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On the versions in the plane $\mathbb{R}^2_{\pi 3}$ of some Euclidean theorems

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Abstract: In this paper, we give the ratios of directed lengths in planes the Euclidean and $\mathbb{R}^2_{\pi 3}$ and the analogues in the plane $\mathbb{R}^2_{\pi 3}$ of the Menelaus and Ceva's Theorems.

Keywords: Menelaus and Ceva's Theorems, $d_{\pi 3}$ distance, non-Euclidean geometry,

1 Introduction and preliminaries

The most interesting and important things that the change of the meaning and definition of distance used in Euclidean geometry for the last 2000 years, came possible by the use of taxicab geometry. The points are the same, the lines are the same, and the angles are measured in the same way. However, the distance function is different. A family of distances, $d_{\pi n}$, that includes Taxicab, Chinese-Checker and Iso-taxi distances, These distances have been studied by some authors [2, 3, 4, 5, 6, 7, 10, 11, 13, 14], as special cases introduced and the group of isometries of the plane with $d_{\pi n}-$metric is the semi-direct product of D_{2n} and $T(2)$ was shown in [1].

Iso-taxicab geometry is a non-Euclidean geometry defined by K. O. Sowell in 1989. In this geometry presented by Sowell three distance functions arise depending upon the relative positions of the points A and B. There are three axes at the origin; the x-axis, the y-axis and the y'-axis. The iso-taxicab trigonometric functions in iso-taxicab plane with three axes were given in [7, 8, 12].

Iso-taxicab functions were defined in terms of $d_{\pi n}$-distances on the plane $\mathbb{R}^2_{\pi 3}$. It was shown that how to obtain what a point on iso-taxicab plane correspond to a point on $\mathbb{R}^2_{\pi 3}$. The classification of the lines of the plane $\mathbb{R}^2_{\pi 3}$ and the shortest distance from a point to a line was obtain on $\mathbb{R}^2_{\pi 3}$. The area formula of triangles were given on $\mathbb{R}^2_{\pi 3}$. In generally, it was mentioned about the plane $\mathbb{R}^2_{\pi 3}$ trigonometry, then unit circle was defined on $\mathbb{R}^2_{\pi 3}$ and correspondingly trigonometric functions, iso-taxicab Pythagorean identity and trigonometric reduction formulas in $\mathbb{R}^2_{\pi 3}$ were given. The measures of angles and reference angle were defined on $\mathbb{R}^2_{\pi 3}$. It was obtained that the change of the length of the line segment under rotations on $\mathbb{R}^2_{\pi 3}$. The measures of angles were introduced by inner-product in [9].

The definition of $d_{\pi n}-$distances family is given as follows;

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Definition 1. *(see 1) Let $A=(x_1,y_1)$ and $B=(x_2,y_2)$ be any two points in $\mathbb{R}^2$, a family of $d_{\pi n}-$distances is defined by;*

$$d_{\pi n}(A,B)=\frac{1}{\sin\frac{\pi}{n}}\left(\left|\sin\frac{k\pi}{n}-\sin\frac{(k-1)\pi}{n}\right||x_1-x_2|+\left|\cos\frac{(k-1)\pi}{n}-\cos\frac{k\pi}{n}\right||y_1-y_2|\right)$$
$$\begin{cases} 1\leq k\leq\left[\frac{n-1}{2}\right],\ k\in\mathbb{Z}, & \text{if } \tan\frac{(k-1)\pi}{n}\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|\leq\tan\frac{k\pi}{n} \\ k=\left[\frac{n+1}{2}\right], & \text{if } \tan\frac{\left[\frac{n-1}{2}\right]\pi}{n}\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|<\infty \text{ or } x_1=x_2 \end{cases}$$

The plane $\mathbb{R}^2$ with the $d_{\pi n}-$distance is denoted by $\mathbb{R}^2_{\pi n}$. For $n=3$ and accordingly $k=1$, $k=2$, we obtain the formula of $d_{\pi 3}-$distance between the points A and B according to the inclination in the plane $\mathbb{R}^2_{\pi 3}$:

$$d_{\pi 3}(A,B)=\begin{cases} |x_1-x_2|+\frac{1}{\sqrt{3}}|y_1-y_2|, & 0\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|\leq\sqrt{3} \\ \frac{2}{\sqrt{3}}|y_1-y_2|, & \sqrt{3}\leq\left|\frac{y_2-y_1}{x_2-x_1}\right|<\infty \text{ or } x_1=x_2 \end{cases}$$

The plane $\mathbb{R}^2_{\pi 3}$ is constructed by simply replacing the Euclidean distance function d_E by the distance function $d_{\pi 3}$ of the plane $\mathbb{R}^2_{\pi 3}$. Therefore it seems to study the plane $\mathbb{R}^2_{\pi 3}$ analogues of the topics which include the concept of distance in the plane $\mathbb{R}^2$. In this paper, we will explain division points, directed lengths, the ratio of directed lengths and Menelaus' Theorem, Ceva's Theorem depending on these concepts.

2 Directed $\mathbb{R}^2_{\pi 3}-$lengths and division point

Let X and Y be any two points on a directed straight line l. We define directed $\mathbb{R}^2_{\pi 3}-$length of the line segment XY as follows:

$$d_{\pi 3}[XY]=\begin{cases} d_{\pi 3}(X,Y), & \text{if } XY \text{ and } l \text{ have the same direction} \\ -d_{\pi 3}(X,Y), & \text{if } XY \text{ and } l \text{ have opposite direction} \end{cases}$$

thus, $d_{\pi 3}[XY]=-d_{\pi 3}[YX]$. Clearly, directed length in the plane $\mathbb{R}^2_{\pi 3}$ can be defined in a similar way. That is

$$d_E[XY]=\begin{cases} d_E(X,Y), & \text{if } XY \text{ and } l \text{ have the same direction} \\ -d_E(X,Y), & \text{if } XY \text{ and } l \text{ have opposite direction} \end{cases}$$

If A,B,C are points on a same directed line and C is between points A and B, we denote this by ACB. If ACB, then C divides the line segment AB internally and the ratio of the $\mathbb{R}^2_{\pi 3}-$lengths is a positive real number. That is

$$\frac{d_{\pi 3}[AC]}{d_{\pi 3}[CB]}=\lambda>0$$

If ABC or CAB, then C divides AB externally and the ratio of the $\mathbb{R}^2_{\pi 3}-$lengths is a negative real number. That is

$$\frac{d_{\pi 3}[AC]}{d_{\pi 3}[CB]}=\lambda<0.$$

In both cases C is called the division point which divide the line segment AB in ratio λ.

Clearly, $C\neq B$. $C=A\Leftrightarrow\lambda=0$ and (C is at infinity $\Leftrightarrow\lambda=-1$).

Let C and C' be two points such that C divides a given line segment AB internally and C' divides AB externally in the same proportion though with opposite signs. Thus, the ratio of the directed lengths,

$$\frac{d_{\pi 3}\,[AC]}{d_{\pi 3}\,[CB]} = -\frac{d_{\pi 3}\,[AC']}{d_{\pi 3}\,[C'B]} = \lambda$$

is the same positive number λ.

Theorem 1. *Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be any distinct points in the plane $\mathbb{R}^2_{\pi 3}$. If $\theta = (x, y)$ is a point on the line passing through P_1 and P_2, then*

$$\frac{d_{\pi 3}\,[P_1\theta]}{d_{\pi 3}\,[\theta P_2]} = \frac{d_E\,[P_1\theta]}{d_E\,[\theta P_2]}.$$

That is; the ratios of directed lengths in the plane Euclidean and the plane $\mathbb{R}^2_{\pi 3}$ are the same.

Proof. The proof of the theorem will be shown in two stages by considering the inclination of the line passing through P_1 and P_2 in the plane $\mathbb{R}^2_{\pi 3}$. Let the slope of line be m;

(i) For $0 \leq |m| \leq \sqrt{3}$;

If $\theta = P_1$ then both ratios are equal to 0. If θ is at infinity then both ratios are equal to -1. Therefore without loss of generality, let $P_1 \neq \theta \neq P_2$. It is enough to show that

$$\frac{|x_1 - x| + \frac{1}{\sqrt{3}}\,|y_1 - y|}{|x - x_2| + \frac{1}{\sqrt{3}}\,|y - y_2|} = \frac{\sqrt{(x_1 - x)^2 + (y_1 - y)^2}}{\sqrt{(x - x_2)^2 + (y - y_2)^2}}. \tag{1}$$

Squaring both sides of the equation (1) one obtains; or simply

$$\frac{|x_1 - x|\,|y_1 - y| - \frac{2}{3}\,|y_1 - y|^2}{|x - x_2|\,|y - y_2| - \frac{2}{3}\,|y - y_2|^2} = \frac{(x_1 - x)^2 + (y_1 - y)^2}{(x - x_2)^2 + (y - y_2)^2}. \tag{2}$$

Examining the left side of the equation (2) one obtains

$$\frac{|x_1 - x|\,|y_1 - y| - \frac{2}{3}\,|y_1 - y|^2}{|x - x_2|\,|y - y_2| - \frac{2}{3}\,|y - y_2|^2} = \frac{(x_1 - x)(y_1 - y) - \frac{2}{3}(y_1 - y)^2}{(x - x_2)(y - y_2) - \frac{2}{3}(y - y_2)^2} \tag{3}$$

for all positions of θ on P_1P_2. Using the equation (3) in the equation (2) one obtains

$$\begin{aligned} &(x_1 - x)(y_1 - y) - \tfrac{2}{3}(y_1 - y)^2 (x - x_2)^2 + (y - y_2)^2 \\ &= \left((x - x_2)(y - y_2) - \tfrac{2}{3}(y - y_2)^2\right)\left((x_1 - x)^2 + (y_1 - y)^2\right). \end{aligned} \tag{4}$$

If $x_1 = x_2$ then $x = x_1 = x_2$ and equation (4) is obvious. If $x_1 \neq x_2$ then

$$y = \frac{(x_2 - x)y_1 - (x_1 - x)y_2}{x_2 - x_1}$$

since θ is on the line P_1P_2. Now, using this value of y in the first bracket of the equation (4) we get the equation (4) is satisfied.

(ii) For $\sqrt{3} \leq |m| \leq \infty$;

If $\theta = P_1$ then both ratios are equal to 0. If θ is at infinity then both ratios are equal to -1. Therefore without loss of generality, let $P_1 \neq \theta \neq P_2$. It is enough to show that is

$$\frac{\frac{2}{\sqrt{3}}\left|y_1-y\right|}{\frac{2}{\sqrt{3}}\left|y-y_2\right|}=\frac{\sqrt{\left(x_1-x\right)^2+\left(y_1-y\right)^2}}{\sqrt{\left(x-x_2\right)^2+\left(y-y_2\right)^2}}. \tag{5}$$

Squaring both sides of the equation (5) one obtains;

$$\frac{\left|y_1-y\right|^2}{\left|y-y_2\right|^2}=\frac{\left(x_1-x\right)^2+\left(y_1-y\right)^2}{\left(x-x_2\right)^2+\left(y-y_2\right)^2}$$

or another way

$$\frac{\left|y_1-y\right|^2\left[\left(x-x_2\right)^2+\left(y-y_2\right)^2\right]}{\left|y-y_2\right|^2\left[\left(x_1-x\right)^2+\left(y_1-y\right)^2\right]}=1$$

which is the equivalent to

$$\frac{\left|y_1-y\right|^2\left(x-x_2\right)^2}{\left|y-y_2\right|^2\left(x_1-x\right)^2}=1$$

or simply

$$\frac{\left|y_1-y\right|^2}{\left|y-y_2\right|^2}=\frac{\left(x_1-x\right)^2}{\left(x-x_2\right)^2}. \tag{6}$$

Examining the left side of the equation (6) one obtains

$$\frac{\left|y_1-y\right|^2}{\left|y-y_2\right|^2}=\frac{\left(y_1-y\right)^2}{\left(y-y_2\right)^2} \tag{7}$$

for all positions of θ on P_1P_2. Using the equation (7) in the equation (6) one obtains

$$\left(y_1-y\right)\left(x-x_2\right)=\left(y-y_2\right)\left(x_1-x\right). \tag{8}$$

If $x_1=x_2$ then $x=x_1=x_2$ and the equation (8) is obvious. If $x_1\neq x_2$ then

$$y=\frac{\left[\left(x_2-x\right)y_1-\left(x_1-x\right)y_2\right]}{\left(x_2-x_1\right)}$$

since θ is on the line P_1P_2. Now, using this value of y in the first bracket of the equation (8) we get

$$\begin{aligned}\left(y_1-\frac{\left[(x_2-x)y_1-(x_1-x)y_2\right]}{(x_2-x_1)}\right)\left(x-x_2\right) &= \left(\frac{\left[(x_2-x)y_1-(x_1-x)y_2\right]}{(x_2-x_1)}-y_2\right)\left(x_1-x\right)\\ \frac{(y_2-y_1)(x-x_2)(x_1-x)}{(x_2-x_1)} &= \frac{(y_2-y_1)(x-x_2)(x_1-x)}{(x_2-x_1)}\end{aligned}$$

which shows that the equation (8) is satisfied.

The following corollary shows how one can find the coordinate of the division point which divides the line segment joining two given points in a given ratio, in the plane $\mathbb{R}^2_{\pi 3}$.

Corollary 1. *Let $P_1=(x_1,y_1)$ and $P_2=(x_2,y_2)$ be two distinct points in the plane $\mathbb{R}^2_{\pi 3}$. If $\theta=(x,y)$ divides the line segment P_1P_2 in ratio λ then,*

$$x=\frac{x_1+\lambda x_2}{1+\lambda},\ y=\frac{y_1+\lambda y_2}{1+\lambda};\ \lambda\in\mathbb{R},\lambda\neq-1$$

as in the plane $\mathbb{R}^2_{\pi 3}$.

Proof. Although the corollary follows from Theorem 2 we prefer to give a direct proof. The given formula is obvious when $\lambda = 0$ or $\lambda = -1$. If $\lambda \neq 0, -1$ and θ divides the line segment P_1P_2 in ratio λ we have

$$\left| \frac{d_{\pi 3}\left[P_1\theta\right]}{d_{\pi 3}\left[\theta P_2\right]} \right| = |\lambda|.$$

The proof of the theorem will be shown in two stages by considering the inclination of the line passing through P_1 and P_2 in the plane $\mathbb{R}^2_{\pi 3}$. Let the slope of line be m;

(i) For $0 \leq |m| \leq \sqrt{3}$;

$$\frac{|x_1 - x| + \frac{1}{\sqrt{3}}|y_1 - y|}{|x - x_2| + \frac{1}{\sqrt{3}}|y - y_2|} = |\lambda| \tag{9}$$

Since $P_1 \neq P_2$

$$|\lambda| = |\lambda| \left(\frac{|x_1 - x_2| + \frac{1}{\sqrt{3}}|y_1 - y_2|}{|x_1 - x_2| + \frac{1}{\sqrt{3}}|y_1 - y_2|} \right) = \frac{|\lambda x_1 - \lambda x_2| + \frac{1}{\sqrt{3}}|\lambda y_1 - \lambda y_2|}{|x_1 - x_2| + \frac{1}{\sqrt{3}}|y_1 - y_2|}.$$

Adding $x_1 - x_1$ and $y_1 - y_1$ to the first and second summands in the numerator and similarly $\lambda x_2 - \lambda x_2$ and $\lambda y_2 - \lambda y_2$ in the denominator respectively, one obtains

$$|\lambda| = \frac{|\lambda x_1 + x_1 - x_1 - \lambda x_2| + \frac{1}{\sqrt{3}}|\lambda y_1 + y_1 - y_1 - \lambda y_2|}{|x_1 + \lambda x_2 - \lambda x_2 - x_2| + \frac{1}{\sqrt{3}}|y_1 + \lambda y_2 - \lambda y_2 - y_2|}.$$

Multiplying the numerator and denominator of the right side of the last statement by $\dfrac{1}{1+\lambda}$, one gets

$$|\lambda| = \frac{\left| x_1 - \frac{x_1 + \lambda x_2}{1+\lambda} \right| + \left| y_1 - \frac{y_1 + \lambda y_2}{1+\lambda} \right|}{\left| \frac{x_1 + \lambda x_2}{1+\lambda} - x_2 \right| + \left| \frac{y_1 + \lambda y_2}{1+\lambda} - y_2 \right|}.$$

Comparing this result with the equation (9) we obtain

$$x = \frac{x_1 + \lambda x_2}{1+\lambda} \text{ and } y = \frac{y_1 + \lambda y_2}{1+\lambda}.$$

(ii) For $\sqrt{3} \leq |m| \leq \infty$;

$$\frac{\frac{2}{\sqrt{3}}|y_1 - y|}{\frac{2}{\sqrt{3}}|y - y_2|} = |\lambda| \tag{10}$$

since $P_1 \neq P_2$

$$|\lambda| = \frac{\frac{2}{\sqrt{3}}|\lambda y_1 - \lambda y_2|}{\frac{2}{\sqrt{3}}|y_1 - y_2|},$$

adding $x_1 - x_1$ and $y_1 - y_1$ to the first and second summands in the numerator and similarly $\lambda x_2 - \lambda x_2$ and $\lambda y_2 - \lambda y_2$ in the denominator respectively, one obtains

$$|\lambda| = \frac{|\lambda y_1 + y_1 - y_1 - \lambda y_2|}{|y_1 + \lambda y_2 - \lambda y_2 - y_2|}.$$

Multiplying the numerator and denominator of the right side of the last statement by $\frac{1}{1+\lambda}$, one gets

$$|\lambda| = \frac{\left|y_1 - \frac{y_1+\lambda y_2}{1+\lambda}\right|}{\left|\frac{y_1+\lambda y_2}{1+\lambda} - y_2\right|}.$$

Comparing this result with the equation (10) we obtain

$$y = \frac{y_1+\lambda y_2}{1+\lambda}.$$

3 Theorems of Menelaus and Ceva in the plane $\mathbb{R}^2_{\pi 3}$

Ceva's and Menelaus theorems are two classic theorems in plane geometry. The main question of these theorems is to determine conditions under which three points are collinear and conditions under which three lines are concurrent. Ceva's theorem characterizes the concurrency of lines and Menelaus's theorem characterizes the collinearity of points.

In this section, we give analogues of the Theorems of Menelaus and Ceva in the plane $\mathbb{R}^2_{\pi 3}$. In fact, the validity of these theorems is clear from the Theorem 2, but we prefer to state and give partial proofs for them.

Theorem 2. *(Menelaus Theorem) Let $\{P_1,P_2,P_3\}$ be a triangle and $\theta_1,\theta_2,\theta_3$ be on the lines that contain the sides P_1P_2, P_2P_3, P_3P_1 in the plane $\mathbb{R}^2_{\pi 3}$. If $\theta_1,\theta_2,\theta_3$ are collinear, then*

$$\frac{d_{\pi 3}[P_1\theta_1]}{d_{\pi 3}[\theta_1 P_2]}\cdot\frac{d_{\pi 3}[P_2\theta_2]}{d_{\pi 3}[\theta_2 P_3]}\cdot\frac{d_{\pi 3}[P_3\theta_3]}{d_{\pi 3}[\theta_3 P_1]} = -1 \tag{11}$$

where none of $\theta_1,\theta_2,\theta_3$ coincide with any of P_1,P_2,P_3.

Proof. Several cases are possible, according to positions P_1,P_2,P_3 and $\theta_1,\theta_2,\theta_3$ We give a proof of the theorem only in the following special case. Let $P_i=(x_i,y_i)$, $i=1,2,3$, $x_i\neq x_{i+1}$ and the points $\theta_1,\theta_2,\theta_3$ be on a line l given by $y=mx+k$ such that $\theta_i = l\wedge P_iP_{i+1}$, (mod3) and l is not parallel to the line P_iP_{i+1} for $i=1,2,3$. Clearly, $mx_i-y_i+k\neq 0$ since $P_i\neq\theta_j$ for $i,j=1,2,3$ and $m\neq (y_{i+1}-y_i)(x_{i+1}-x_i)^{-1}$. The equation of the line P_iP_{i+1} is given by $y=(y_{i+1}-y_i)(x_{i+1}-x_i)^{-1}x-(x_iy_{i+1}-x_{i+1}y_i)(x_{i+1}-x_i)^{-1}$. It follows from a simple calculation that

$$\theta_i = \left(\frac{x_iy_{i+1}-x_{i+1}y_i-kx_i-kx_{i+1}}{mx_i-mx_{i+1}-y_i+y_{i+1}}, \frac{mx_iy_{i+1}-mx_{i+1}y_i-ky_i+ky_{i+1}}{mx_i-mx_{i+1}-y_i+y_{i+1}}\right).$$

Now, let's find $\frac{d_{\pi 3}[P_i\theta_i]}{d_{\pi 3}[\theta_iP_{i+1}]}$. Let's the proof it, the position of the line segment in the plane $\mathbb{R}^2_{\pi 3}$ Let the slope of line segment be m;

(i) For $0\leqslant |m|\leqslant\sqrt{3}$;

$$\frac{d_{\pi 3}[P_1\theta_1]}{d_{\pi 3}[\theta_1P_2]} = -\frac{d_{\pi 3}(P_1,\theta_1)}{d_{\pi 3}(\theta_1,P_2)} = -\frac{|mx_1-y_1+k|}{|mx_2-y_2+k|}$$

similarly,

$$\frac{d_{\pi 3}[P_2\theta_2]}{d_{\pi 3}[\theta_2P_3]} = \frac{d_{\pi 3}(P_2,\theta_2)}{d_{\pi 3}(\theta_2,P_3)} = \frac{|mx_2-y_2+k|}{|mx_3-y_3+k|}$$

and

$$\frac{d_{\pi 3}[P_3\theta_3]}{d_{\pi 3}[\theta_3P_1]} = \frac{d_{\pi 3}(P_3,\theta_3)}{d_{\pi 3}(\theta_3,P_1)} = \frac{|mx_3-y_3+k|}{|mx_1-y_1+k|}$$

and consequently,

$$\frac{d_{\pi 3}\left[P_i\theta_i\right]}{d_{\pi 3}\left[\theta_i P_{i+1}\right]} = s\frac{|mx_i - y_i + k|}{|mx_{i+1} - y_{i+1} + k|}, \quad s = \begin{cases} -1, & \text{if } i = 1 \\ 1, & \text{if } i = 2,3 \end{cases}$$

now, it can be easily computed that

$$\prod_{i=1}^{3}\left(\frac{d_{\pi 3}\left[P_i\theta_i\right]}{d_{\pi 3}\left[\theta_i P_{i+1}\right]}\right) = -1$$

and therefore, we obtain

$$\frac{d_{\pi 3}\left[P_1\theta_1\right]}{d_{\pi 3}\left[\theta_1 P_2\right]}\cdot\frac{d_{\pi 3}\left[P_2\theta_2\right]}{d_{\pi 3}\left[\theta_2 P_3\right]}\cdot\frac{d_{\pi 3}\left[P_3\theta_3\right]}{d_{\pi 3}\left[\theta_3 P_1\right]} = -1.$$

(ii) For $\sqrt{3} \leqslant |m| \leqslant \infty$;

$$\begin{aligned}
\frac{d_{\pi 3}[P_1\theta_1]}{d_{\pi 3}[\theta_1 P_2]} &= -\frac{d_{\pi 3}(P_1,\theta_1)}{d_{\pi 3}(\theta_1,P_2)} \\
&= -\frac{\frac{2}{\sqrt{3}}\left|y_1 - \frac{mx_1y_2 - mx_2y_1 - ky_1 + ky_2}{mx_1 - mx_2 - y_1 + y_2}\right|}{\frac{2}{\sqrt{3}}\left|\frac{mx_1y_2 - mx_2y_1 - ky_1 + ky_2}{mx_1 - mx_2 - y_1 + y_2} - y_2\right|} \\
&= -\frac{\frac{2}{\sqrt{3}}\left|mx_1y_1 - mx_1y_2 + y_1y_2 - y_1^2 + ky_1 - ky_2\right|}{\frac{2}{\sqrt{3}}\left|mx_2y_2 - mx_2y_1 + y_1y_2 - y_2^2 + ky_2 - ky_1\right|} \\
&= -\frac{\frac{2}{\sqrt{3}}\left|y_1(mx_1 - y_1 + k) - y_2(mx_1 - y_1 + k)\right|}{\frac{2}{\sqrt{3}}\left|y_2(mx_2 - y_2 + k) - y_1(mx_2 - y_2 + k)\right|} \\
&= -\frac{\frac{2}{\sqrt{3}}|y_1 - y_2||mx_1 - y_1 + k|}{\frac{2}{\sqrt{3}}|y_1 - y_2||mx_2 - y_2 + k|} \\
&= -\frac{|mx_1 - y_1 + k|}{|mx_2 - y_2 + k|}
\end{aligned}$$

similarly,

$$\frac{d_{\pi 3}\left[P_2\theta_2\right]}{d_{\pi 3}\left[\theta_2 P_3\right]} = \frac{d_{\pi 3}(P_2,\theta_2)}{d_{\pi 3}(\theta_2,P_3)} = \frac{|mx_2 - y_2 + k|}{|mx_3 - y_3 + k|}$$

and

$$\frac{d_{\pi 3}\left[P_3\theta_3\right]}{d_{\pi 3}\left[\theta_3 P_1\right]} = \frac{d_{\pi 3}(P_3,\theta_3)}{d_{\pi 3}(\theta_3,P_1)} = \frac{|mx_3 - y_3 + k|}{|mx_1 - y_1 + k|}$$

and consequently,

$$\frac{d_{\pi 3}\left[P_i\theta_i\right]}{d_{\pi 3}\left[\theta_i P_{i+1}\right]} = s\frac{|mx_i - y_i + k|}{|mx_{i+1} - y_{i+1} + k|}, \quad s = \begin{cases} -1, & \text{if } i = 1 \\ 1, & \text{if } i = 2,3 \end{cases}.$$

now, it can be easily computed that

$$\prod_{i=1}^{3}\left(\frac{d_{\pi 3}\left[P_i\theta_i\right]}{d_{\pi 3}\left[\theta_i P_{i+1}\right]}\right) = -1$$

and therefore, we obtain

$$\frac{d_{\pi 3}\left[P_1\theta_3\right]}{d_{\pi 3}\left[\theta_3 P_2\right]}\cdot\frac{d_{\pi 3}\left[P_2\theta_1\right]}{d_{\pi 3}\left[\theta_1 P_3\right]}\cdot\frac{d_{\pi 3}\left[P_3\theta_2\right]}{d_{\pi 3}\left[\theta_2 P_1\right]} = -1.$$

Theorem 3. *(Ceva's Theorem) Let $\{P_1,P_2,P_3\}$ be a triangle, P is any point inside of $\{P_1,P_2,P_3\}$ and lines l_1,l_2,l_3 pass through the vertices P_1,P_2,P_3, respectively, and intersect lines containing the opposite sides at points $\theta_1,\theta_2,\theta_3$, The lines l_1,l_2,l_3 are concurrent (or parallel) if and only if*

$$\frac{d_{\pi 3}\left[P_1\theta_1\right]}{d_{\pi 3}\left[\theta_1 P_2\right]}\cdot\frac{d_{\pi 3}\left[P_2\theta_2\right]}{d_{\pi 3}\left[\theta_2 P_3\right]}\cdot\frac{d_{\pi 3}\left[P_3\theta_3\right]}{d_{\pi 3}\left[\theta_3 P_1\right]} = 1.$$

Note that none of $\theta_1,\theta_2,\theta_3$ are P_1,P_2,P_3.

Proof. Let's apply the Menelaus' Theorem separately by considering the triangle $\{P_1, P_2, \theta_1\}$ and line segment $\theta_3 PP_3$; the triangle $\{P_1, \theta_1, P_3\}$ and line segment $P_2 P \theta_2$ for both situation in the plane $\mathbb{R}^2_{\pi 3}$. We obtain

$$\frac{d_{\pi 3}[P_3\theta_1]}{d_{\pi 3}[P_3P_2]} \cdot \frac{d_{\pi 3}[P_2\theta_3]}{d_{\pi 3}[\theta_3P_1]} \cdot \frac{d_{\pi 3}[P_1P]}{d_{\pi 3}[P\theta_1]} = -1$$

and

$$\frac{d_{\pi 3}[P_2\theta_1]}{d_{\pi 3}[P_2P_3]} \cdot \frac{d_{\pi 3}[P_3\theta_2]}{d_{\pi 3}[\theta_2P_1]} \cdot \frac{d_{\pi 3}[P_1P]}{d_{\pi 3}[P\theta_1]} = -1 \tag{12}$$

when (11) and (12) equations are divided by side to side we obtain

$$\frac{d_{\pi 3}[P_1\theta_2]}{d_{\pi 3}[\theta_2P_3]} \cdot \frac{d_{\pi 3}[P_3\theta_1]}{d_{\pi 3}[\theta_1P_2]} \cdot \frac{d_{\pi 3}[P_2\theta_3]}{d_{\pi 3}[\theta_3P_1]} = 1.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] A. Bayar, R. Kaya, *On Isometries of* $\mathbb{R}^2_{\pi n}$, Hacettepe Journal of Mathematics and Statistics, Vol. 40 (5) (2011), 673-679.

[2] A. Bayar, S. Ekmekci, M. Ozcan, *On Trigonometric Functions and Cosine and Sine Rules in Taxicab Plane*, International Electronic Journal of Geometry, Vol. 2 , No 1, (2009),17–24.

[3] A. Bayar, S. Ekmekci, *On The Chinese Checker Sine and Cosine Functions, International Journal of Mathematics and Analysis*, Vol. 1, No 3, (2006), 249–259.

[4] S. Ekmekci, Z. Akca, A. Altintas, *On Trigonometric Functions and Norm in The Generalized Taxicab Metric*, Math. Sciences and Applications E-Notes, Vol. 3, No 2, (2015), 27–33.

[5] S. Ekmekci, A. Bayar, Z. Akca, *On The Plane Geometry with Generalized Absolute Value Metric*, Math. Probl. Eng., Vol. 1, (2008), 1–8.

[6] R. Kaya, H. Barış Çolakoğlu, *The taxicab versions of Some EuclideanTheorems,* International Journal of Pure and Applied Mathematics, Volume 26 No 1, (2006), 69-80.

[7] I. Kocayusufoglu, *Trigonometry on Iso-taxicab Geometry*, Mathematical and Computational Applications, Vol. 5, No 3, (2002), 201-212.

[8] I. Kocayusufoglu, T. Ada, On The Iso-taxicab Trigonometry, Applied Sciences, Vol. 8 (2006), 101-111.

[9] S. Nazlı and Z. Akça, On The Trigonometric Functions in $\mathbb{R}^2_{\pi 3}$, Konuralp Journal of Mathematics, 8 (2), (2020), 429-437.

[10] M. Ozcan, S. Ekmekci, A. Bayar, *A Note on The Variation of The Taxicab Lengths Under Rotations,* Pi Mu Epsilon Journal, Vol. 11, No 7, (2002), 381-384.

[11] M. Ozcan, R. Kaya , On the Ratio of Directed Lengths in the Taxicab Plane and Related Properties, Missouri Journal of Mathematical Sciences, Vol.14, No 2, (2002), 107-117.

[12] K. O. Sowell, *Taxicab Geometry-a New Slant*, Mathematics Magazine, 62, (1989), 238-248.

[13] K. Thompson, T. Dray, *Taxicab Angles and Trigonometry*, Pi Mu Epsilon J., Vol. 11, (2000), 87-97.

[14] Z. Akça, R. Kaya, *On The Taxicab Trigonometry*, J. of Inst. Math. Comput. Sci. Math. Ser., Vol.10, (1997), 151-159.