New Trends in Mathematical Sciences

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On s-convex functions in the third sense and new integral inequalities

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Abstract: In this paper, some new integral inequalities are obtained for functions whose n-th derivative is s-convex function in the third sense with the help of an integral identity.

Keywords: Convex function, s-convex functions in the third sense, hölder integral inequality and power mean integral inequality.

1 Introduction

The classical definition of convex functions on a convex subset U of a vector space X is the statement that $f : U \to \mathbb{R}$ is said to be convex function if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in U$ and $\lambda \in [0,1]$. If this inequality reverse, then f is said to be concave on U. In the definition of convexity, new classes of abstract convex functions can be produced with the conditions imposed on the coefficients [1,2,18,11,12,7,9,8,10,19]. Some of these classes are as follows; Let $s \in (0,1]$ and $U \subseteq \mathbb{R}^n$ be a s-convex set. A function $f : U \to \mathbb{R}$ is said to be s-convex in the first sense if

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y)$$

for all $x, y \in U$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s = 1$ [16]. Let $s \in (0,1]$ and $U \subseteq \mathbb{R}^n$ be a convex set. A function $f : U \to \mathbb{R}$ is said to be s-convex in the second sense if the inequality

$$f(\lambda x + \mu y) \leq \lambda^s f(x) + \mu^s f(y)$$

holds for all $x, y \in U$ and all $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$[3]. s-Convex functions in the third sense were introduced to the literature by Kemali et al. [12]. For this new convexity class, Hermite-Hadamard type inequalities and some integral inequalities have been studied [19,6].The functions have been stated as the following, Let $s \in (0,1]$ and $U \subseteq \mathbb{R}^n$ be a s-convex set. A function $f : U \to \mathbb{R}$ is said to be s-convex function in the third sense if the inequality

$$f(\lambda x + \mu y) \leq \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y) \tag{1}$$

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is satisfied for all $x,y\in U$ and $\lambda,\mu\geq 0$ such that $\lambda^s+\mu^s=1$. The inequality (1) is equivalent to the following inequalities:

$$f(\lambda^{\frac{1}{s}}x+(1-\lambda)^{\frac{1}{s}}y)\leq \lambda^{\frac{1}{s^2}}f(x)+(1-\lambda)^{\frac{1}{s^2}}f(y)$$

or

$$f(\lambda x+(1-\lambda^s)^{\frac{1}{s}}y)\leq \lambda^{\frac{1}{s}}f(x)+(1-\lambda^s)^{\frac{1}{s^2}}f(y)$$

where $\lambda\in[0,1]$ and $x,y\in U$. In this paper, $U\subseteq\mathbb{R}^n$ will be taken as a s-convex set. The classes of s-convex functions in first, second and third senses are denoted by K_s^1, K_s^2 and K_s^3 respectively. It can be easily seen that in the case $s=1$, each type of s-convexity is reduced to the ordinary convexity of functions. There is a large number of studies on s-convex functions and their properties, relevant inequalities mainly including Hermite-Hadamard type inequalities (see [15,5,4, 13,17,16,?] and the references therein).

Lemma 1. *[12]. $I\subseteq\mathbb{R}$ is a convex set and also a s-convex set.*

Sarikaya *et* al. proved the following Lemma in [15] and some new integral inequalities for convex and concave functions were establish by this useful lemma. In this study, the same Lemma and new Lemma will be used to get the main results.

Lemma 2.*[15] Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be n-times differentiable function on I, $a,b\in I^o$ with $a<b$. If $f^{(n)}\in L[a,b]$, then we have the identity,*

$$\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx=\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx$$

where an empty sum is understood to be nil.

Throughout this paper, the following notations and conventions are used.

Let I and I^o be the subset of $\mathbb{R}$ and interior of I, respectively and

$$A(a,b)=\frac{a+b}{2},\ L_p(a,b)=\begin{cases} a, & \text{if } a=b\\ \left(\frac{a^{p+1}-b^{p+1}}{(p+1)(a-b)}\right)^{1/p}, & \text{else}\end{cases}$$

be Arithmetic mean and Stolarsky mean (Generalized Logarithmic mean) for a,b,p be positive number with $a\neq b$, respectively.

2 Main results

In this section, some new integral inequalities for functions whose absolute value of n-th derivative are s-convex functions in the third sense are given.

Lemma 3. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be differentiable function on $[a,b]$. If $f'\in L[a,b]$, then*

$$bf(b)-af(a)-\int_a^b f(x)dx=\int_0^1\left[b^2t-a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1}-abt^s(1-t^s)^{\frac{1}{s}-1}+ab(1-t^s)^{\frac{1}{s}}\right]f'(tb+(1-t^s)^{\frac{1}{s}}a)dt$$

is holds.

Proof. If the following inequality is calculated by the partial integration method, then

$$\int_a^b xf'(x)dx = bf(b) - af(a) - \int_a^b f(x)dx,$$

here, if $x = tb + (1-t^s)^{\frac{1}{s}}a$ variable replacement is done,

$$\int_0^1 (tb + (1-t^s)^{\frac{1}{s}}a).(b - t^{s-1}.(1-t^s)^{\frac{1}{s}}a)f'(tb + (1-t^s)^{\frac{1}{s}}a)dt$$
$$= \int_0^1 \left[b^2t - a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1} - abt^s(1-t^s)^{\frac{1}{s}-1} + ab(1-t^s)^{\frac{1}{s}}\right] f'(tb + (1-t^s)^{\frac{1}{s}}a)dt$$
$$= (tb + (1-t^s)^{\frac{1}{s}}a)f(tb + (1-t^s)^{\frac{1}{s}}a) \mid_0^1 - \int_0^1 f(tb + (1-t^s)^{\frac{1}{s}}a)(b - a(1-t^s)^{\frac{1}{s}-1}t^{s-1})dt$$
$$= (b)f(b) - (a)f(a) - \int_a^b f(x)dx.$$

Using the Lemma 3 we get the following integral inequalities,

Theorem 1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be differentiable function on $[a,b]$. If $|f'|$ is integrable on $[a,b]$ and s-convex function in the third sense, then the following inequality is holds*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \le (|a| + |b|)^2 \left[\frac{s}{1+s}\left|f'(b) + \frac{1}{s}\beta\left(\frac{1}{s^2} + 1, \frac{1}{s}\right)|f'(a)|\right|\right].$$

Proof. Using the Lemma 3 we get,

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right|$$
$$\le \left|\int_0^1 \left(b^2t - a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1} - abt^s(1-t^s)^{\frac{1}{s}-1} + ab(1-t^s)^{\frac{1}{s}}\right) f'\left(tb + (1-t^s)^{\frac{1}{s}}a\right)dt\right|$$
$$\le \int_0^1 \left|\left(b^2t - a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1} - abt^s(1-t^s)^{\frac{1}{s}-1} + ab(1-t^s)^{\frac{1}{s}}\right)\right|\left|f'\left(tb + (1-t^s)^{\frac{1}{s}}a\right)\right|dt$$
$$\le \int_0^1 \left[\left|b^2t\right| + \left|a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1}\right| + \left|abt^s(1-t^s)^{\frac{1}{s}-1}\right| + \left|ab(1-t^s)^{\frac{1}{s}}\right|\right]\left[t^{\frac{1}{s}}\left|f'(b)\right| + (1-t^s)^{\frac{1}{s^2}}\left|f'(a)\right|\right]dt$$
$$\le \int_0^1 (|a| + |b|)^2 \left[t^{\frac{1}{s}}\left|f'(b)\right| + (1-t^s)^{\frac{1}{s^2}}\left|f'(a)\right|\right]dt$$
$$= (|a| + |b|)^2 \left[\left|f'(b)\right|\frac{t^{\frac{1}{s}+1}}{\frac{1}{s}+1}\mid_0^1 + \left|f'(a)\right|\frac{1}{s}\beta\left(\frac{1}{s^2} + 1, \frac{1}{s}\right)\right]$$
$$= (|a| + |b|)^2 \left[\frac{s}{1+s}\left|f'(b)\right| + \frac{1}{s}\beta\left(\frac{1}{s^2} + 1, \frac{1}{s}\right)\left|f'(a)\right|\right].$$

Corollary 1. *In addition to the conditions in the Theorem 1, if we take $s = 1$, we get*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \le (|a| + |b|)^2 \frac{|f'(a)| + |f'(b)|}{2}.$$

Corollary 2. *In addition to the conditions in the Theorem 1, if we take $|f'(x)| \le M$ for all $x \in [a,b]$, then we have the following inequality,*

$$\left|bf(b) - af(a) - \int_a^b f(x)dx\right| \le M(|a| + |b|)^2.$$

Theorem 2. *Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable function on $[a,b]$, $a,b\in\mathbb{R}$ with $a<b$ and $p\in(1,\infty)$ such that $\frac{1}{p}<s$. If $|f'|^p$ is s-convex function in the third sense on $\mathbb{R}$, then we get*

$$\left|bf(b)-af(a)-\int_a^b f(x)dx\right|\leq\left(\tfrac{1}{1+s}\right)^{\frac{1}{p}}(|a|+|b|)^2\left[s\left|f'(b)\right|^p+(1+s)\beta\left(\tfrac{1}{s^2}+1,\tfrac{1}{s}\right)\left|f'(a)\right|^p\right]^{\frac{1}{p}}.$$

Proof. Using Lemma 3, Hölder integral inequality and s-convexity of $|f'|^p$ we have the following inequality,

$$\begin{aligned}
&\left|bf(b)-af(a)-\int_a^b f(x)dx\right|\\
&\leq\left|\int_0^1\left(b^2t-a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1}-abt^s(1-t^s)^{\frac{1}{s}-1}+ab(1-t^s)^{\frac{1}{s}}\right)f'\left(tb+(1-t^s)^{\frac{1}{s}}a\right)dt\right|\\
&\leq\int_0^1\left|b^2t-a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1}-abt^s(1-t^s)^{\frac{1}{s}-1}+ab(1-t^s)^{\frac{1}{s}}\right|\left|f'\left(tb+(1-t^s)^{\frac{1}{s}}a\right)\right|dt\\
&\leq\left[\int_0^1\left[b^2t+a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1}+abt^s(1-t^s)^{\frac{1}{s}-1}+ab(1-t^s)^{\frac{1}{s}}\right]^{\frac{p}{p-1}}dt\right]^{\frac{p-1}{p}}\left[\int_0^1\left|f'\left(tb+(1-t^s)^{\frac{1}{s}}a\right)\right|^p dt\right]^{\frac{1}{p}}\\
&\leq\left[\int_0^1\left[b^2t+a^2t^{s-1}(1-t^s)^{\frac{2}{s}-1}+|ab|t^s(1-t^s)^{\frac{1}{s}-1}+|ab|(1-t^s)^{\frac{1}{s}}\right]^{\frac{p}{p-1}}dt\right]^{\frac{p-1}{p}}\left[\int_0^1\left(t^{\frac{1}{s}}\left|f'(b)\right|^p+(1-t^s)^{\frac{1}{s}}\left|f'(a)\right|^p dt\right)\right]^{\frac{1}{p}}\\
&\leq\left[\int_0^1\left[b^2+2|ab|+a^2\right]^{\frac{p}{p-1}}dt\right]^{\frac{p-1}{p}}\left[\int_0^1\left(t^{\frac{1}{s}}\left|f'(b)\right|^p+(1-t^s)^{\frac{1}{s}}\left|f'(a)\right|^p\right)dt\right]^{\frac{1}{p}}\\
&=\left([|a|+|b|]^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}\left(\left|f'(b)\right|^p\frac{t^{\frac{1}{s}+1}}{\frac{1}{s}+1}\Big|_0^1+\left|f'(a)\right|^p\frac{1}{s}\beta\left(\frac{1}{s^2}+1,\frac{1}{s}\right)\right)^{\frac{1}{p}}\\
&=(|a|+|b|)^2\left(\frac{s}{1+s}\left|f'(b)\right|^p+\frac{1}{s}\beta\left(\frac{1}{s^2}+1,\frac{1}{s}\right)\left|f'(a)\right|^p\right)^{\frac{1}{p}}\\
&=(|a|+|b|)^2\left(\frac{1}{1+s}\right)^{\frac{1}{p}}\left(s\left|f'(b)\right|^p+\frac{s+1}{s}\beta\left(\frac{1}{s^2}+1,\frac{1}{s}\right)\left|f'(a)\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Corollary 3. *In addition to the conditions in the Theorem 2, if we take $s=1$, we get*

$$\left|bf(b)-af(a)-\int_a^b f(x)dx\right|\leq(|a|+|b|)^2\left(\frac{|f'(b)|^p+|f'(a)|^p}{2}\right)^{\frac{1}{p}}.$$

Corollary 4. *In addition to the conditions in the Theorem 2, if we take $|f'(x)|^p\leq M$ for all $x\in\mathbb{R}$, then we have the following inequality*

$$\left|bf(b)-af(a)-\int_a^b f(x)dx\right|\leq M^{\frac{1}{p}}(|a|+|b|)^2.$$

For the integral inequalities obtained below, Lemma 2 is used.

Theorem 3. *For $n\in\mathbb{N}$, let $f:I\subset[0,\infty)\longrightarrow\mathbb{R}$ be $n-$times differentiable function on I^o and let $a,b\in I^o$ with $a<b$. If $\left|f^{(n)}\right|^q$ for $q>1$ is s-convex in the third sense function on $[a,b]$ and $\left|f^{(n)}\right|^q\in L[a,b]$ for some fixed $s\in[0,1)$, then the following inequality holds;*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|\leq\frac{1}{n!}(b-a)L_{np}^n(a,b)\left[\frac{s\left(\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right)}{1+s}\right]^{1/q}. \tag{2}$$

Proof. Let $x = \frac{x-a}{b-a}b + \frac{b-x}{b-a}a$. From s-Convexity of $\left|f^{(n)}\right|^q$, we get

$$\left|f^{(n)}(x)\right|^q = \left|f^{(n)}\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right)\right|^q \leq \left(\frac{x-a}{b-a}\right)^{1/s}\left|f^{(n)}(b)\right|^q + \left[1-\left(\frac{x-a}{b-a}\right)\right]^{1/s}\left|f^{(n)}(a)\right|^q.$$

From Lemma 2 and since $\left|f^{(n)}\right|^q$ is a s-convex function in the third sense and using the well known Hölder integral inequality, we get

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right| \\
&\leq \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx \\
&\leq \frac{1}{n!}\left[\int_a^b x^{np}dx\right]^{1/p}\left[\int_a^b\left|f^{(n)}(x)\right|^q dx\right]^{1/q} \\
&\leq \frac{1}{n!}\left[\int_a^b x^{np}dx\right]^{1/p}\left[\int_a^b\left[\left(\frac{x-a}{b-a}\right)^{1/s}\left|f^{(n)}(b)\right|^q+\left(1-\left(\frac{x-a}{b-a}\right)^{1/s}\right)\left|f^{(n)}(a)\right|^q\right]dx\right]^{1/q} \\
&= \frac{1}{n!}\left[\int_a^b x^{np}dx\right]^{1/p}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}\int_a^b(x-a)^{1/s}dx+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\int_a^b\left[(b-a)^{1/s}-(x-a)^{1/s}\right]dx\right]^{1/q} \\
&= \frac{1}{n!}\left[\frac{x^{np+1}}{np+1}\Big|_a^b\right]^{1/p}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}\left[\frac{(x-a)^{\frac{1}{s}+1}}{\frac{1}{s}+1}\right]|_a^b+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\left[(b-a)^{1/s}x-\frac{(x-a)^{\frac{1}{s}+1}}{\frac{1}{s}+1}]|_a^b\right]\right]^{1/q} \\
&= \frac{1}{n!}\left[\frac{b^{np+1}-a^{np+1}}{np+1}\right]^{1/p}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}\frac{(b-a)^{\frac{1}{s}+1}}{\frac{1}{s}+1}+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\left[(b-a)^{1/s}b-\frac{(b-a)^{\frac{1}{s}+1}}{\frac{1}{s}+1}-(b-a)^{1/s}a\right]\right]^{1/q} \\
&= \frac{1}{n!}(b-a)^{1/p}\left[\frac{b^{np+1}-a^{np+1}}{np+1}\right]^{1/p}\left[(b-a)\frac{\left|f^{(n)}(b)\right|^q}{\frac{1+s}{s}}+\left|f^{(n)}(a)\right|^q\left[b-\frac{(b-a)}{\frac{1+s}{s}}-a\right]\right]^{1/q} \\
&= \frac{1}{n!}(b-a)^{1/p}L_{np}^n(a,b)\left[\frac{s(b-a)\left|f^{(n)}(b)\right|^q}{1+s}+\frac{s(b-a)\left|f^{(n)}(a)\right|^q}{1+s}\right]^{1/q} \\
&= \frac{1}{n!}(b-a)^{1/p}L_{np}^n(a,b)(b-a)^{1/q}\left(\frac{s}{1+s}\right)^{1/q}\left(\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right)^{1/q} \\
&= \frac{1}{n!}(b-a)L_{np}^n(a,b)\left[\frac{s\left(\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right)}{1+s}\right]^{1/q}.
\end{aligned}$$

Corollary 5. *Under the same assumptions given in Theorem 3,*

(i) *If we choose $s=1$, then the inequality (2) becomes the following inequality,*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right| \leq \frac{b-a}{n!}L_{np}^n(a,b)A^{1/q}\left(\left|f^{(n)}(b)\right|^q+\left|f^{(n)}(a)\right|^q\right).$$

(ii) *If we choose $s=1$, $n=1$, then the inequality (2) becomes the following inequality,*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le L_p(a,b)A^{1/q}(\left|f'(a)\right|^q,\left|f'(b)\right|^q).$$

(iii) *If we choose $s=1$, $n=1$, $q=1$,then the inequality (2) becomes the following inequality,*

$$\left|\frac{f(b)b-f(a)a}{b-a}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le L_p(a,b)A(\left|f'(a)\right|,\left|f'(b)\right|).$$

Theorem 4. *For $n\in\mathbb{N}$, let $f: I\subset[0,\infty)\longrightarrow\mathbb{R}$ be $n-$times differentiable function on I^o and let $a,b\in I^o$ with $a<b$. If $\left|f^{(n)}\right|^q$ for $q>1$ is s-convex in the third sense function on $[a,b]$ and $\left|f^{(n)}\right|^q\in L[a,b]$ for some fixed $s\in[0,1)$, then the following inequality holds;*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|$$
$$\le \frac{(b-a)}{n!}L_n^{n\frac{q-1}{q}}(a,b)\left[\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{\frac{1}{s}+1}}F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q L_n^n(a,b)\right]^{1/q} \tag{3}$$

where, $F\left(\frac{1}{s},n,x\right)=\int_a^b(x-a)^{1/s}x^n dx.$

Proof. From Lemma 2, using the well known Power mean integral inequality and the s-convexity in the third sense of $\left|f^{(n)}\right|^q$, we get

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right|$$
$$\le \frac{1}{n!}\left[\int_a^b x^n dx\right]^{1-1/q}\left[\int_a^b x^n\left(\frac{x-a}{b-a}\right)^{1/s}\left|f^{(n)}(b)\right|^q+\left[\left(1-\left(\frac{x-a}{b-a}\right)^{1/s}\right)\left|f^{(n)}(a)\right|^q\right]dx\right]^{1/q}$$
$$=\frac{1}{n!}\left[\int_a^b x^n dx\right]^{1-1/q}\left[\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}\int_a^b x^n(x-a)^{1/s}dx+\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\int_a^b\left[(b-a)^{1/s}-(x-a)^{1/s}\right]x^n dx\right]^{1/q}$$
$$=\frac{1}{n!}\left[\int_a^b x^n dx\right]^{1-1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}-\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\right)\int_a^b(x-a)^{1/s}x^n dx+\left|f^{(n)}(a)\right|^q\int_a^b x^n dx\right]^{1/q}$$
$$=\frac{1}{n!}\left[\left.\frac{x^{n+1}}{n+1}\right|_a^b\right]^{1-1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}-\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\right)F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q\left.\frac{x^{n+1}}{n+1}\right|_a^b\right]^{1/q}$$
$$=\frac{1}{n!}\left[\frac{b^{n+1}-a^{n+1}}{n+1}\right]^{1-1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q}{(b-a)^{1/s}}-\frac{\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\right)F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q\frac{b^{n+1}-a^{n+1}}{n+1}\right]^{1/q}$$
$$=\frac{1}{n!}(b-a)^{1-1/q}\left[\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right]^{1-1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q}{(b-a)^{1/s+1}}\right)F\left(\frac{1}{s},n,x\right)+\left|f^{(n)}(a)\right|^q(b-a)\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right]^{1/q}$$

$$= \frac{1}{n!}(b-a)^{1-1/q} L_n^{n\frac{q-1}{q}}(a,b)(b-a)^{1/q}\left[\left(\frac{\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q}{(b-a)^{1/s+1}}\right) F\left(\frac{1}{s},n,x\right) + \left|f^{(n)}(a)\right|^q L_n^n(a,b)\right]^{1/q}$$

$$= \frac{b-a}{n!} L_n^{n\frac{q-1}{q}}(a,b)\left[\left(\frac{\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q}{(b-a)^{1/s+1}}\right) F\left(\frac{1}{s},n,x\right) + \left|f^{(n)}(a)\right|^q L_n^n(a,b)\right]^{1/q}.$$

Corollary 6. *Under the same assumptions given in Theorem 4,*

(i) *If we choose $s=1$, then the inequality (3) becomes the following inequality,*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right] - \int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{n!} L_n^{n\frac{q-1}{q}}(a,b)\left[\frac{\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q}{(b-a)^2}\left(L_{n+1}^{n+1}(a,b) - aL_n^n(a,b)\right) + \left|f^{(n)}(a)\right|^q L_n^n(a,b)\right]^{1/q}.$$

(ii) *If we choose $s=1, n=1$, then the inequality (3) becomes the following inequality,*

$$\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \left(\frac{1}{b}\right)^{1/q} A^{1-1/q}(a,b)\left[(2b+a)\left|f'(b)\right|^q + (2a+b)\left|f'(a)\right|^q\right]^{1/q}.$$

(iii) *If we choose $s=1, n=1$ ve $q=1$, then the inequality (3) becomes the following inequality,*

$$\left|\frac{f(b)b - f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le (2b+a)\left|f'(b)\right| + (2a+b)\left|f'(a)\right|.$$

Theorem 5. *For $n \in \mathbb{N}$, let $f : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be $n-$times differentiable function on I^o and let $a,b \in I^o$ with $a<b$. If $\left|f^{(n)}\right|^q$ for $q>1$ is s-convex in the third sense function on $[a,b]$ and $\left|f^{(n)}\right|^q \in L[a,b]$ for some fixed $s \in [0,1)$, then the following inequality holds;*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right] - \int_a^b f(x)dx\right| \le \frac{b-a}{n!}\left[\frac{\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q}{(b-a)^{\frac{1}{s}+1}} G\left(\frac{1}{s},n,q,x\right) + \left|f^{(n)}(a)\right|^q L_{nq}^n(a,b)\right]^{1/q},$$

where $G\left(\frac{1}{s},n,q,x\right) = \int_a^b x^{nq}(x-a)^{1/s}dx$.

Proof. From Lemma 2, using the well known Hölder integral inequality and the s-convexity in the third sense of $\left|f^{(n)}\right|^q$, we find,

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1} - f^{(k)}(a)a^{k+1}}{(k+1)!}\right] - \int_a^b f(x)dx\right|$$
$$\le \left|\frac{(-1)^{n+1}}{n!}\int_a^b x^n f^{(n)}(x)dx\right|$$
$$\le \frac{1}{n!}\left(\int_a^b 1^p dx\right)^{1/p}\left(\int_a^b x^{nq}\left|f^{(n)}(x)\right|^q dx\right)^{1/q}$$
$$\le \frac{1}{n!}\left(\int_a^b 1dx\right)^{1/p}\left(\int_a^b x^{nq}\left[\left(\frac{x-a}{b-a}\right)^{1/s}\left|f^{(n)}(b)\right|^q + \left(1 - \left(\frac{x-a}{b-a}\right)^{1/s}\right)\left|f^{(n)}(a)\right|^q\right]dx\right)^{1/q}$$

$$= \frac{1}{n!}\left(\int_a^b dx\right)^{1/p}\left(\frac{\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q}{(b-a)^{1/s}}\int_a^b x^{nq}(x-a)^{1/s}dx + \left|f^{(n)}(a)\right|^q\int_a^b x^{nq}dx\right)^{1/q}$$

$$= \frac{1}{n!}(b-a)^{1/p}\left((b-a)\frac{\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q}{(b-a)^{\frac{1}{s}+1}}G\left(\frac{1}{s},n,q,x\right) + \left|f^{(n)}(a)\right|^q(b-a)\left(\frac{b^{nq+1}-a^{nq+1}}{(b-a)(nq+1)}\right)\right)^{1/q}$$

$$= \frac{1}{n!}(b-a)^{1/p}(b-a)^{1/q}\left(\frac{\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q}{(b-a)^{\frac{1}{s}+1}}G\left(\frac{1}{s},n,q,x\right) + \left|f^{(n)}(a)\right|^q L_{nq}^n(a,b)\right)^{1/q}.$$

Corollary 7. *Under the same assumptions given in Theorem 5,*

(i) *If we choose $s=1$, then the inequality (4) becomes the following inequality,*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right] - \int_a^b f(x)dx\right|$$
$$\leq \left[\frac{\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q}{n!}\left[L_{nq+1}^{nq+1}(a,b) - aL_{nq}^{nq}(a,b)\right] + \left|f^n(a)\right|^q L_{nq}^{nq}(a,b)\right]^{1/q}.$$

(ii) *If we choose $s=1, n=1$, then the inequality (4) becomes the following inequality,*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \left[\left(\left|f'(b)\right|^q - \left|f'(a)\right|^q\right)\left[L_{q+1}^{q+1}(a,b) - aL_q^q(a,b)\right] + \left|f'(a)\right|^q L_q^q(a,b)\right]^{1/q}.$$

(iii) *If we choose $s=1, n=1$ ve $q=1$, then the inequality (4) becomes the following inequality,*

$$\left|\frac{f(b)b-f(a)a}{b-a} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{b}\left[(2b+a)\left|f'(b)\right| + (2a+b)\left|f'(a)\right|\right].$$

Theorem 6. *For $n\in\mathbb{N}$, let $f: I\subset[0,\infty)\longrightarrow\mathbb{R}$ be $n-$times differentiable function on I^o and let $a,b\in I^o$ with $a<b$. If $\left|f^{(n)}\right|^q$ for $q>1$ is s-convex in the third sense function on $[a,b]$ and $\left|f^{(n)}\right|^q \in L[a,b]$ for some fixed $s\in[0,1)$, then the following inequality holds;*

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right] - \int_a^b f(x)dx\right| \leq \frac{(b-a)^{1/p}}{n!}2^{\frac{1-s^2}{qs^2}}L_{np}^n(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|. \quad (4)$$

Proof. From Lemma 2, the s-concavity in the third sense of $\left|f^{(n)}\right|^q$ and using the Hermite-Hadamard inequality for s-convex functions in the third sense [19] , we get

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right] - \int_a^b f(x)dx\right| \leq \frac{1}{n!}\int_a^b x^n\left|f^{(n)}(x)\right|dx$$
$$\leq \frac{1}{n!}\left(\int_a^b x^{np}dx\right)^{1/p}\left(\int_a^b\left|f^{(n)}(x)\right|^q dx\right)^{1/q}$$
$$\leq \frac{1}{n!}\left[\frac{x^{np+1}}{np+1}\Big|_a^b\right]^{1/p}\left[2^{\frac{1}{s^2}-1}\left|f^{(n)}\left(\frac{a+b}{2^{1/s}}\right)\right|^q\right]^{1/q}$$
$$= \frac{1}{n!}\left[\frac{b^{np+1}-a^{np+1}}{np+1}\right]^{1/p}2^{\frac{1}{qs^2}-\frac{1}{q}}\left|f^{(n)}\left(\frac{a+b}{2^{1/s}}\right)\right|$$
$$= \frac{(b-a)^{1/p}}{n!}2^{\frac{1-s^2}{qs^2}}L_{np}^n(a,b)\left|f^{(n)}\left(\frac{a+b}{2^{1/s}}\right)\right|.$$

Corollary 8. *Under the same assumptions given in Theorem 6,*

i) If we choose $s = 1$, then the inequality (4) becomes the following inequality,

$$\left|\sum_{k=0}^{n-1}(-1)^k\left[\frac{f^{(k)}(b)b^{k+1}-f^{(k)}(a)a^{k+1}}{(k+1)!}\right]-\int_a^b f(x)dx\right| \leq \frac{(b-a)^{1/p}}{n!}L_{np}^n(a,b)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|.$$

ii) If we choose $s = 1, n = 1$, then the inequality (4) becomes the following inequality,

$$\left|\frac{f(b)b-f(a)a}{(b-a)^{1/p}}-\frac{1}{(b-a)^{1/p}}\int_a^b f(x)dx\right| \leq L_p(a,b)\left|f'\left(\frac{a+b}{2}\right)\right|.$$

3 Conclusion

The article considers some new integral inequalities that differ from the current results for n-times differentiable s-convex functions in the third sense. While obtaining these integral inequalities, a new integral identity (Lemma 3) and an existing integral identity (Lemma 2) were used. Similar studies have been done for different classes of abstract convex functions. Researchers interested in convexity can obtain new inequalities for this new class by proving a new integral identity.

References

[1] Adilov, G., Yesilce, I., On generalizations of the concept of convexity, Hacettepe Journal of Mathematics and Statistics, 41 (5), 723–730, (2012).

[2] Adilov, G. R., Tinaztepe, G., Tinaztepe, R., On the global minimization of increasing positively homogeneous functions over the unit simplex, International Journal of Computer Mathematics, 87 (12), 2733-2746, (2010).

[3] Breckner, W.W., Stetigkeitsaussagen für eine Klasse verallgemekterter konvexer Funktionen in topologischen linearen Raumen, Publ. Inst. Math., 23, 13–20, (1978).

[4] Chen, F., Wu, S., Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions. Journal of Nonlinear Science Applications, 9, 705–716, (2016). Retrieved from www.tjnsa.com

[5] Dragomir, S.S., Pecaric, J., Persson, L.E., Some inequalities of Hadamard type, Soochow J. Math, 21 (3), 335–341, (1995).

[6] Eken, Z., Hermite-Hadamard Type Inequalities related s-convex functions in the third sense. Turkish Journal of Mathematics and Computer Science, 13 (2), (2021).

[7] Iscan, I., Kunt, M., Yazici, N., Hermite-Hadamard-Fejer type inequalities for harmonically convex functions via fractional integrals. New Trends in Mathematical Sciences, 4 (3), 239, (2016).

[8] Iscan, I., Kadakal, H., Kadakal, M., Some new integral inequalities for n-times differentiable log-convex functions. New Trends in Mathematical Sciences, 5 (2), 10-15, (2017).

[9] Iscan, I., Kadakal, H., Kadakal, M., Some new integral inequalities for functions whose nth derivatives in absolute value are (a, m)-convex functions. New Trends in Mathematical Sciences, 5 (2), 180-185, (2017).

[10] Iscan, I.and Aydin, M., On new general integral inequalities for h-convex function. New Trends in Mathematical Sciences, 4 (3), 73-87, (2016).

[11] Kemali, S., Yesilce, I., Adilov, G., B-convexity, B^{-1}-convexity, and their comparison, Numerical Functional Analysis and Optimization, 36 (2), 133–146, (2015).

[12] Kemali, S., Sezer, S., Tinaztepe, G., Adilov, G., s-Convex functions in the third sense. The Korean Journal of Mathematics, 29 (3), 593-602, (2021).

[13] Khan, M.A., Chu, Y., Khan, T.U., Khan, J., Some new inequalities of Hermite-Hadamard type for s-convex functions with applications. Open Mathematics, 15 (1), 1414–1430, (2017). https://doi.org/10.1515/math-2017-0121

[14] Kirmaci, U., Bakula, M., Ozdemir, M., Pecaric, J., Hadamard-tpye inequalities for s-convex functions, Appl. Math. Comp., 193, 26-35, (2007).

[15] Maden, S., Kadakal, H., Kadakal, M., Iscan, I., Some new integral inequalities for n-times differentiable convex and concave functions. Journal of Nonlinear Sciences and Applications, 10 (12), 6141-6148, (2017).

[16] Orlicz, W., A note on modular spaces I, Bull. Acad. Polon. Soi., Ser. Math. Astronom Phys., 9, 157–162, (1961).

[17] Sarikaya, M. Z., Kiris, M. E., Some new inequalities of Hermite-Hadamard type for s-convex functions. 16 (1), 491–501, (2015). https://doi.org/10.18514/MMN.2015.1099.

[18] Sezer, S., Eken, Z., Tinaztepe, G., Adilov, G., p-Convex functions and some of theirp properties, Numerical Functional Analysis and Optimization, 42 (4), 443-459, (2021).

[19] Sezer, S., The Hermite-Hadamard inequality for s-Convex functions in the third sense. AIMS Mathematics, 6(7), 7719-7732, (2021).