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n-exterior isoclinic Lie crossed modules

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Abstract: Exterior isoclinism gives an alternative classification method for finite Lie algebras and Lie crossed modules. In this paper, we define the notions of exterior isoclinism of Lie algebras and Lie crossed modules. Also, we generalize these notions for n-th dimension.

Keywords: Isoclinism, Exterior isoclinism, Exterior degree, Lie algebras, Lie crossed modules.

1 Introduction

The notion of isoclinism was firstly given in [10]. This notion is a classification method for finite groups and also an equivalence relation weaker than isomorphism. The main idea of isoclinism is that isomorphism of central quotients and commutator rather than isomorphism of whole groups. This notion was studied in [2,9,13,20]. On the other hand, Lie algebra version of isoclinism was firstly introduced by Moneyhun in [14].

Crossed modules were defined by Whitehead [23] and considered as 2-dimensional group in [3]. For details about this notion, one can see [16,19]. The main notion of this paper is Lie crossed modules which were introduced in [4,5,6]. Some properties of Lie crossed modules can be found in [1,22]. However, isoclinism of this notion, that is isoclinic Lie crossed modules and n-isoclinic Lie crossed modules were studied in [11,12].

Given two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$, the nonabelian exterior product is the quotient of the nonabelian tensor product $\mathfrak{a} \star \mathfrak{b}$ by the elements of the form $c * c$, where $c \in \mathfrak{a} \cap \mathfrak{b}$. This makes sense in Lie theory since these are elements of the kernel of the homomorphism $\mathfrak{g} \star \mathfrak{g} \to \mathfrak{g}$, $g \star g\prime \longmapsto [g, g\prime]$ for all $g, g\prime \in \mathfrak{g}$. In this paper, we define a new equivalence relation called exterior isoclinism similar to isoclinism using the nonabelian exterior product, exterior square and exterior center of Lie algebras defined in [7]. Also, we generalize this notion by using Lie crossed modules. The group version of these notions can be found in [8,17,18].

In order to our purpose, we plan this paper as follows:

In Section 2, we recall some needed notions about isoclinism and lie crossed modules. Also, give some fundamental properties of these notions.

In Section 3, we introduce some new definitions such as exterior isoclinism, exterior degree and n-exterior isoclinism of Lie algebras.

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In Section 4, we define the exterior and n-exterior isoclinism of Lie crossed modules. Also, we obtain the relation between exterior isoclinic Lie crossed modules and exterior isoclinic Lie algebras.

2 Preliminaries

In this section, we recall the notions of isoclinism for Lie algebras and Lie crossed modules.
We fix a field $\mathbb{K}$ and assume all Lie algebras to be over $\mathbb{K}$.

2.1 Isoclinism of Lie algebras

Definition 1. *[14] Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. $\mathfrak{g}$ and $\mathfrak{h}$ are isoclinic if there exist isomorphisms $\eta : \mathfrak{g}/Z(\mathfrak{g}) \longrightarrow \mathfrak{h}/Z(\mathfrak{h})$ and $\xi : [\mathfrak{g},\mathfrak{g}] \longrightarrow [\mathfrak{h},\mathfrak{h}]$ between central quotients and commutator subalgebras, respectively, such that the following diagram*

$$\begin{array}{ccc} \mathfrak{g}/Z(\mathfrak{g}) \times \mathfrak{g}/Z(\mathfrak{g}) & \xrightarrow{c_{\mathfrak{g}}} & [\mathfrak{g},\mathfrak{g}] \\ {\scriptstyle \eta\times\eta}\downarrow & & \downarrow{\scriptstyle \xi} \\ \mathfrak{h}/Z(\mathfrak{h}) \times \mathfrak{h}/Z(\mathfrak{h}) & \xrightarrow[c_{\mathfrak{h}}]{} & [\mathfrak{h},\mathfrak{h}] \end{array}$$

is commutative where $c_{\mathfrak{g}}, c_{\mathfrak{h}}$ are commutator maps of Lie algebras. The pair (η,ξ) is called an isoclinism from $\mathfrak{g}$ to $\mathfrak{h}$, and denoted by $(\eta,\xi) : \mathfrak{g} \sim \mathfrak{h}$.

Remark. As expected, isoclinism is an equivalence relation.

Example 1 *(1) All abelian Lie algebras are isoclinic to each other. The commutator maps and the pairs (η,ξ) consist of trivial homomorphisms.*
(2) Every Lie algebra is isoclinic to a stem Lie algebra whose center is contained in its commutator subalgebra.

2.2 Isoclinism of Lie crossed modules

Firstly, we recall the some preliminaries of Lie crossed modules from [4,5,6].

A Lie crossed module consists of a Lie algebra homomorphism

$$\partial : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$$

with a Lie action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, written $(x,y) \longmapsto [x,y]$, satisfying the following conditions:

$$\begin{aligned} 1)\ \partial([x,y]) &= [x,\partial(y)], \\ 2)\ [\partial(y),y'] &= [y,y'], \end{aligned}$$

for all $x \in \mathfrak{g}_0$, $y,y' \in \mathfrak{g}_1$. We will denote such a Lie crossed module by $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$.

Example 2

(1) $\mathfrak{g} \xrightarrow{ad} Der(\mathfrak{g})$ *is a Lie crossed module for any Lie algebra* $\mathfrak{g}$.

(2) If $\mathfrak{h}$ *is an ideal of* $\mathfrak{g}$*, then* $\mathfrak{g}$ *acts on* $\mathfrak{h}$ *via adjoint representation and* $\mathfrak{h} \xhookrightarrow{inc.} \mathfrak{g}$ *is a Lie crossed module.*

(3) $\mathfrak{h} \xrightarrow{0} \mathfrak{g}$ *is a crossed module, where* $\mathfrak{h}$ *is a* $\mathfrak{g}$*-module.*

A morphism between Lie crossed modules $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$ and $\mathfrak{g}' : \mathfrak{g}'_1 \xrightarrow{\partial'} \mathfrak{g}'_0$ is a pair (α, β) of Lie algebra homomorphisms $\alpha : \mathfrak{g}_1 \longrightarrow \mathfrak{g}'_1$, $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{g}'_0$ such that $\beta\partial = \partial'\alpha$ and $\alpha([x,y]) = [\beta(x), \alpha(y)]$, for all $x \in \mathfrak{g}_0$, $y \in \mathfrak{g}_1$. So, we get a category **XLie** whose objects are the Lie crossed modules. A Lie crossed module $\mathfrak{h} : \mathfrak{h}_1 \xrightarrow{\partial_{\mathfrak{h}}} \mathfrak{h}_0$ is a *subcrossed module* of a crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0$ if $\mathfrak{h}_1, \mathfrak{h}_0$ are Lie subalgebras of $\mathfrak{g}_1$, $\mathfrak{g}_0$, respectively, $\partial_{\mathfrak{h}} = \partial_{\mathfrak{g}}|_{\mathfrak{h}_1}$ and the action of $\mathfrak{h}_0$ on $\mathfrak{h}_1$ is induced from the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$. If $\mathfrak{h}_1$and $\mathfrak{h}_0$ are ideals of $\mathfrak{g}_1$ and $\mathfrak{g}_0$, respectively, $[x,y'] \in \mathfrak{h}_1$ and $[x',y] \in \mathfrak{h}_1$, for all $x \in \mathfrak{g}_0$, $y \in \mathfrak{g}_1$, $x' \in \mathfrak{h}_0$, $y' \in \mathfrak{h}_1$ then $\mathfrak{h}$ is called an *ideal* of $\mathfrak{g}$. Hence, we get the *quotient crossed module* $\mathfrak{g}/\mathfrak{h} : \mathfrak{g}_1/\mathfrak{h}_1 \xrightarrow{\bar{\partial}} \mathfrak{g}_0/\mathfrak{h}_0$.

Given any Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$, the *center* of $\mathfrak{g}$ is the Lie crossed module $Z(\mathfrak{g}) : \mathfrak{g}_1^{\mathfrak{g}_0} \xrightarrow{\partial|} (St_{\mathfrak{g}_0}(\mathfrak{g}_1) \cap Z(\mathfrak{g}_0))$ where

$$\mathfrak{g}_1^{\mathfrak{g}_0} = \{y \in \mathfrak{g}_1 : \ [x,y] = 0, \text{ for all } x \in \mathfrak{g}_0\}$$

and

$$St_{\mathfrak{g}_0}(\mathfrak{g}_1) = \{x \in \mathfrak{g}_0 : \ [x,y] = 0, \text{ for all } y \in \mathfrak{g}_1\}.$$

The *commutator subcrossed module* $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is defined by

$$[\mathfrak{g}, \mathfrak{g}] : D_{\mathfrak{g}_0}(\mathfrak{g}_1) \xrightarrow{\partial} [\mathfrak{g}_0, \mathfrak{g}_0]$$

where $D_{\mathfrak{g}_0}(\mathfrak{g}_1) = \{[x,y] : x \in \mathfrak{g}_0, y \in \mathfrak{g}_1\}$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is the commutator subalgebra of $\mathfrak{g}_0$.

A Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$ is called *aspherical* if $\ker(\partial) = 0$ and *simply connected* if $\mathrm{coker}(\partial) = 0$.

A Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$ is called *finite dimensional* if $\mathfrak{g}_1$ and $\mathfrak{g}_0$ are finite dimensional Lie algebras.

Now we are going to define the notion of isoclinic Lie crossed modules from [11,12].

Notation In the rest of the paper, for a given Lie crossed module $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial} \mathfrak{g}_0$, we denote $\mathfrak{g}/Z(\mathfrak{g})$ by $\overline{\mathfrak{g}_1} \xrightarrow{\bar{\partial}} \overline{\mathfrak{g}_0}$ where $\overline{\mathfrak{g}_1} = \mathfrak{g}_1/\mathfrak{g}_1^{\mathfrak{g}_0}$ and $\overline{\mathfrak{g}_0} = \mathfrak{g}_0/(St_{\mathfrak{g}_0}(\mathfrak{g}_1) \cap Z(\mathfrak{g}_0))$, for shortness.

Definition 2. *Given any Lie crossed modules* $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0$ *and* $\mathfrak{h} : \mathfrak{h}_1 \xrightarrow{\partial_{\mathfrak{h}}} \mathfrak{h}_0$ *are isoclinic if there exist isomorphisms*

$$(\eta_1, \eta_0) : (\overline{\mathfrak{g}_1} \xrightarrow{\overline{\partial_{\mathfrak{g}}}} \overline{\mathfrak{g}_0}) \longrightarrow (\overline{\mathfrak{h}_1} \xrightarrow{\overline{\partial_{\mathfrak{h}}}} \overline{\mathfrak{h}_0})$$

and

$$(\xi_1, \xi_0) : (D_{\mathfrak{g}_0}(\mathfrak{g}_1) \xrightarrow{\partial_{\mathfrak{g}}} [\mathfrak{g}_0, \mathfrak{g}_0]) \longrightarrow (D_{\mathfrak{h}_0}(\mathfrak{h}_1) \xrightarrow{\partial_{\mathfrak{h}}} [\mathfrak{h}_0, \mathfrak{h}_0])$$

such that the diagrams

$$\begin{array}{ccc} \overline{\mathfrak{g}_1} \times \overline{\mathfrak{g}_0} & \xrightarrow{c_1} & D_{\mathfrak{g}_0}(\mathfrak{g}_1) \\ \downarrow{\scriptstyle \eta_1 \times \eta_0} & & \downarrow{\scriptstyle \xi_1} \\ \overline{\mathfrak{h}_1} \times \overline{\mathfrak{h}_0} & \xrightarrow[c_1']{} & D_{\mathfrak{h}_0}(\mathfrak{h}_1) \end{array} \qquad (1)$$

and

$$\begin{array}{ccc} \overline{\mathfrak{g}_0}\times\overline{\mathfrak{g}_0} & \xrightarrow{c_0} & [\mathfrak{g}_0,\mathfrak{g}_0] \\ {\scriptstyle \eta_0\times\eta_0}\downarrow & & \downarrow{\scriptstyle \xi_0} \\ \overline{\mathfrak{h}_0}\times\overline{\mathfrak{h}_0} & \xrightarrow[c_0{}']{} & [\mathfrak{h}_0,\mathfrak{h}_0] \end{array} \qquad (2)$$

are commutative where (c_1,c_0) *and* (c_1',c_0') *are commutator maps, defined in Proposition 14 in [11], of the Lie crossed modules* $\mathfrak{g}$ *and* $\mathfrak{h}$ *, respectively.*

The pair $((\eta_1,\eta_0),(\xi_1,\xi_0))$ is called an *isoclinism* from $\mathfrak{g}$ to $\mathfrak{h}$ and this situation is denoted by $((\eta_1,\eta_0),(\xi_1,\xi_0)):\mathfrak{g}\sim\mathfrak{h}$.

Example 3
(1) All Abelian Lie crossed modules (crossed modules that coincide with their center) are isoclinic. The pairs $((\eta_1,\eta_0),(\xi_1,\xi_0))$ *consist of trivial homomorphisms.*
(2) Let (η,ξ) *be an isoclinism between the Lie algebras* $\mathfrak{g}$ *to* $\mathfrak{h}$*. Then* $\mathfrak{g}\xrightarrow{id}\mathfrak{g}$ *is isoclinic to* $\mathfrak{h}\xrightarrow{id}\mathfrak{h}$ *where* $(\eta_1,\eta_0)=(\eta,\eta)$*,* $(\xi_1,\xi_0)=(\xi,\xi)$*.*
(3) Let $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{h}$ *be an ideal of* $\mathfrak{g}$ *with* $\mathfrak{h}+Z(\mathfrak{g})=\mathfrak{g}$*. Then* $\mathfrak{h}\xhookrightarrow{inc.}\mathfrak{g}$ *is isoclinic to* $\mathfrak{g}\xrightarrow{id}\mathfrak{g}$*. Here* (η_1,η_0) *and* (ξ_1,ξ_0) *are defined by* $(inc.,inc.)$*,* (id,id)*, respectively.*

*Remark.*If Lie crossed modules $\mathfrak{g}$ and $\mathfrak{h}$ are simply connected or finite dimensional, then the commutativity of diagrams (1) as (2) in Definition 2 are equivalent to the commutativity of following diagram.

$$\begin{array}{ccc} \mathfrak{g}/Z\wedge(\mathfrak{g})\times\mathfrak{g}/Z\wedge(\mathfrak{g}) & \longrightarrow & [\mathfrak{g},\mathfrak{g}] \\ {\scriptstyle (\eta_1,\eta_0)\times(\eta_1,\eta_0)}\downarrow & & \downarrow{\scriptstyle (\xi_1,\xi_0)} \\ \mathfrak{h}/Z\wedge(\mathfrak{h})\times\mathfrak{h}/Z\wedge(\mathfrak{h}) & \longrightarrow & [\mathfrak{h},\mathfrak{h}] \end{array}$$

3 Exterior isoclinism of Lie algebras

In this section, we define the notion of exterior isoclinism similar to isoclinism using the nonabelian exterior product of Lie algebras from [7]. Also, we introduce the concepts of exterior degree and n-exterior isoclinism of Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{m}$, $\mathfrak{n}$ be ideals of $\mathfrak{g}$. The *exterior product* $\mathfrak{m}\wedge\mathfrak{n}$ is a Lie algebra generated by the elements $m\wedge n$ satisfying the following relations:

1.$\alpha(m\wedge n)=\alpha m\wedge n=m\wedge\alpha n$,
2.$(m+m')\wedge n=m\wedge n+m'\wedge n$,
3.$m\wedge(n+n')=m\wedge n+m\wedge n'$,
4.$[m,m']\wedge n=m\wedge[m',n]-m'\wedge[m,n]$,
5.$m\wedge[n,n']=[n',m]\wedge n-[n,m]\wedge n'$,
6.$[m\wedge n,m'\wedge n']=-[n,m]\wedge[m',n']$,
7.If $m=n$, then $m\wedge n=0$,

for all $\alpha\in\mathbb{K}$, $m,m'\in\mathfrak{m}$ and $n,n'\in\mathfrak{n}$.

The exterior product $\mathfrak{m}\wedge\mathfrak{n}$ can also be defined by its universal property: Given a Lie algebra $\mathfrak{g}$ and a function $\phi:\mathfrak{m}\times\mathfrak{n}\longrightarrow\mathfrak{g}$, we say that ϕ is an *exterior pairing* if we have:

1.$\alpha\phi(m,n) = \phi(\alpha m,n) = \phi(m,\alpha n)$,
2.$\phi((m+m'),n) = \phi(m,n)+\phi(m',n)$,
3.$\phi(m,(n+n')) = \phi(m,n)+\phi(m,n')$,
4.$\phi([m,m'],n) = \phi(m,[m',n])-\phi(m',[m,n])$,
5.$\phi(m,[n,n']) = \phi([n',m],n)-\phi([n,m],n')$,
6.$[\phi(m,n),\phi(m',n')] = \phi([n,m],[m',n'])$,
7.If $m = n$, then $\phi(m,n) = 0$,

for all $\alpha \in \mathbb{K}$, $m,m' \in \mathfrak{m}$ and $n,n' \in \mathfrak{n}$. Note that the function $\mathfrak{m}\times\mathfrak{n} \longrightarrow \mathfrak{m}\wedge\mathfrak{n}$ given by $(m,n) \longmapsto m\wedge n$ is the *universal exterior pairing*. The *exterior center* of a Lie algebra $\mathfrak{g}$ can be defined as

$$Z\wedge(\mathfrak{g}) = \{x \in \mathfrak{g} | 0 = x\wedge a \in \mathfrak{g}\wedge\mathfrak{g} \text{ for all } a \in \mathfrak{g}\}.$$

The exterior square of a Lie algebra $\mathfrak{g}$ (sometimes also called the nonabelian exterior square), denoted by $\mathfrak{g}\wedge\mathfrak{g}$. (For details see [7].)

Definition 3. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. $\mathfrak{g}$ and $\mathfrak{h}$ are exterior isoclinic if there exist isomorphisms $\eta : \mathfrak{g}/Z\wedge(\mathfrak{g}) \longrightarrow \mathfrak{h}/Z\wedge(\mathfrak{h})$ and $\xi : \mathfrak{g}\wedge\mathfrak{g} \longrightarrow \mathfrak{h}\wedge\mathfrak{h}$ between exterior central quotients and exterior squares, respectively, such that the following diagram*

$$\begin{array}{ccc} \mathfrak{g}/Z\wedge(\mathfrak{g})\times\mathfrak{g}/Z\wedge(\mathfrak{g}) & \xrightarrow{c_{\mathfrak{g}}} & \mathfrak{g}\wedge\mathfrak{g} \\ \Big\downarrow{\scriptstyle \eta\times\eta} & & \Big\downarrow{\scriptstyle \xi} \\ \mathfrak{h}/Z\wedge(\mathfrak{h})\times\mathfrak{h}/Z\wedge(\mathfrak{h}) & \xrightarrow[c_{\mathfrak{h}}]{} & \mathfrak{h}\wedge\mathfrak{h} \end{array}$$

is commutative where $c_{\mathfrak{g}}, c_{\mathfrak{h}}$ are universal exterior pairings. The pair (η,ξ) is called an exterior isoclinism from $\mathfrak{g}$ to $\mathfrak{h}$, and denoted by $(\eta,\xi) : \mathfrak{g} \approx \mathfrak{h}$.

*Remark.*Exterior isoclinism is an equivalence relation.

Example 4
(1) Isomorphic Lie algebras are also exterior isoclinic.
(2) All abelian Lie algebras are exterior isoclinic to each other.
(3) Every Lie algebra is exterior isoclinic to a exterior stem Lie algebra whose exterior center is contained in its exterior squares.

3.1 Exterior degree of finite Lie algebras

Let $\mathfrak{g}$ be a finite Lie algebra. $\mathfrak{g}$ is commutative if $[x,y] = 0$ for $x,y \in \mathfrak{g}$. The probability of two elements of $\mathfrak{g}$ commute is called *commutativity degree* of $\mathfrak{g}$ and denoted by $d(\mathfrak{g})$. Formally, this probability can be calculated by

$$d(\mathfrak{g}) = \frac{|\{(x,y) \in \mathfrak{g}\times\mathfrak{g} \mid [x,y] = 0\}|}{|\mathfrak{g}|^2}.$$

Now, we can define the *exterior degree* of $\mathfrak{g}$ analogous to exterior degree of a finite group given in [15] as:

$$d\wedge(\mathfrak{g}) = \frac{|\{(x,y) \in \mathfrak{g}\times\mathfrak{g} \mid x\wedge y = 0\}|}{|\mathfrak{g}|^2}.$$

Proposition 1. *Every exterior isoclinic finite Lie algebras has same exterior degree.*

Proof. If $\mathfrak{g}$ and $\mathfrak{h}$ are two exterior isoclinic finite Lie algebras, then we will show that $d\wedge(\mathfrak{g}) = d\wedge(\mathfrak{h})$.

Let (η,ξ) be an exterior isoclinism from $\mathfrak{g}$ to $\mathfrak{h}$; we can write

$$\begin{aligned}
|\mathfrak{g}/Z\wedge(\mathfrak{g})|^2 d\wedge(\mathfrak{g}) &= \frac{1}{|Z\wedge(\mathfrak{g})|^2}|\mathfrak{g}|^2 d\wedge(\mathfrak{g})\\
&= \frac{1}{|Z\wedge(\mathfrak{g})|^2}|\{(x,y)\in\mathfrak{g}\times\mathfrak{g} : x\wedge y = 0\}|\\
&= \frac{1}{|Z\wedge(\mathfrak{g})|^2}|\{(x,y)\in\mathfrak{g}\times\mathfrak{g} : c_{\mathfrak{g}}(x+Z\wedge(\mathfrak{g}), y+Z\wedge(\mathfrak{g})) = 0\}|\\
&= |\{(\alpha,\beta)\in\mathfrak{g}/Z\wedge(\mathfrak{g})\times\mathfrak{g}/Z\wedge(\mathfrak{g}) : c_{\mathfrak{g}}(\alpha,\beta) = 0\}|\\
&= |\{(\alpha,\beta)\in\mathfrak{g}/Z\wedge(\mathfrak{g})\times\mathfrak{g}/Z\wedge(\mathfrak{g}) : \zeta(c_{\mathfrak{g}}(\alpha,\beta)) = 0\}|\\
&= \left|\left\{(\alpha,\beta)\in\mathfrak{g}/Z\wedge(\mathfrak{g})\times\mathfrak{g}/Z\wedge(\mathfrak{g}) : c_{\mathfrak{h}}(\mu(\alpha),\mu(\beta)) = 0\right\}\right|\\
&= \left|\left\{(\gamma,\delta)\in\mathfrak{h}/Z\wedge(\mathfrak{h})\times\mathfrak{g}/Z\wedge(\mathfrak{h}) : c_{\mathfrak{h}}(\gamma,\delta) = 0\right\}\right|\\
&= |\mathfrak{h}/Z\wedge(\mathfrak{h})|^2 d\wedge(\mathfrak{h})
\end{aligned}$$

thus we have $d\wedge(\mathfrak{g}) = d\wedge(\mathfrak{h})$.

3.2 n-Exterior isoclinic Lie algebras

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and n be a nonnegative integer. Then $\mathfrak{g}$ and $\mathfrak{h}$ are said to be *n-exterior isoclinic*, $\mathfrak{g}\underset{n}{\approx}\mathfrak{h}$, if there exist isomorphisms $\eta:\mathfrak{g}/Z_n\wedge(\mathfrak{g})\longrightarrow\mathfrak{h}/Z_n\wedge(\mathfrak{h})$ and $\xi:\mathfrak{g}\wedge_{n+1}\mathfrak{g}\longrightarrow\mathfrak{h}\wedge_{n+1}\mathfrak{h}$ in such a way that ξ is compatible with η, that is, the $(n+1)$-fold exterior squares $\cdots\wedge(b_1\wedge b_2)\wedge b_3\wedge\cdots\wedge b_{n+1}$ equals $\xi(\cdots\wedge(a_1\wedge a_2)\wedge a_3\wedge\cdots\wedge a_{n+1})$ for any $b_i\in\eta(a_i+Z_n\wedge(\mathfrak{g}))$ and $a_i\in\mathfrak{g}$ for $i=1,\ldots,n+1$. The pair (η,ξ) is called an *n-exterior isoclinism* between $\mathfrak{g}$ and $\mathfrak{h}$.

4 Exterior isoclinism of Lie crossed modules

In this section, we define the exterior center and exterior commutator subalgebra to introduce the exterior isoclinism of Lie crossed modules. A more general construction $\mathfrak{g}\wedge\mathfrak{h}$ is given in [21] for arbitrary Lie crossed modules $\mathfrak{g},\mathfrak{h}$.

Definition 4. *Let* $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_{\mathfrak{g}}}\mathfrak{g}_0$ *be a Lie crossed module. Then*

$$Z\wedge(\mathfrak{g}) = \mathfrak{g}_1{}^{\mathfrak{g}_0}\to St_{\mathfrak{g}_0}(\mathfrak{g}_1)\cap Z\wedge(\mathfrak{g}_0)$$

is a Lie crossed module. $Z\wedge(\mathfrak{g})$ *is called exterior center of the Lie crossed module* $\mathfrak{g}$*. Moreover,*

$$\mathfrak{g}\wedge\mathfrak{g} = D_{\mathfrak{g}_0}(\mathfrak{g}_1)\to\mathfrak{g}_0\wedge\mathfrak{g}_0$$

is exterior commutator subalgebra of the Lie crossed module $\mathfrak{g}$.

Given an ideal $\mathfrak{h}:\mathfrak{h}_1\xrightarrow{\partial_{\mathfrak{h}}}\mathfrak{h}_0$ of a Lie crossed module $\mathfrak{g}:\mathfrak{g}_1\xrightarrow{\partial_{\mathfrak{g}}}\mathfrak{g}_0$, there is a quotient Lie crossed module

$$\mathfrak{g}/\mathfrak{h} \;=\; \mathfrak{g}_1/\mathfrak{h}_1\to\mathfrak{g}_0/\mathfrak{h}_0$$

where the action is defined by

$$[x_0+\mathfrak{h}_0, x_1+\mathfrak{h}_1] \;:=\; [x_0,x_1]+\mathfrak{h}_1$$

and the boundary map is given by

$$\overline{\partial}(x_1+\mathfrak{h}_1) \ := \ \partial(x_1)+\mathfrak{h}_0.$$

So, we can get a Lie crossed module

$$\mathfrak{g}/Z\wedge(\mathfrak{g}) \ = \ (\mathfrak{g}_1/{\mathfrak{g}_1}^{\mathfrak{g}_0} \to \mathfrak{g}_0/(St_{\mathfrak{g}_0}(\mathfrak{g}_1)\cap Z\wedge(\mathfrak{g}_0)))$$

This Lie crossed module is called *exterior central quotient* Lie crossed module.

Notation In the sequel of the paper, for a given Lie crossed module $\mathfrak{g}:\mathfrak{g}_1 \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0$, we denote $\mathfrak{g}/Z\wedge(\mathfrak{g})$ by $\overline{\mathfrak{g}_1} \xrightarrow{\overline{\partial_{\mathfrak{g}}}} \overline{\mathfrak{g}_0}$ where $\overline{\mathfrak{g}_1}=\mathfrak{g}_1/\mathfrak{g}_1^{\mathfrak{g}_0}$ and $\overline{\mathfrak{g}_0}=\mathfrak{g}_0/(St_{\mathfrak{g}_0}(\mathfrak{g}_1)\cap Z\wedge(\mathfrak{g}_0))$, for shortness.

Proposition 2. *Let $\mathfrak{g}:\mathfrak{g}_1 \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0$ be a Lie crossed module. Define the maps*

$$\begin{array}{rcl} c_1: & \overline{\mathfrak{g}_1}\times\overline{\mathfrak{g}_0} & \longrightarrow D_{\mathfrak{g}_0}(\mathfrak{g}_1) \\ & (x_1+\mathfrak{g}_1^{\mathfrak{g}_0}, x_0+(St_{\mathfrak{g}_0}(\mathfrak{g}_1)\cap Z\wedge(\mathfrak{g}_0))) & \longmapsto \ [x_0,x_1] \end{array}$$

and

$$\begin{array}{rcl} c_0: & \overline{\mathfrak{g}_0}\times\overline{\mathfrak{g}_0} & \longrightarrow \mathfrak{g}_0\wedge\mathfrak{g}_0 \\ & x_0+(St_{\mathfrak{g}_0}(\mathfrak{g}_1))\cap Z\wedge(\mathfrak{g}_0)), x_0'+(St_{\mathfrak{g}_0}(\mathfrak{g}_1)\cap Z\wedge(\mathfrak{g}_0))) & \longmapsto x_0\wedge x_0', \end{array}$$

for all $x_1\in\mathfrak{g}_1$, $x_0,x_0'\in\mathfrak{g}_0$. Then the maps c_1 and c_0 are well-defined.

Proof. It can be easily checked by a similar way of Proposition 14 in [11].

Definition 5. *The Lie crossed modules $\mathfrak{g}:\mathfrak{g}_1 \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0$ and $\mathfrak{h}:\mathfrak{h}_1 \xrightarrow{\partial_{\mathfrak{h}}} \mathfrak{h}_0$ are isoclinic if there exist isomorphisms*

$$(\eta_1,\eta_0):(\overline{\mathfrak{g}_1} \xrightarrow{\overline{\partial_{\mathfrak{g}}}} \overline{\mathfrak{g}_0}) \longrightarrow (\overline{\mathfrak{h}_1} \xrightarrow{\overline{\partial_{\mathfrak{h}}}} \overline{\mathfrak{h}_0})$$

and

$$(\xi_1,\xi_0):(D_{\mathfrak{g}_0}(\mathfrak{g}_1) \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0\wedge\mathfrak{g}_0 \longrightarrow (D_{\mathfrak{h}_0}(\mathfrak{h}_1) \xrightarrow{\partial_{\mathfrak{h}}} \mathfrak{h}_0\wedge\mathfrak{h}_0$$

such that the diagrams

$$\begin{array}{ccc} \overline{\mathfrak{g}_1}\times\overline{\mathfrak{g}_0} & \xrightarrow{c_1} & D_{\mathfrak{g}_0}(\mathfrak{g}_1) \\ {\scriptstyle \eta_1\times\eta_0}\downarrow & & \downarrow{\scriptstyle \xi_1} \\ \overline{\mathfrak{h}_1}\times\overline{\mathfrak{h}_0} & \xrightarrow[c_1{}']{} & D_{\mathfrak{h}_0}(\mathfrak{h}_1) \end{array} \qquad (3)$$

and

$$\begin{array}{ccc} \overline{\mathfrak{g}_0}\times\overline{\mathfrak{g}_0} & \xrightarrow{c_0} & \mathfrak{g}_0\wedge\mathfrak{g}_0 \\ {\scriptstyle \eta_0\times\eta_0}\downarrow & & \downarrow{\scriptstyle \xi_0} \\ \overline{\mathfrak{h}_0}\times\overline{\mathfrak{h}_0} & \xrightarrow[c_0{}']{} & \mathfrak{h}_0\wedge\mathfrak{h}_0 \end{array} \qquad (4)$$

are commutative where (c_1,c_0) and (c_1',c_0') are universal exterior pairings, defined in Proposition 2, of the Lie crossed modules $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

The pair $((\eta_1,\eta_0),(\xi_1,\xi_0))$ is called an *exterior isoclinism* from $\mathfrak{g}$ to $\mathfrak{h}$.

Example 5

(1) All abelian Lie crossed modules (crossed modules that coincide with their exterior center) are isoclinic.

(2) Let (η,ξ) be an exterior isoclinism from $\mathfrak{g}$ to $\mathfrak{h}$. Then $\mathfrak{g} \xrightarrow{id} \mathfrak{g}$ is exterior isoclinic to $\mathfrak{h} \xrightarrow{id} \mathfrak{h}$ where $(\eta_1,\eta_0) = (\eta,\eta)$, $(\xi_1,\xi_0) = (\xi,\xi)$.

Remark. If the Lie crossed modules L and M are simply connected or finite dimensional, then the commutativity of diagrams (1) as (2) in Definition 5 are equivalent to the commutativity of the following diagram.

$$\begin{array}{ccc} \mathfrak{g}/Z\wedge(\mathfrak{g}) \times \mathfrak{g}/Z\wedge(\mathfrak{g}) & \longrightarrow & \mathfrak{g}\wedge\mathfrak{g} \\ \Big\downarrow{\scriptstyle (\eta_1,\eta_0)\times(\eta_1,\eta_0)} & & \Big\downarrow{\scriptstyle (\xi_1,\xi_0)} \\ \mathfrak{h}/Z\wedge(\mathfrak{h}) \times \mathfrak{h}/Z\wedge(\mathfrak{h}) & \longrightarrow & \mathfrak{h}\wedge\mathfrak{h} \end{array}$$

Proposition 3. *Let $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{d_{\mathfrak{g}}} \mathfrak{g}_0$ and $\mathfrak{h} : \mathfrak{h}_1 \xrightarrow{d_{\mathfrak{h}}} \mathfrak{h}_0$ be isoclinic finite crossed modules. Then $\mathfrak{g}_1$ and $\mathfrak{g}_0$ are isoclinic to $\mathfrak{h}_1$ and $\mathfrak{h}_0$, respectively.*

*Proof.*Let $\mathfrak{g}$ and $\mathfrak{h}$ be isoclinic crossed module. Then we have the crossed module isomorphisms

$$(\eta_1,\eta_0) : (\overline{\mathfrak{g}_1} \xrightarrow{\overline{d_{\mathfrak{g}}}} \overline{\mathfrak{g}_0}) \longrightarrow (\overline{\mathfrak{h}_1} \xrightarrow{\overline{d_{\mathfrak{h}}}} \overline{\mathfrak{h}_0})$$
$$(\xi_1,\xi_0) : (D_{\mathfrak{g}_0}(\mathfrak{g}_1) \xrightarrow{d_{\mathfrak{g}}|} [\mathfrak{g}_0,\mathfrak{g}_0]) \longrightarrow (D_{\mathfrak{h}_0}(\mathfrak{h}_1) \xrightarrow{d_{\mathfrak{h}}|} [\mathfrak{h}_0,\mathfrak{h}_0])$$

which makes diagrams (1) and (2) commutative. Since $\mathfrak{g}_1$ and $\mathfrak{h}_1$ are finite, the restriction $\xi_1| : [\mathfrak{g}_1,\mathfrak{g}_1] \longrightarrow [\mathfrak{h}_1,\mathfrak{h}_1]$ is also an isomorphism. Similarly, we have the isomorphisms $\eta_1' : \mathfrak{g}_1/Z(\mathfrak{g}_1) \longrightarrow \mathfrak{h}_1/Z(\mathfrak{h}_1), \eta_1'(x_1+Z(\mathfrak{g}_1)) = y_1+Z(\mathfrak{h}_1)$, $\eta_0' : \mathfrak{g}_0/Z(\mathfrak{g}_0) \longrightarrow \mathfrak{h}_0/Z(\mathfrak{h}_0), \eta_0'(x_0+Z(\mathfrak{g}_0)) = y_0+Z(\mathfrak{h}_0)$ where $x_0,x_1 \in \mathfrak{g}_0, y_0,y_1 \in \mathfrak{h}_0$, and ξ_0 which make $\mathfrak{g}_1$ and $\mathfrak{g}_0$ isoclinic to $\mathfrak{h}_1$ and $\mathfrak{h}_0$, respectively.

5 n-Exterior isoclinism of Lie crossed modules

Let $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0$ be a Lie crossed module. We use the following notations in this section:

- $\mathfrak{g}\wedge_n\mathfrak{g}$ denotes the n-th term of the exterior lower central series of $\mathfrak{g}$ defined inductively by $\mathfrak{g}\wedge\mathfrak{g} = \mathfrak{g}$ and $\mathfrak{g}\wedge_{n+1}\mathfrak{g} = (\mathfrak{g}\wedge_n\mathfrak{g})\wedge\mathfrak{g}$, for $n \geq 1$.
- $Z_n\wedge(\mathfrak{g})$ denotes the n-th term of the exterior upper central series of $\mathfrak{g}$ defined inductively by $Z_0\wedge(\mathfrak{g}) = 0$ and $Z_{n+1}\wedge(\mathfrak{g})/Z_n\wedge(\mathfrak{g})$ is the exterior centre of $\mathfrak{g}/Z_n\wedge(\mathfrak{g})$, for $n \geq 0$.
- $\zeta_n\wedge(\mathfrak{g}_1) = \{x_1 \in \mathfrak{g}_1 \mid {}_n\mathfrak{g}_0\wedge x_1 = 0\}$, where $\mathfrak{g}_0\wedge x_1 =< x_0\wedge x_1 \mid x_0 \in \mathfrak{g}_0 >$ and inductively ${}_{n+1}\mathfrak{g}_0\wedge x_1 = \mathfrak{g}_0\wedge({}_n\mathfrak{g}_0\wedge x_1)$.
- $\kappa_n\wedge(\mathfrak{g}_0) = Z_n\wedge(\mathfrak{g}_0)\cap\{x_0 \in \mathfrak{g}_0 \mid {}_i\mathfrak{g}_0\wedge({}_{n-1-i}\mathfrak{g}_0\wedge x_0)\wedge\mathfrak{g}_1 = 0$ for all $0 < i < n-1\}$, where $\mathfrak{g}_0\wedge\mathfrak{g}_1' = \mathfrak{g}_1'$ for each subalgebra $\mathfrak{g}_1'$ of $\mathfrak{g}_1$, $x_0\wedge\mathfrak{g}_1 =< x_0\wedge x_1 \mid x_1 \in \mathfrak{g}_1 >$, $\mathfrak{g}_0\wedge x_0 = x_0$ and inductively ${}_n\mathfrak{g}_0\wedge x_0 = \mathfrak{g}_0\wedge({}_{n-1}\mathfrak{g}_0\wedge x_0)$.
- $\Gamma_n(\mathfrak{g}_1,\mathfrak{g}_0) = {}_{n-1}\,\mathfrak{g}_0\wedge\mathfrak{g}_1$ where $\mathfrak{g}_0\wedge\mathfrak{g}_1 = \mathfrak{g}_1$ and inductively, ${}_n\mathfrak{g}_0\wedge\mathfrak{g}_1 = \mathfrak{g}_0\wedge({}_{n-1}\mathfrak{g}_0\wedge\mathfrak{g}_1)$.

Definition 6. *Lie crossed modules $\mathfrak{g} : \mathfrak{g}_1 \xrightarrow{\partial_{\mathfrak{g}}} \mathfrak{g}_0$ and $\mathfrak{h} : \mathfrak{h}_1 \xrightarrow{\partial_h} \mathfrak{h}_0$ are said to be n-exterior isoclinic $(n \geq 0)$, $\mathfrak{g} \underset{n}{\approx} \mathfrak{h}$, if there exists a pair of isomorphisms of Lie crossed modules*

$$\alpha = (\alpha_1,\alpha_2) : \frac{\mathfrak{g}}{Z_n\wedge(\mathfrak{g})} \longrightarrow \frac{\mathfrak{h}}{Z_n\wedge(\mathfrak{h})},$$
$$\beta = (\beta_1,\beta_2) : \mathfrak{g}\wedge_{n+1}\mathfrak{g} \longrightarrow \mathfrak{h}\wedge_{n+1}\mathfrak{h},$$

such that the following diagrams are commutative

$$\begin{array}{ccc} \frac{\mathfrak{g}_1}{\zeta_n\wedge(\mathfrak{g}_1)} \times \frac{\mathfrak{g}_0}{\kappa_n\wedge(\mathfrak{g}_0)} \times \cdots \times \frac{\mathfrak{g}_0}{\kappa_n\wedge(\mathfrak{g}_0)} & \xrightarrow{\eta_{\mathfrak{g}}^{n+1}} & \Gamma_{n+1}\wedge(\mathfrak{g}_1,\mathfrak{g}_0) \\ \alpha_1\times\alpha_2^n \downarrow & & \downarrow \beta_1 \\ \frac{\mathfrak{h}_1}{\zeta_n\wedge(\mathfrak{h}_1)} \times \frac{\mathfrak{h}_0}{\kappa_n\wedge(\mathfrak{h}_0)} \times \cdots \times \frac{\mathfrak{h}_0}{\kappa_n\wedge(\mathfrak{h}\wedge_0)} & \xrightarrow{\eta_{\mathfrak{h}}^{n+1}} & \Gamma_{n+1}\wedge(\mathfrak{h}_1,\mathfrak{h}_0) \end{array}$$

and

$$\begin{array}{ccc} \frac{\mathfrak{g}_0}{\kappa_n\wedge(\mathfrak{g}_0)} \times \cdots \times \frac{\mathfrak{g}_0}{\kappa_n\wedge(\mathfrak{g}_0)} & \xrightarrow{\theta_{\mathfrak{g}}^{n+1}} & \mathfrak{g}_0\wedge_{n+1}\mathfrak{g}_0 \\ \alpha_2^{n+1} \downarrow & & \downarrow \beta_2 \\ \frac{\mathfrak{h}_0}{\kappa_n\wedge(\mathfrak{h}_0)} \times \cdots \times \frac{\mathfrak{h}_0}{\kappa_n\wedge(\mathfrak{h}_0)} & \xrightarrow{\theta_{\mathfrak{h}}^{n+1}} & \mathfrak{h}_0\wedge_{n+1}\mathfrak{h}_0. \end{array}$$

where

$$\eta_{\mathfrak{g}}^{n+1}(a+\zeta_n\wedge(\mathfrak{g}), b_1+\kappa_n\wedge(\mathfrak{g}_0),\ldots,b_n+\kappa_n\wedge(\mathfrak{g}_0)) = b_n\wedge\ldots\wedge b_2\wedge(b_1\wedge a)\wedge\ldots,$$
$$\theta_{\mathfrak{g}}^{n+1}(b_1+\kappa_n\wedge(\mathfrak{g}_0),\ldots,b_n+\kappa_n\wedge(\mathfrak{g}_0), b_{n+1}+\kappa_{n+1}\wedge(\mathfrak{g}_0)) = \ldots\wedge(b_1\wedge b_2)\wedge b_3\wedge\ldots\wedge b_{n+1},$$

for all $a\in\mathfrak{g}_1$, $b_i\in\mathfrak{g}_0$.

As Lie algebras are considered as Lie crossed modules, we obtain the definition of n-exterior isoclinic Lie algebras. Since n-exterior isoclinism between Lie crossed modules is an equivalence relation, we can say that it divides the class of all Lie crossed modules into n-exterior isoclinism equivalence classes.

6 Conclusion

We obtain a new equivalence relation called exterior isoclinism similar to isoclinism using the nonabelian exterior product of Lie algebras.By using this notion, we get a classification method for Lie crossed modules.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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