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On Some Types of Hesitant Fuzzy Mappings

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Abstract: In this paper, we provide a new mapping class termed hesitant fuzzy weakly continuous, which includes hesitant fuzzy continuous mappings, and obtain several characterizations of hesitant fuzzy weakly continuous mappings. Furthermore, we propose the concepts of hesitant fuzzy preopen and hesitant fuzzy semiopen mappings in hesitant fuzzy topological spaces, as well as various features of these mappings.

Keywords: hesitant fuzzy open set, hesitant fuzzy weakly continuous, hesitant fuzzy preopen mapping, hesitant fuzzy semiopen mapping.

1 Introduction

As a generalization of a crisp set, Zadeh [14] developed the concept of a fuzzy set. Fuzzy set theory provides a new method for dealing with fundamentally ambiguous language notions. Chang [1] defined the notion of fuzzy topological spaces, and then, many researchers investigated various properties, for example neighborhood systems, continuities, compactness, and separation axioms in fuzzy topological spaces. Torra [12] introduced the notion of a hesitant fuzzy set as an extension of a fuzzy set, and the motivation for this extension is the hesitancy arising in the determination of the membership value of an element. Hesitancy does not arise just because of an error margin or a possibility distribution, but because there are some possible values of which there is a hesitation about which one would be the right one. These situations mainly arise in decision-making problems where there are a group of decision-makers to consider the evaluation of a scenario. In a hesitant fuzzy set, the membership function takes values from the power set of $[0,1]$. Hesitant fuzzy set theory has a wide range of applications in various fields such as multi-criteria decision-making, group decision making, decision support systems, evaluation processes, and clustering algorithms. Xia and Xu [13] applied a hesitant fuzzy set to decision making by defining "hesitant fuzzy information aggregation". Jun et al. [3] studied hesitant fuzzy bi-ideals in semigroups. Divakaran and John [2] introduced a basic version of hesitant fuzzy rough sets through hesitant fuzzy relations. On the other hand, Jun and Ahn [4] applied hesitant fuzzy sets to BCK/BCI-algebras. Kim et al. [5] gave characterizations of a hesitant fuzzy positive implicative ideal, a hesitant fuzzy implicative ideal, and a hesitant fuzzy commutative ideal, respectively in BCK-algebras. Recently, Lee and Hur [8] defined a hesitant fuzzy topology and introduced the concepts of a hesitant fuzzy neighborhood, closure, interior, hesitant fuzzy subspace and obtained some of their properties. Also, they introduced the hesitant fuzzy open and hesitant fuzzy continuous mappings. Ibrahim [9] studied the concepts of a hesitant fuzzy preopen and hesitant fuzzy semiopen sets in a hesitant fuzzy topological space. Also, he introduced the notion of almost γ-continuous [10] and weakly γ-continuous functions [11].

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2 Preliminaries

Definition 1. *[12] Let X be a reference set, and $P[0,1]$ denote the power set of $[0,1]$. Then, a mapping $h: X \to P[0,1]$ is called a hesitant fuzzy set in X. The hesitant fuzzy empty (resp. whole) set, denoted by h^0 (resp. h^1), is a hesitant fuzzy set in X defined as $h^0(x) = \phi$ (resp. $h^1(x) = [0,1]$), for each $x \in X$. Especially, we will denote the set of all hesitant fuzzy sets in X as $HS(X)$ [6].*

Definition 2. *Assume that X is a nonempty set and $h, h_i \in HS(X)$ for i belong to the set of natural numbers $\mathbf{N}$. Then,*

(1) *h_1 is a subset of h_2, denoted by $h_1 \subseteq h_2$, if $h_1(x) \subseteq h_2(x)$, for each $x \in X$ [2].*
(2) *h_1 is equal to h_2, denoted by $h_1 = h_2$, if $h_1(x) \subseteq h_2(x)$ and $h_2(x) \subseteq h_1(x)$ [2].*
(3) *the intersection of h_1 and h_2, denoted by $h_1 \widetilde{\cap} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$, $(h_1 \widetilde{\cap} h_2)(x) = h_1(x) \cap h_2(x)$ [6].*
(4) *the union of h_1 and h_2, denoted by $h_1 \widetilde{\cup} h_2$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$, $(h_1 \widetilde{\cup} h_2)(x) = h_1(x) \cup h_2(x)$ [6].*
(5) *the complement of h, denoted by h^c, is a hesitant fuzzy set in X defined as: for each $x \in X$, $h^c(x) = h(x)^c = [0,1] \setminus h(x)$ [6].*
(6) *the intersection of $\{h_i\}_{i \in \mathbf{N}}$, denoted by $\widetilde{\bigcap}_{i \in \mathbf{N}} h_i$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$, $(\widetilde{\bigcap}_{i \in \mathbf{N}} h_i)(x) = \bigcap_{i \in \mathbf{N}} h_i(x)$ [6].*
(7) *the union of $\{h_i\}_{i \in \mathbf{N}}$, denoted by $\widetilde{\bigcup}_{i \in \mathbf{N}} h_i$, is a hesitant fuzzy set in X defined as follows: for each $x \in X$, $(\widetilde{\bigcup}_{i \in \mathbf{N}} h_i)(x) = \bigcup_{i \in \mathbf{N}} h_i(x)$ [6].*

Definition 3. *[8] Let X be a nonempty set, and $\tau \subseteq HS(X)$. Then, τ is called a hesitant topology (HFT) on X, if it satisfies the following axioms:*

(1) *$h^0, h^1 \in \tau$.*
(2) *For any $h_1, h_2 \in \tau$, we have $h_1 \widetilde{\cap} h_2 \in \tau$.*
(3) *For each $h_i \in \tau$, we have $\widetilde{\cup}_{i \in \mathbf{N}} h_i \in \tau$.*

The pair (X, τ) is called a hesitant fuzzy topological space. Each member of τ is called a hesitant fuzzy open set (HFOS) in X. A hesitant fuzzy set h in X is called a hesitant fuzzy closed set (HFCS) in (X, τ), if $h^c \in \tau$. The set of all hesitant fuzzy closed sets is denoted by $HFC(X)$.

Definition 4. *[8] Let (X, τ) be a hesitant fuzzy topological space, and $h_A \in HS(X)$. Then:*

(1) *$int_H(h_A) = \widetilde{\bigcup}\{h_U \in \tau : h_U \subseteq h_A\}$.*
(2) *$cl_H(h_A) = \widetilde{\bigcap}\{h_F \in HFC(X) : h_A \subseteq h_F\}$.*

Definition 5. *[9] Let (X, τ) be a hesitant fuzzy topological space. A subset h of $HS(X)$ is called:*

(1) *hesitant fuzzy preopen if $h \subseteq int_H(cl_H(h))$.*
(2) *hesitant fuzzy semiopen if $h \subseteq cl_H(int_H(h))$.*

Remark. [9]The concepts of hesitant fuzzy preopen and hesitant fuzzy semiopen are independent.

Theorem 1. *[9]*Let (X, τ) be a hesitant fuzzy topological space, then the following statements are hold:

(1) Every hesitant fuzzy open set is hesitant fuzzy preopen.
(2) Every hesitant fuzzy open set is hesitant fuzzy semiopen.

Theorem 2. *[9]*Let (X, τ) be a hesitant fuzzy topological space and $h_A \in HS(X)$. Then, h_A is hesitant fuzzy semiopen if and only if $cl_H(h_A) = cl_H(int_H(h_A))$.

Definition 6. *[6] Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then,*

(1) *the image of $h_X \in HS(X)$ under f, denoted by $f(h_X)$ is a hesitant fuzzy set in Y defined as follows: for each $y \in Y$,*

$$f(h_X)(y) = \begin{cases} \widetilde{\bigcup}_{x\in f^{-1}(y)} h_X(x) & \text{if } f^{-1}(y) \neq \phi, \\ \phi & \text{otherwise.} \end{cases}$$

(2) *the preimage of $h_Y \in HS(Y)$ under f, denoted by $f^{-1}(h_Y)$, is a hesitant fuzzy set in Y defined as follows: for each $x \in X$,*

$$f^{-1}(h_Y)(x) = h_Y(f(x)).$$

Definition 7. *[8] Let (X,τ) and (Y,σ) be hesitant fuzzy topological spaces. Then, a mapping $f : (X,\tau) \to (Y,\sigma)$ is called*

(1) *hesitant fuzzy open if $f(h_U) \in \sigma$, for each $h_U \in \tau$.*
(2) *hesitant fuzzy continuous if $f^{-1}(h_V) \in \tau$, for each $h_V \in \sigma$.*

Theorem 3. *[8]*A mapping $f : (X,\tau) \to (Y,\sigma)$ is hesitant fuzzy continuous if and only if $cl_H(f^{-1}(h_V)) \subseteq f^{-1}(cl_H(h_V))$, for every hesitant fuzzy set h_V in Y.

Proposition 1. *[8]*Let (X,τ) be a hesitant fuzzy topological space, and $h \in HS(X)$. Then, $int_H(h) = (cl_H(h^c))^c$, that is, $(int_H(h))^c = cl_H(h^c)$.

Definition 8. *[7]*Let $h \in HS(X)$. Then, h is called a hesitant fuzzy point with the support $x \in X$ and the value δ, denoted by x_δ, if $x_\delta : X \to P[0,1]$ is the mapping given by: for each $y \in X$,

$$x_\delta(y) = \begin{cases} \delta \subseteq [0,1] & \text{if } y = x, \\ \phi, & \text{otherwise.} \end{cases}$$

In particular, $H_P(X)$ is called the set of all hesitant fuzzy points in X. If $\delta \subseteq h(x)$, then x_δ is said to belong to h, denoted by $x_\delta \in h$. It is obvious that $h = \widetilde{\bigcup}_{x_\delta \in h} x_\delta$.

Definition 9. *[8]*Let (X,τ) be a hesitant fuzzy topological space, $h_N \in HS(X)$ and $x_\delta \in H_P(X)$. Then, h_N is called a hesitant fuzzy neighborhood (HFN) of x_δ, if there is $h_U \in \tau$ such that $x_\delta \in h_U \subseteq h_N$. The set of all HFNs of x_δ in (X,τ) is denoted by $HFN(x_\delta)$.

Theorem 4. *[8]*Let (X,τ) be a hesitant fuzzy topological space, $h_A \in HS(X)$ and $x_\delta \in H_P(X)$. Then, $x_\delta \in int_H(h_A)$ if and only if there is $h_N \in HFN(x_\delta)$ such that $h_N \subseteq h_A$.

3 Hesitant Fuzzy Weakly Continuous

Definition 10. *A mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be hesitant fuzzy weakly continuous at $x_\delta \in H_P(X)$ if for each hesitant fuzzy open set h_V in Y containing $f(x_\delta)$, there exists a hesitant fuzzy open set h_U in X containing x_δ such that $f(h_U) \subseteq cl_H(h_V)$. If f is hesitant fuzzy weakly continuous at every $x_\delta \in H_P(X)$, then it is called hesitant fuzzy weakly continuous.*

Remark. It is obvious from the above definition that hesitant fuzzy continuous implies hesitant fuzzy weakly continuous. However, the converse is not true in general as it is shown in the following example.

Example 1. Consider $X=\{a,b,c\}$ with the hesitant topology $\tau=\{h^0,h^1,h_X\}$ and $Y=\{n,m,k\}$ with the hesitant topology $\sigma=\{h^0,h^1,h_{Y_1},h_{Y_2}\}$, where

$$h_X(x)=\begin{cases}\{0.1\} & \text{if } x=a,\\ \{0.2\} & \text{if } x=b,\\ \{0.3\} & \text{if } x=c,\end{cases}$$

$$h_{Y_1}(y)=\begin{cases}\{0.1,0.2,0.3\} & \text{if } y=n,\\ \phi & \text{if } y=m,\\ \phi & \text{if } y=k,\end{cases}$$

and

$$h_{Y_2}(y)=\begin{cases}[0,1] & \text{if } y=n,\\ \phi & \text{if } y=m,\\ \phi & \text{if } y=k.\end{cases}$$

Let $f:X\to Y$ defined as follows:

$$f(x)=\begin{cases}n & \text{if } x=a,\\ n & \text{if } x=b,\\ n & \text{if } x=c.\end{cases}$$

Then, f is hesitant fuzzy weakly continuous, but it is not hesitant fuzzy continuous, because h_{Y_1} is a hesitant fuzzy open set in Y, but $f^{-1}(h_{Y_1})$ is not hesitant fuzzy open in X, where

$$f^{-1}(h_{Y_1})(x)=\begin{cases}\{0.1,0.2,0.3\} & \text{if } x=a,\\ \{0.1,0.2,0.3\} & \text{if } x=b,\\ \{0.1,0.2,0.3\} & \text{if } x=c.\end{cases}$$

Remark. A mapping $f:(X,\tau)\to(Y,\sigma)$ is hesitant fuzzy weakly continuous if and only if for each hesitant fuzzy open set h_V in Y containing $f(x_\delta)$, there exists a hesitant fuzzy open set h_U in X containing x_δ such that $f(h_U)\subseteq cl_H(int_H(h_V))$.

Theorem 5. For a mapping $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent:

(1) f is hesitant fuzzy weakly continuous.
(2) $f^{-1}(h_V)\subseteq int_H(f^{-1}(cl_H(h_V)))$, for every hesitant fuzzy open set h_V in Y.
(3) $cl_H(f^{-1}(int_H(h_F)))\subseteq f^{-1}(h_F)$, for every hesitant fuzzy closed set h_F in Y.
(4) $cl_H(f^{-1}(int_H(cl_H(h_B))))\subseteq f^{-1}(cl_H(h_B))$, for every hesitant fuzzy subset B of Y.
(5) $f^{-1}(int_H(h_B))\subseteq int_H(f^{-1}(cl_H(int_H(h_B))))$, for every hesitant fuzzy subset h_B of Y.
(6) $cl_H(f^{-1}(h_V))\subseteq f^{-1}(cl_H(h_V))$, for every hesitant fuzzy open set h_V in Y.

Proof. $(1)\Rightarrow(2)$. Let h_V be a hesitant fuzzy open set in Y such that $x_\delta\in f^{-1}(h_V)$. Then $f(x_\delta)\in h_V$. There exists hesitant fuzzy open set h_U containing x_δ such that $f(h_U)\subseteq cl_H(h_V)$. Thus, we obtain $x_\delta\in h_U\subseteq f^{-1}(cl_H(h_V))$. This implies that $x_\delta\in int_H(f^{-1}(cl_H(h_V)))$.
$(2)\Rightarrow(3)$. Let h_F be any hesitant fuzzy closed set in Y. Suppose that $x_\delta\notin f^{-1}(h_F)$. Then, h_F^c is hesitant fuzzy open in Y and $x_\delta\in f^{-1}(h_F)^c=f^{-1}(h_F^c)$. By (2) and Proposition 1, we have
$x_\delta\in int_H(f^{-1}(cl_H(h_F^c)))=int_H(f^{-1}(int_H(h_F)^c))=int_H(f^{-1}(int_H(h_F))^c)$
$=(cl_H(f^{-1}(int_H(h_F))))^c$. Therefore, we obtain $x_\delta\notin cl_H(f^{-1}(int_H(h_F)))$.
$(3)\Rightarrow(4)$. Let h_B be any hesitant fuzzy subset of Y. Then $cl_H(h_B)$ is hesitant fuzzy closed in Y and by (3), we have that if $x_\delta\in cl_H(f^{-1}(int_H(cl_H(h_B))))$ then $x_\delta\in f^{-1}(cl_H(h_B))$.
$(4)\Rightarrow(5)$. Let h_B be any hesitant fuzzy subset of Y and $x_\delta\in f^{-1}(int_H(h_B))$. Then, we have

$x_\delta \in f^{-1}(int_H(h_B)) = f^{-1}(cl_H(h_B^c))^c$. Then, $x_\delta \notin f^{-1}(cl_H(h_B^c))$ and by (4), $x_\delta \in cl_H(f^{-1}(int_H(cl_H(h_B^c))))^c = int_H(f^{-1}(cl_H(int_H(h_B))))$.
(5) $\Rightarrow$ (6). Let h_V be any hesitant fuzzy open set in Y. Suppose that $x_\delta \notin f^{-1}(cl_H(h_V))$. Then, $f(x_\delta) \notin cl_H(h_V)$ and there exists a hesitant fuzzy open set h_W containing $f(x_\delta)$ such that $h_W \widetilde{\cap} h_V = h^0$; hence $cl_H(h_W)\widetilde{\cap} h_V = h^0$. By (5), we have $x_\delta \in int_H(f^{-1}(cl_H(h_W)))$ and hence there exists hesitant fuzzy open h_U such that $x_\delta \in h_U \subseteq f^{-1}(cl_H(h_W))$. Since $cl_H(h_W)\widetilde{\cap} h_V = h^0$; $h_U \widetilde{\cap} f^{-1}(h_V) = h^0$ implies that $x_\delta \notin cl_H(f^{-1}(h_V))$. Therefore, if $x_\delta \in cl_H(f^{-1}(h_V))$, then $x_\delta \in f^{-1}(cl_H(h_V))$.
(6) $\Rightarrow$ (1). Let $x \in X$ and h_V any hesitant fuzzy open set in Y containing $f(x_\delta)$. Then, we have $x_\delta \in f^{-1}(h_V) \subseteq f^{-1}(int_H(cl_H(h_V))) = f^{-1}(cl_H(cl_H(h_V)^c))^c$. By (6), $x_\delta \notin cl_H(f^{-1}(cl_H(h_V)^c))$ and hence $x_\delta \in int_H(f^{-1}(cl_H(h_V)))$. Therefore, there exists a hesitant fuzzy open set h_U such that $x_\delta \in h_U \subseteq f^{-1}(cl_H(h_V))$; hence $f(h_U) \subseteq cl_H(h_V)$. This shows that f is hesitant fuzzy weakly continuous.

Theorem 6. For a mapping $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:

(1) f is hesitant fuzzy weakly continuous.
(2) $cl_H(f^{-1}(h_V)) \subseteq f^{-1}(cl_H(h_V))$, for every hesitant fuzzy preopen set h_V in Y.
(3) $f^{-1}(h_V) \subseteq int_H(f^{-1}(cl_H(h_V)))$, for every hesitant fuzzy preopen set h_V in Y.

Proof. (1) $\Rightarrow$ (2). Let h_V be any hesitant fuzzy preopen set in Y such that $x_\delta \in cl_H(f^{-1}(h_V))$. Suppose that $x_\delta \notin f^{-1}(cl_H(h_V))$. Then, there exists a hesitant fuzzy open set h_W containing $f(x_\delta)$ such that $h_W \widetilde{\cap} h_V = h^0$. Hence, we have $h_W \widetilde{\cap} cl_H(h_V) = h^0$ and hence $cl_H(h_W)\widetilde{\cap} int_H(cl_H(h_V)) = h^0$. Since h_V is hesitant fuzzy preopen, then $h_V \subseteq int_H(cl_H(h_V))$ and we have $h_V \widetilde{\cap} cl_H(h_W) = h^0$. Since f is hesitant fuzzy weakly continuous at x_δ and h_W is a hesitant fuzzy open set containing $f(x_\delta)$, there exists hesitant fuzzy open h_U containing x_δ such that $f(h_U) \subseteq cl_H(h_W)$. Then, $f(h_U)\widetilde{\cap} h_V = h^0$ and hence $h_U \widetilde{\cap} f^{-1}(h_V) = h^0$. This shows that $x_\delta \notin cl_H(f^{-1}(h_V))$. This is a contradiction. Therefore, we have $x_\delta \in f^{-1}(cl_H(h_V))$.
(2) $\Rightarrow$ (3). Let h_V be a hesitant fuzzy preopen set in Y and $x_\delta \in f^{-1}(h_V)$. Then, we have $f^{-1}(h_V) \subseteq f^{-1}(int_H(cl_H(h_V))) = f^{-1}(cl_H(cl_H(h_V)^c))^c$. Therefore, $x_\delta \notin f^{-1}(cl_H(cl_H(h_V)^c))$. Then by (2), $x_\delta \notin cl_H(f^{-1}(cl_H(h_V)^c))$. Hence, $x_\delta \in cl_H(f^{-1}(cl_H(h_V)^c))^c = int_H(f^{-1}(cl_H(h_V)))$.
(3) $\Rightarrow$ (1). This follows from Theorem 5, since every hesitant fuzzy open set is hesitant fuzzy preopen.

Theorem 7. Let (X,τ), (Y,σ) and (Z,τ_1) be hesitant fuzzy topological spaces. If $f : (X,\tau) \to (Y,\sigma)$ is hesitant fuzzy weakly continuous and $g : (Y,\sigma) \to (Z,\tau_1)$ is hesitant fuzzy continuous, then the composition $gof : (X,\tau) \to (Z,\tau_1)$ is hesitant fuzzy weakly continuous.

Proof. Let $x \in X$ and h_A be a hesitant fuzzy open set in Z containing $g(f(x_\delta))$. Since g is hesitant fuzzy continuous, then $g^{-1}(h_A)$ is a hesitant fuzzy open set in Y containing $f(x_\delta)$. But f is hesitant fuzzy weakly continuous, then there exists a hesitant fuzzy open set h_B in X containing x_δ such that $f(h_B) \subseteq cl_H(g^{-1}(h_A))$. By Theorem 3, we obtain $(gof)(h_B) \subseteq g(cl_H(g^{-1}(h_A))) \subseteq cl_H(h_A)$. Therefore, gof is hesitant fuzzy weakly continuous.

Definition 11. *A mapping $f : (X,\tau) \to (Y,\sigma)$ is called hesitant fuzzy preopen (resp. semiopen) if $f(h_U) \in \sigma$, for each hesitant fuzzy preopen (resp. semiopen) h_U in X.*

The proof of the following corollaries follows directly from Theorem 1 and is thus omitted.

Corollary 1. Every hesitant fuzzy preopen mapping is hesitant fuzzy open.

Corollary 2. Every hesitant fuzzy semiopen mapping is hesitant fuzzy open.

The converse of Corollaries 1 and 2 are not true in general as it is shown in the following example.

Example 2. Consider $X=\{a,b,c\}$ with the hesitant topology $\tau=\{h^0,h^1,h_X\}$ and $Y=\{n,m,k\}$ with the hesitant topology $\sigma=\{h^0,h^1,h_{Y_1},h_{Y_2}\}$, where

$$h_X(x)=\begin{cases}\{0.3\} & \text{if } x=a,\\ \{0.2\} & \text{if } x=b,\\ \{0.6\} & \text{if } x=c,\end{cases}$$

$$h_{Y_1}(y)=\begin{cases}\{0.2,0.3,0.6\} & \text{if } y=n,\\ \phi & \text{if } y=m,\\ \phi & \text{if } y=k,\end{cases}$$

and

$$h_{Y_2}(y)=\begin{cases}[0,1] & \text{if } y=n,\\ \phi & \text{if } y=m,\\ \phi & \text{if } y=k.\end{cases}$$

Let $f:X\to Y$ defined as follows:

$$f(x)=\begin{cases}n & \text{if } x=a,\\ n & \text{if } x=b,\\ n & \text{if } x=c.\end{cases}$$

Then, f is hesitant fuzzy open, but it is neither hesitant fuzzy preopen nor hesitant fuzzy semiopen, because h_A is both hesitant fuzzy preopen and hesitant fuzzy semiopen set in X but $f(h_A)$ is not hesitant fuzzy open in Y, where

$$h_A(x)=\begin{cases}\{0.1,0.3\} & \text{if } x=a,\\ \{0.2,0.4\} & \text{if } x=b,\\ \{0.5,0.6\} & \text{if } x=c,\end{cases}$$

and

$$f(h_A)(y)=\begin{cases}\{0.1,0.2,0.3,0.4,0.5,0.6\} & \text{if } y=n,\\ \phi & \text{if } y=m,\\ \phi & \text{if } y=k.\end{cases}$$

Remark. From Remark 2, the hesitant fuzzy semiopen mapping need not be hesitant fuzzy preopen in general as it is shown in the following example.

Example 3. Consider $X=\{a,b,c\}$ with the hesitant topology $\tau=\{h^0,h^1\}$ and $Y=\{n,m,k\}$ with the hesitant topology $\sigma=\{h^0,h^1\}$. Let $f:X\to Y$ defined as follows:

$$f(x)=\begin{cases}n & \text{if } x=a,\\ m & \text{if } x=b,\\ k & \text{if } x=c.\end{cases}$$

Then, f is hesitant fuzzy semiopen, but it is not hesitant fuzzy preopen, because h_A is hesitant fuzzy preopen set in X, but $f(h_A)$ is not hesitant fuzzy open in Y. Where

$$h_A(x)=\begin{cases}(0.1,0.5) & \text{if } x=a,\\ \{0.1,0.5\} & \text{if } x=b,\\ (0.6,0.9) & \text{if } x=c,\end{cases}$$

and

$$f(h_A)(y) = \begin{cases} (0.1, 0.5) & \text{if } y = n, \\ \{0.1, 0.5\} & \text{if } y = m, \\ (0.6, 0.9) & \text{if } y = k. \end{cases}$$

Theorem 8. Let (Z, τ_1) be hesitant fuzzy topological space. If

(1) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy preopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy open, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy preopen.
(2) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy preopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy open, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy open.
(3) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy preopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy preopen, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy preopen.
(4) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy preopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy preopen, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy open.

Proof.

(1) Let h_A be a hesitant fuzzy preopen set in X. Since f is hesitant fuzzy preopen, then $f(h_A) \in \sigma$. Since g is hesitant fuzzy open, then $g(f(h_A)) \in \tau_1$ and so $(gof)(h_A) \in \tau_1$. Therefore, gof is hesitant fuzzy preopen.

The proofs of (2), (3) and (4) are similar to (1).

Theorem 9. Let (Z, τ_1) be hesitant fuzzy topological space. If

(1) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy semiopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy open, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy semiopen.
(2) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy semiopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy open, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy open.
(3) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy semiopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy semiopen, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy semiopen.
(4) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy semiopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy semiopen, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy open.

Proof.

(1) Let h_A be a hesitant fuzzy semiopen set in X. Since f is hesitant fuzzy semiopen, then $f(h_A) \in \sigma$. Since g is hesitant fuzzy open, then $g(f(h_A)) \in \tau_1$ and so $(gof)(h_A) \in \tau_1$. Therefore, gof is hesitant fuzzy semiopen.

The proofs of (2), (3) and (4) are similar to (1).

Theorem 10. Let (Z, τ_1) be hesitant fuzzy topological space. If

(1) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy preopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy semiopen, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy open.
(2) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy preopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy semiopen, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy preopen.
(3) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy semiopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy preopen, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy semiopen.
(4) $f : (X, \tau) \to (Y, \sigma)$ is hesitant fuzzy semiopen and $g : (Y, \sigma) \to (Z, \tau_1)$ is hesitant fuzzy preopen, then $gof : (X, \tau) \to (Z, \tau_1)$ is hesitant fuzzy open.

Proof.

(1) Let $h_A \in \tau$. Since f is hesitant fuzzy preopen and every hesitant fuzzy open is hesitant fuzzy preopen, then $f(h_A) \in \sigma$. Since g is hesitant fuzzy semiopen, then $g(f(h_A)) \in \tau_1$ and so $(gof)(h_A) \in \tau_1$. Therefore, gof is hesitant fuzzy open.

The proofs of (2), (3) and (4) are similar to (1).

Theorem 11. If a mapping $f:(X,\tau) \to (Y,\sigma)$ is

(1) hesitant fuzzy semiopen, then $f(int_H(h_A)) \subseteq int_H(f(h_A))$, for each $h_A \in HS(X)$.
(2) hesitant fuzzy preopen, then $f(int_H(h_A)) \subseteq int_H(f(h_A))$, for each $h_A \in HS(X)$.

Proof.

(1) Let f be hesitant fuzzy semiopen and $h_A \in HS(X)$. Since every hesitant fuzzy open is hesitant fuzzy semiopen and $int_H(h_A)$ is hesitant fuzzy open, then by the hypothesis, $f(int_H(h_A))$ is hesitant fuzzy open in Y. Since $int_H(h_A) \subseteq h_A$, so $f(int_H(h_A)) \subseteq f(h_A)$ and hence $f(int_H(h_A)) = int_H(f(int_H(h_A))) \subseteq int_H(f(h_A))$.
(2) The proof is similar to (1).

4 Conclusions

The terms hesitant fuzzy weakly continuous, hesitant fuzzy preopen, and hesitant fuzzy semiopen mappings were introduced. Moreover, several of their characteristics are studied. Meanwhile, we discussed the relationship between hesitant fuzzy weakly continuous and hesitant fuzzy continuous, and also relationship between hesitant fuzzy open, hesitant fuzzy preopen and hesitant fuzzy semiopen mappings.

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