y'] & a\in\mathfrak{h}_0-(St_{\mathfrak{h}_0}(\mathfrak{h}_1)\cap Z(\mathfrak{h}_0))\\ 0 & a\in St_{\mathfrak{h}_0}(\mathfrak{h}_1)\cap Z(\mathfrak{h}_0),\end{cases}$$

the result follows.

Proposition 7. *If $\mathfrak{g}$ and $\mathfrak{h}$ are finite dimensional non-abelian isoclinic Lie crossed modules, then $Der_{\mathscr{C}}(\mathfrak{g})\cong Der_{\mathscr{C}}(\mathfrak{h})$.*

Proof. See [22], for details.

Corollary 1. *Let $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_{\mathfrak{g}}}\mathfrak{g}_0$ and $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_{\mathfrak{h}}}\mathfrak{h}_0$ be two finite dimensional non-abelian isoclinic Lie crossed modules. Then $Act_{\mathscr{C}}(\mathfrak{g})\cong Act_{\mathscr{C}}(\mathfrak{h})$.*

*Proof.*Follows from Theorem 1 and Proposition 7.

5 Commutativity degree of isoclinic Lie crossed modules

In this section, first we define the commutativity degree of Lie algebras analogous to groups. Then we get a definition of commutativity degree of Lie crossed modules. Consequently, we give a relation of commutativity degrees of isoclinic Lie crossed modules.

Recall that, given any finite Lie algebra $\mathfrak{g}$, $\mathfrak{g}$ is a *commutative* Lie algebra if $[x,y]=0$ for $x,y\in\mathfrak{g}$.

It is well known that the probability of the event that two elements of G commute is called *commutativity degree* and denoted by $d(G)$. We can define commutativity degree of Lie algebras analogously as follows:

Definition 4. *The probability of the event that two elements of $\mathfrak{g}$ commute is called commutativity degree of $\mathfrak{g}$ and denoted by $d(\mathfrak{g})$. Formally, this probability can be calculated by*

$$d(\mathfrak{g})=\frac{|\{(x,y)\in\mathfrak{g}\times\mathfrak{g}\mid[x,y]=0\}|}{|\mathfrak{g}|^2}.$$

Obviously, $\mathfrak{g}$ is Abelian if and only if $d(\mathfrak{g})=1$.

In order to define the commutativity degree of a Lie crossed module, we need to calculate the probability of the event that a randomly chosen pair of Lie crossed modules between Lie algebras $\mathfrak{g}_1$and $\mathfrak{g}_0$ commute.

Definition 5. *Let $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial}\mathfrak{g}_0$ be a finite Lie crossed module. The commutativity degree of $\mathfrak{g}$ is defined by*

$$d(\mathfrak{g})=\left[\frac{|\{(x,y)\in\mathfrak{g}_0\times\mathfrak{g}_1\mid[x,y]=y\}|}{|\mathfrak{g}_0|\,|\mathfrak{g}_1|},\frac{|\{(x,x')\in\mathfrak{g}_0\times\mathfrak{g}_0\mid[x,x']=0\}|}{|\mathfrak{g}_0|\,|\mathfrak{g}_0|}\right].$$

Now, we will show that isoclinic Lie crossed modules have the same commutativity degree.

Theorem 2. *If* $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial}\mathfrak{g}_0$ *and* $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_\mathfrak{h}}\mathfrak{h}_0$ *are two isoclinic finite crossed modules, then* $d(\mathfrak{g})=d(\mathfrak{h}')$.

Proof. Suppose that $\mathfrak{g}$ and $\mathfrak{h}$ are isoclinic. From Proposition 12 in [14], $\mathfrak{g}_1$ and $\mathfrak{g}_0$ are isoclinic to $\mathfrak{h}_1$ and $\mathfrak{h}_0$, respectively. So, we can get $d(\mathfrak{g}_1)=d(\mathfrak{h}_1)$ and $d(\mathfrak{g}_0)=d(\mathfrak{h}_0)$. Furthermore, we have

$$\frac{|\mathfrak{g}_1\times\mathfrak{g}_0|}{|\mathfrak{g}_1{}^{\mathfrak{g}_0}\times A|}\frac{|\{(x,y)\in\mathfrak{g}_0\times\mathfrak{g}_1\mid[x,y]=y\}|}{|\mathfrak{g}_0\times\mathfrak{g}_1|}=\frac{1}{|\mathfrak{g}_1{}^{\mathfrak{g}_0}\times A|}\left|\{(x,y)\in\mathfrak{g}_0\times\mathfrak{g}_1\mid[x,y]=y\}\right|$$

$$=\frac{1}{\left|\mathfrak{g}_1^{\mathfrak{g}_0}\times A\right|}\left|\{(x,y)\in\mathfrak{g}_0\times\mathfrak{g}_1\mid c_1(y\mathfrak{g}_1^{\mathfrak{g}_0},xA)=1\}\right|$$

$$=\left|\left\{(\alpha,\beta)\in\frac{\mathfrak{g}_1}{\mathfrak{g}_1^{\mathfrak{g}_0}}\times\frac{\mathfrak{g}_0}{A}\middle|\xi_1(c_1(\alpha,\beta))=1\right\}\right|(\because\xi_1\text{ iso.})$$

$$=\left|\left\{(\alpha,\beta)\in\frac{\mathfrak{g}_1}{\mathfrak{g}_1^{\mathfrak{g}_0}}\times\frac{\mathfrak{g}_0}{A}\middle|c_1'(\eta_1\times\eta_0)(\alpha,\beta)=1\right\}\right|(\because com.diag)$$

$$=\left|\left\{(\alpha,\beta)\in\frac{\mathfrak{g}_1}{\mathfrak{g}_1^{\mathfrak{g}_0}}\times\frac{\mathfrak{g}_0}{A}\middle|c_1'(\eta_1(\alpha),\eta_0(\beta))=1\right\}\right|$$

$$=\left|\left\{(\gamma,\delta)\in\frac{\mathfrak{h}_1}{\mathfrak{h}_1^{\mathfrak{h}_0}}\times\frac{\mathfrak{h}_0}{A'}\middle|c_1'(\gamma,\delta)=1\right\}\right|(\because\eta_1,\eta_0\text{ iso.})$$

$$=\frac{1}{\left|\mathfrak{h}_1^{\mathfrak{h}_0}\times A'\right|}\left|\{(x',y')\in R'\times S'\mid[x',y']=y'\}\right|$$

where $A=St_{\mathfrak{g}_0}(\mathfrak{g}_1)\cap Z(\mathfrak{g}_0)$ and $A'=St_{\mathfrak{h}_0}(\mathfrak{h}_1)\cap Z(\mathfrak{h}_0)$. Since $\mathfrak{g}_1/\mathfrak{g}_1^{\mathfrak{g}_0}\cong\mathfrak{h}_1/\mathfrak{h}_1^{\mathfrak{h}_0}$ and $\mathfrak{g}_0/A\cong\mathfrak{h}_0/A'$, we can write

$$\frac{|\mathfrak{g}_1\times\mathfrak{g}_0|}{\left|\mathfrak{g}_1^{\mathfrak{g}_0}\times A\right|}=\frac{|\mathfrak{h}_1\times\mathfrak{h}_0|}{\left|\mathfrak{h}_1^{\mathfrak{h}_0}\times A'\right|}.$$

That is $d(\mathfrak{g})=d(\mathfrak{h})$.

Corollary 2. *Let* $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial}\mathfrak{h}_0$ *be a Lie subcrossed module of* $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial}\mathfrak{g}_0$ *and* $\mathfrak{g}=\mathfrak{h}+Z(\mathfrak{g})$. *Then* $d(\mathfrak{g})=d(\mathfrak{h})$.

Proof. It is clear from Proposition 9 of [14].

We can generalize Theorem 2 by using the analogous definition of n-th commutativity degree defined in [4] and the notion n-isoclinic Lie crossed module given in [14]. Now, we define the n-th commutativity degree of a Lie crossed module.

Definition 6. *Given a Lie crossed module* $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_\mathfrak{g}}\mathfrak{g}_0$, *the n-th commutativity degree of* $\mathfrak{g}$ *is defined by*

$$d_n(\mathfrak{g})=\left[\frac{\frac{\left|\{(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n)\in\mathfrak{g}_0^n\times\mathfrak{g}_1^n\mid[(x_1,x_2,\dots,x_n),(y_1,y_2,\dots,y_n)]=(y_1,y_2,\dots,y_n)\}\right|}{|\mathfrak{g}_0|^n|\mathfrak{g}_1|^n}}{\frac{\left|\left\{(x_1,x_2,\dots,x_n,x_{n+1})\in\mathfrak{g}_0^{n+1}\mid[x_i,x_j]=0\text{ where }i,j\in\{1,2,\dots,n+1\}\right\}\right|}{|\mathfrak{g}_0|^{n+1}}},\right]$$

.

Corollary 3. *If* $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_\mathfrak{g}}\mathfrak{g}_0$ *and* $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_\mathfrak{h}}\mathfrak{h}_0$ *are two finite n -th isoclinic finite crossed modules, then* $d_n(\mathfrak{g})=d_n(\mathfrak{h})$.

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