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The Solution of Linear Volterra Integral Equation of the First Kind with Aboodh Transform

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Abstract: In this paper, we apply Aboodh transform to solve linear Volterra integral equation of the first kind. A few examples solved by Aboodh Transform. Aboodh transform is a powerful method for solving linear Volterra integral equations of the first kind. The convolution theorem for the Aboodh transform has been proved.

Keywords: Integral Equations, Linear Volterra Integral Equation of the First Kind, Aboodh Transform, Convolution Theorem, Inverse Aboodh Transform

1 Introduction

Integral transforms have wide applications in the various disciplines of engineering and science to solve the problems of heat transfer, springs, mixing problems, electrical networks, bending of beams, carbon dating problems, Newton's second law of motion, signal processing, exponential growth and decay problems. In the advanced time, researchers are interested in solving the advance problems of research, science, space, engineering and real life by introducing new integral transforms. The Aboodh Transform is integral transform. There are many integral transforms in the literature. Some of these transformations are Laplace Transform, Fourier Transform, Sumudu Transform, Elzaki Transform, Aboodh Transform, Kamal Transform, ZZ-Transform [1–9, 14]. These transformations are used to solve for differential equations and integral equations. The Aboodh transform was first presented by Khalid Aboodh in 2013 [9–12]. This transformation has also been applied to the solution of ordinary differential equations and partial differential equations.

The linear Volterra integral equation of the first kind is given by

$$f(t) = \int_0^x K(x,t)\, u(t)\, dt.$$

Where the unknown function $u(x)$, that will be determined, occurs only inside the integral sign. The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions [11, 13].

The Aboodh transform defined for $t \geq 0$. Let $f(t)$ be an exponential order function in the set A as

$$A = \left\{ f(t) : \exists\, M,\ k_1, k_2 > 0,\ \ |f(t)| < Me^{-vt} \right\},$$

where, the constant M is finite number and k_1, k_2 are finite or may be infinite numbers. Then the Aboodh transform is,

$$\mathscr{A}[f(t)] = A(v) = \frac{1}{v}\int_0^\infty f(t)\, e^{-vt} dt\ ,\ t \geq 0, k_1 \leq v \leq k_2. \tag{1}$$

The unique function $f(t)$ in (1) is called the inverse transform of $A(v)$ is indicated by

$$f(t) = \mathscr{A}^{-1}[A(v)].$$

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1.1 Linearity Property of Aboodh Transform:

If $\mathscr{A}\{f(t)\} = F(v)$ and $\mathscr{A}\{g(t)\} = G(v)$ then
$\mathscr{A}\{af(t) + bg(t)\} = a\mathscr{A}\{f(t)\} + b\mathscr{A}\{g(t)\} = aF(v) + bG(v)$, where a, b are arbitrary constants.

1.2 Aboodh Transform of Some Elementary Functions:

	$\mathscr{A}\{f(t)\}$	$A(v)$
1	1	$\frac{1}{v^2}$
2	t	$\frac{1}{v^3}$
3	e^{at}	$\frac{1}{v^2-av}$, $v > a$
4	$sinat$	$\frac{a}{v(v^2+a^2)}$
5	$cosat$	$\frac{1}{v^2+a^2}$
6	t^n	$\frac{n!}{v^{n+2}}$
7	te^{at}	$\frac{1}{v(v-a)^2}$, $v > a$
8	$e^{at}\sin bt$	$\frac{b}{v[(v-a)^2+b^2]}$, $v > a$
9	$e^{at}\cos bt$	$\frac{v-a}{v[(v-a)^2+b^2]}$, $v > a$
10	$t\cos at$	$\frac{v^2-a^2}{(v^2+a^2)^2}$

1.3 Existence of Aboodh Transform

Theorem 1. *If $f(t)$ is piecewise continuous in interval $0 \le t \le K$ and of exponential order α for $t > K$, then its Aboodh transform $\mathscr{A}[f(t)]$ exists for all $v > \alpha$.*

Proof 1: We have for every positive number K,

$$\mathscr{A}[f(t)] = \frac{1}{v}\int_0^\infty e^{-vt} f(t)dt = \frac{1}{v}\int_0^K e^{-vt} f(t)\,dt + \frac{1}{v}\int_K^\infty e^{-vt} f(t)dt. \tag{2}$$

Since $f(t)$ is piecewise continuous in every finite interval $0 \le t \le K$, the first integral on the right side exists. Also the second integral on the right side exists. To see this we have only to observe that in such case:

$$\left|\frac{1}{v}\int_K^\infty e^{-vt} f(t)dt\right| \le \frac{1}{v}\int_K^\infty \left|f(t)e^{-vt}\right| dt \le \frac{1}{v}\int_K^\infty e^{-vt}\ |f(t)|\, dt \le \frac{1}{v}\int_K^\infty e^{-vt} M\, e^{\alpha t}\ dt$$
$$\le \frac{M}{v}\int_K^\infty e^{-vt} e^{\alpha t} \le \frac{M}{v}\int_K^\infty e^{-(v-\alpha)t} dt = \frac{M}{v}\left.\frac{e^{-(v-\alpha)t}}{(-(v-\gamma))}\right|_K^\infty = \frac{Me^{-(v-\alpha)K}}{v(v-\alpha)}.$$

So $f(t)$ is of exponential order α for $t > K$.

1.4 Convolution of two Functions:

Convolution of $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by:

$$F(t) * G(t) = F * G = \int_0^t F(x)\, G(t-x)\, dx = \int_0^t F(t-x)\, G(x)\, dx.$$

1.5 Convolution Theorem for Aboodh Transforms:

Theorem 2. *If* $\mathscr{A}\{F(t)\} = \mathscr{A}(v)$ *and* $\mathscr{A}\{G(t)\} = B(v)$ *then*

$$\mathscr{A}[f*g] = v\mathscr{A}[f]\mathscr{A}[g],$$
$$\mathscr{A}\{F(t)*G(t)\} = v\mathscr{A}\{F(t)\}\mathscr{A}\{G(t)\} = A(v)B(v),$$
$$\mathscr{A}[f*g] = v\mathscr{A}[f]\mathscr{A}[g]$$
$$or$$
$$\frac{1}{v}\mathscr{A}[f*g] = \mathscr{A}[f]\mathscr{A}[g].$$

Proof:

$$\mathscr{A}[f]\mathscr{A}[g] = \frac{1}{v}\int_0^\infty e^{-v\tau}f(\tau)\,d\tau\,.\frac{1}{v}\int_0^\infty e^{-vu}g(u)du,$$
$$= \frac{1}{v^2}\int_0^\infty e^{-v\tau}f(\tau)\,d\tau\int_0^\infty e^{-vu}g(u)du.$$
$$t = \tau + u \; and \; t - \tau \; = u.$$
$$\mathscr{A}[g] = \int_\tau^\infty e^{-v(t-\tau)}g(t-\tau)dt$$
$$= \int_\tau^\infty e^{-vt+v\tau}g(t-\tau)dt,$$
$$= e^{v\tau}\int_\tau^\infty e^{-vt}g(p-\tau)dt,$$
$$thus$$
$$\mathscr{A}[f]\mathscr{A}[g] = \frac{1}{v^2}\int_0^\infty e^{-v\tau}f(\tau)\,d\tau\;.\;e^{v\tau}\int_\tau^\infty g(t-\tau)e^{-vt}dt,$$
$$= \frac{1}{v^2}\int_0^\infty f(\tau)\;.\int_\tau^\infty g(t-\tau)e^{-vt}dtd\tau,$$
$$= \frac{1}{v^2}\int_0^\infty e^{-vt}\int_0^\infty f(\tau)g(t-\tau)\,d\tau dt,$$
$$= \frac{1}{v^2}\int_0^\infty e^{-vt}(f*g)(t)\,dt.$$
$$\mathscr{A}[f]\mathscr{A}[g] = \frac{1}{v}\mathscr{A}(f*g),$$
$$v\mathscr{A}[f]\mathscr{A}[g] = \mathscr{A}(f*g).$$

1.6 Inverse of Aboodh Transforms:

If $\mathscr{A}\{F(t)\} = F(v)$ then $F(t)$ is called the inverse Aboodh transform of $F(v)$ and it is defined as $F(t) = \mathscr{A}^{-1}\{F(v)\}$, where $\mathscr{A}^{-1}$ is the inverse Aboodh transform operator.

1.7 Applications

In this section, some applications are given in order to demostrate the effectiveness of Aboodh transform for solving of linear Volterra integral equation of the first kind.

Application 1 [11]

Consider Linear Volterra Integral Equation of the First Kind:

$$x = \int_0^x u(t)\,dt. \tag{3}$$

Applying the Aboodh transform to both sides of (3), we have:

$$\mathscr{A}\{x\} = \mathscr{A}\left\{\int_0^x u(t)\,dt\right\}. \tag{4}$$

Using convolution theorem of Aboodh transform on (4), we have:

$$\mathscr{A}\{x\} = v\mathscr{A}\{1\}\mathscr{A}\{u(x)\},$$
$$\frac{1}{v^3} = v\frac{1}{v^2}\mathscr{A}\{u(x)\}.$$

$$\mathscr{A}\{u(x)\} = \frac{1}{v^2}. \tag{5}$$

Operating inverse Aboodh transform on both sides of (5), we have:

$$u(x) = \mathscr{A}^{-1}\left(\frac{1}{v^2}\right),$$
$$u(x) = 1.$$

which is the required exact solution of (3).

Application 2 [11]

Consider Linear Volterra Integral Equation of the First Kind:

$$x^2 = \frac{1}{2}\int_0^x (x-t)\,u(t)\,dt. \tag{6}$$

Applying the Aboodh transform to both sides of (6), we have:

$$\mathscr{A}\{x^2\} = \mathscr{A}\left\{\frac{1}{2}\int_0^x (x-t)\,u(t)\,dt\right\}. \tag{7}$$

Using convolution theorem of Aboodh transform on (8), we have:

$$\frac{2}{v^4} = \frac{1}{2}v\mathscr{A}\{x\}\mathscr{A}\{u(x)\},$$
$$\frac{2}{v^4} = \frac{1}{2}v\frac{1}{v^3}\mathscr{A}\{u(x)\}.$$

$$\mathscr{A}\{u(x)\} = \frac{4}{v^2}. \tag{8}$$

operating inverse Aboodh transform on both sides of (7), we have:

$$u(x) = \mathscr{A}^{-1}\left\{\frac{4}{v^2}\right\}$$
$$u(x) = 4,$$

which is the required exact solution of (6).

Application 3 [11]

Consider Linear Volterra Integral Equation of the First Kind:

$$x = \int_0^x e^{-(x-t)}u(t)\,dt. \tag{9}$$

Applying the Aboodh transform to both sides of (9), we have:

$$\mathscr{A}\{x\} = \mathscr{A}\left\{\int_0^x e^{-(x-t)} u(t)\, dt\right\}. \tag{10}$$

Using convolution theorem of Aboodh transform on (10), we have:

$$\mathscr{A}\{x\} = v\mathscr{A}\left\{e^{-x}\right\}\mathscr{A}\{u(x)\},$$
$$\frac{1}{v^3} = v\frac{1}{(v^2+v)}\mathscr{A}\{u(x)\},$$
$$\frac{v+1}{v^3} = \mathscr{A}\{u(x)\}.$$

$$\frac{1}{v^2} + \frac{1}{v^3} = \mathscr{A}\{u(x)\}. \tag{11}$$

$$\mathscr{A}\{u(x)\} = \frac{1}{v^2} + \frac{1}{v^3}.$$

operating inverse Aboodh transform on both sides of (11), we have:

$$\mathscr{A}^{-1}\{\mathscr{A}(u(x))\} = \mathscr{A}^{-1}\left\{\frac{1}{v^2} + \frac{1}{v^3}\right\},$$

from the linearity property of the inverse Aboodh transform

$$\mathscr{A}^{-1}\{\mathscr{A}(u(x))\} = \mathscr{A}^{-1}\left\{\frac{1}{v^2}\right\} + \mathscr{A}^{-1}\left\{\frac{1}{v^3}\right\},$$
$$u(x) = 1 + x.$$

which is the required exact solution of (9).

Application 4 [11]

Consider Linear Volterra Integral Equation of the First Kind:

$$sinx = \int_0^x e^{(x-t)} u(t)\, dt. \tag{12}$$

Applying the Aboodh transform to both sides of (13), we have:

$$\mathscr{A}\{sinx\} = \mathscr{A}\left\{\int_0^x e^{(x-t)} u(t)\, dt\right\}. \tag{13}$$

Using convolution theorem of Aboodh transform on (13), we have:

$$\mathscr{A}\{sinx\} = v\mathscr{A}\{e^x\}\mathscr{A}\{u(x)\},$$
$$\frac{1}{v(v^2+1)} = v\frac{1}{v^2-v}\mathscr{A}\{u(x)\},$$
$$\mathscr{A}\{u(x)\} = \frac{v-1}{v(v^2+1)}.$$

$$\mathscr{A}\{u(x)\} = \frac{1}{(v^2+1)} - \frac{1}{v(v^2+1)}, \tag{14}$$

operating inverse Aboodh transform on both sides of (14), we have:

$$\mathscr{A}^{-1}\mathscr{A}\{u(x)\} = \mathscr{A}^{-1}\left\{\frac{1}{(v^2+1)} - \frac{1}{v(v^2+1)}\right\},$$

from the linearity property of the inverse Aboodh transform

$$u(x) = \mathscr{A}^{-1}\left\{\frac{1}{(v^2+1)}\right\} - \mathscr{A}^{-1}\left\{\frac{1}{v(v^2+1)}\right\},$$
$$u(x) = cosx - sinx,$$

which is the required exact solution of (12).

Application 5 [11]

Consider Linear Volterra Integral Equation of the First Kind:

$$x = \int_0^x e^{(x-t)}u(t)\,dt. \tag{15}$$

Applying the Aboodh transform to both sides of (15), we have:

$$\mathscr{A}\{x\} = \mathscr{A}\left\{\int_0^x e^{(x-t)}u(t)\,dt\right\}. \tag{16}$$

Using convolution theorem of Aboodh transform on (16), we have:

$$\mathscr{A}\{x\} = v\mathscr{A}(e^x)Z\{u(x)\},$$
$$\frac{1}{v^3} = v\frac{1}{v^2-v}\mathscr{A}\{u(x)\},$$
$$\frac{v-1}{v^3} = \mathscr{A}\{u(x)\},$$
$$\frac{1}{v^2} - \frac{1}{v^3} = \mathscr{A}\{u(x)\}.$$

$$\mathscr{A}\{u(x)\} = \frac{1}{v^2} - \frac{1}{v^3}, \tag{17}$$

operating inverse Aboodh transform on both sides of (17), we have:

$$\mathscr{A}^{-1}\mathscr{A}(\{u(x)\}) = \mathscr{A}^{-1}\left\{\frac{1}{v^2} - \frac{1}{v^3}\right\},$$

from the linearity property of the inverse Aboodh transform

$$\mathscr{A}^{-1}(\mathscr{A}\{u(x)\}) = \mathscr{A}^{-1}\left\{\frac{1}{v^2}\right\} - \mathscr{A}^{-1}\left\{\frac{1}{v^3}\right\},$$
$$u(x) = 1 - x,$$

which is the required exact solution of (15).

Application 6 [11]

Consider Linear Volterra Integral Equation of the First Kind:

$$y(t) = t^2 + \int_0^t y(u)\sin(t-u)du. \tag{18}$$

Applying the Aboodh transform to both sides of (18), we have:

$$\mathscr{A}\{y(t)\} = \mathscr{A}\left\{t^2 + \int_0^t y(u)\,sin(t-u)du\right\},$$

from the linearity property of the inverse Aboodh transform

$$\mathscr{A}\{y(t)\} = \mathscr{A}\left\{t^2\right\} + \mathscr{A}\left\{\int_0^t y(u)\,sin(t-u)du\right\}. \tag{19}$$

Using convolution theorem of Aboodh transform on (19), we have:

$$\mathscr{A}\{y(t)\} = \mathscr{A}\left\{t^2\right\} + v\mathscr{A}\{y(t)\}\mathscr{A}\{sint\},$$

$$\mathscr{A}\ \{y(t)\} = \frac{2}{v^4} + v\mathscr{A}\{y(t)\}\frac{1}{v(v^2+1)},$$

$$\mathscr{A}\{y(t)\} - \mathscr{A}\{y(t)\}\left(\frac{1}{(v^2+1)}\right) = \frac{2}{v^4},$$

$$\mathscr{A}\{y(t)\}\left(1 - \frac{1}{(v^2+1)}\right) = \frac{1}{v^4},$$

$$\mathscr{A}\ \{y(t)\}\left(\frac{v^2}{v^2+1}\right) = \frac{2}{v^4},$$

$$\mathscr{A}\{y(t)\} = \frac{2}{v^4}\frac{v^2+1}{v^2},$$

$$\mathscr{A}\{y(t)\} = \frac{2}{v^4}.\frac{v^2+1}{v^2},$$

$$\mathscr{A}\{y(t)\} = \frac{2}{v^4} + \frac{2}{v^6}, \tag{20}$$

operating inverse Aboodh transform on both sides of (20), we have:

$$\mathscr{A}^{-1}(\mathscr{A}\{y(t)\}) = \mathscr{A}^{-1}\left\{\frac{2}{v^4} + \frac{2}{v^6}\right\},$$

from the linearity property of the inverse Aboodh transform

$$y(t) = \mathscr{A}^{-1}\left\{\frac{2}{v^4}\right\} + \mathscr{A}^{-1}\left\{\frac{2}{v^6}\right\},$$

$$y(t) = t^2 + \frac{t^4}{12},$$

which is the required exact solution of (19).

Application 7 [12]

Consider Linear Volterra Integral Equation of the First Kind:

$$tsint = \int_0^t cos(t-u)\,u(t)dt. \tag{21}$$

Applying the Aboodh transform to both sides of (21), we have

$$\mathscr{A}\{tsint\} = \mathscr{A}\left\{\int_0^t cos(t-u)\,u(t)dt\right\}. \tag{22}$$

Using convolution theorem of Aboodh transform on (22), we have:

$$\mathscr{A}\{tsint\} = v\mathscr{A}\{cost\}\mathscr{A}\{u(t)\}, \tag{23}$$

$$\frac{2}{(v^2+1)^2} = v\frac{1}{v^2+1}\mathscr{A}\{u(t)\},$$
$$\mathscr{A}\{u(t)\} = \frac{2}{v(v^2+1)},$$

operating inverse Aboodh transform on both sides of (23), we have:

$$\mathscr{A}^{-1}(\mathscr{A}\{u(t)\}) = \mathscr{A}^{-1}(\frac{2}{v(v^2+1)}),$$
$$u(t) = \mathscr{A}^{-1}\left\{\frac{2}{v(v^2+1)}\right\},$$
$$u(t) = 2sint,$$

which is the required exact solution of (21).

Application 8 [12]

Consider linear Volterra integral equation of the first kind:

$$t = \int_0^t 3^{t-u}u(t)\,dt\ ,\ \ 0 \le t \le 1. \tag{24}$$

Applying the Aboodh transform to both sides of (24), we have

$$\mathscr{A}\{t\} = \mathscr{A}\left\{\int_0^t 3^{t-u}u(t)\,dt\right\}. \tag{25}$$

Using convolution theorem of Aboodh transform on (25), we have:

$$\mathscr{A}\{t\} = v\mathscr{A}\left\{3^t\right\}\mathscr{A}\{u(t)\},$$
$$\frac{1}{v^3} = v\left\{-\frac{1}{-v^2+ln3}\right\}\mathscr{A}\{u(t)\},$$
$$\mathscr{A}\{u(t)\} = \frac{v-ln3}{v^3},$$

$$\mathscr{A}\{u(t)\} = \frac{1}{v^2} - \frac{ln3}{v^3}, \tag{26}$$

operating inverse Aboodh transform on both sides of (26), we have:

$$\mathscr{A}^{-1}(\mathscr{A}\{u(t)\}) = \mathscr{A}^{-1}(\frac{1}{v^2}) - \mathscr{A}^{-1}(\frac{ln3}{v^3}),$$
$$u(t) = 1 - tln3,$$

which is the required exact solution of (24).

2 Conclusion

In this study, we have successfully discussed the Aboodh transform for the solution of Linear Volterra Integral Equation of the First Kind. The given applications show that the exact solution have been obtained using very less computational work and spending a very little time.

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