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Classification of complete (k,2)-arcs in NFPG(2,9) using Veronesean

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Abstract: In this paper, we investigate the Veronesean arc, the non-Veronesean arc by converting a point expressed in Cartesian coordinates to homogeneous coordinates in left nearfield plane of order 9 where a $k-arc$ in a finite projective or affine plane is a set of k points no three of which are collinear. And also, we examine that whether founded complete $(7,2)$-Non-Veronesean arc satisfy Pascal's Theorem in the left nearfield projective plane of order 9. Six of $(7,2)$-Non-Veronesean arc's all points which are $\{1,4,11,21,35,38\}$ points line on same conic. But it is determined that $(7,2)$-Non-Veronesean arc does not satisfy Pascal's Theorem.

Keywords: (k,2)-arc, Pascal′s Theorem, Projective Plane, Veronesean arc.

1 Introduction

A projective plane π consist of a set $\mathscr{P}$ of points and a set $\mathscr{L}$ of subsets of $\mathscr{P}$, called lines, such that every pair of points is contained in exactly one line, every two distinct lines intersect in exactly one point, and there exist four points in such a position that they pairwise define six distinct lines. A subplane of a projective plane π is a set $\mathscr{B}$ of points and lines which is itself a projective plane, relative to the incidence relation given in π.

It is well known that there exist at least four non-isomorphic projective planes of order 9. The known four distinct projective planes of order 9 are extensively studied by Room et. al [15]. These are Desarguesian plane, the left nearfield plane, the right nearfield plane and Hughes plane. The last three planes of order 9 are called "miniquaternion planes" because they can be coordinatized by the miniquaternion near field. O. Veblen et. al. discovered these miniquaternion planes in 1907 [20] .

In a finite projective plane π (not necessarily Desarguesian) a set K of k $(k \geq 3)$ points such that no three points of K are collinear (on a line) is called a k-arc. If the plane π has order p then $k \leq p+2$, however the maximum value of k can only be achieved if p is even. In a plane of order p , a $(p+1)$-arc is called an oval and, if p is even, a $(p+2)$-arc is called a hyperoval. A general reference for ovals is Hirschfeld's study [17]. There are known plenty of examples of arcs in projective planes; see [3,7].

In [2], an algorithm (implemented in *C#*) to determine and classify Fano subplanes of the projective plane of order 9 coordinatized by elements of a left nearfield of order 9 is given. In [3,7], all complete $(k,2)$-arcs containing complete quadrangles which generate the Fano planes in the projective plane whose algebraic structure is the left nearfield of order 9 are examined. The Veronesean varieties have been studied by many scientists for years, while the beginning has been

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studied as classical real or complex varieties. In finite fields, they have proven to be very useful tools in finite geometry. The simplest Veronesean varieties are the quadric Veroneseans V_n of index n. Four different types of representations of finite quadratic Veroneseans can be found in the literature. In 1976, Tallini introduced a characterization which uses the the intersection properties of the so-called conic planes (planes of $PG(5,q)$ meeting V_2 in a conic) of V_2 [16] . It was only valid for q odd. Then Thas and Van Maldeghem [18] studied on this and generalized this to arbitrary $q>2$ and arbitrary index n. In meanwhile Ferri studied on the characterization such that for q odd and $q \geq 5$, the sizes of the intersections of V_2 with hyperplanes and planes of $PG(5,q)$ [8]. For this case, Hirschfeld and Thas [11] showed that its valid, when $q=3$ and then Thas and Van Maldeghem [18] generalized this situation to $q \neq 2$. The set of all conics contained in V_n and the tangent lines to these to axiomatize V_n in $PG(n(n+3)/2,q)$ such that q is odd [14]. Hirschfeld and Thas generalized Mazzocca and Melone's results to arbitrary q [13]. In the meantime, Thas et. al. continued to work on it, generalized and completed several characterizations of the finite quadric Veronesean [18]. Also, they classified the objects which satisfy the original set of Mazzocca and Melone's axioms in 2004 [17] by useing the unique representation of $PG(n,q)$ in $PG(d,q), d \geq n(n+3)/2$, such that points and lines of $PG(n,q)$ correspond to points and plane ovals of $PG(d,q)$, respectively, with $q \neq 2$, and with the condition that the point set of $PG(d,q)$ corresponding to the point set of $PG(n,q)$ generates $PG(d,q)$ [17]. In 2005, they characterized the finite Veronesean $H_n \subseteq PG(n(n+2),q)$ of all Hermitian varieties of $PG(n,q^2)$ as the unique representation of $PG(n,q^2)$ in $PG(d,q), d?n(n+2)$, where points and lines of $PG(n,q^2)$ are represented by points and ovoids of solids, respectively, of $PG(d,q)$, with the only condition that the point set of $PG(d,q)$ corresponding to the point set of $PG(n,q^2)$ generates $PG(d,q)$ [19]. In 2008, Dentice et. al. extended analogous theorems of Mazzocca and Melone [6; 14] and Thas et. al. for finite projective spaces by classifying all Veronesean caps of projective spaces of finite dimension over skewfields [17]. In 2012, Akca et al. generalized and classified lax Veronesean embeddings of projective spaces [1]. Pascal's theorem which is named as the hexagrammum mysticum theorem, states that if six arbitrary points are selected on the cone (in the appropriate affine plane, it can be an ellipse, parabola, or hyperbola), and the sequence is connected by line segments to form a hexagon. The three pairs of opposite sides of the hexagon (which can be extended if necessary) meet at three points on a straight line. These straight lines are called the Pascal lines of the hexagon [5]. Let A,B,C,D,E and F be the six vertices of a hexagon such that no three of them are collinear. Then, the intersection points of opposite sides $P=\overline{AB}\cap\overline{DE}$, $Q=\overline{BC}\cap\overline{EF}$, $R=\overline{CD}\cap\overline{AF}$ are collinear if and only if the points A,B,C,D,E and F lie on an oval. The if-statement is Pascal's Theorem , the converse is known as the Braikenridge-Maclaurin Theorem. The line through P, Q, R is called Pascal's line. Our aim is to determine whether the arcs are Veronesean and satify Pascal's Theorem between complete $(k,2)$-arcs in $NFPG(2,q)$.

2 Constructing of NFPG(2,9)

The set F with the binary operations $+$ and $\cdot$ is called a Left Nearfield if the following conditions hold:

(i) $(F,+)$ is an abelian group
(ii) For $\forall a,b,c \in F$, $(a\cdot b)\cdot c = a\cdot(b\cdot c)$
(iii) For $\forall a,b,c \in F$, $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$
(iv) For $\forall a \in F$, F contains an element 1 such that $1\cdot a = a\cdot 1 = a$
(v) For every non-zero element a of F, there exist an element a^{-1} such that $a\cdot a^{-1} = a^{-1}\cdot a = 1$.

The original construction of Hall planes was based on a Hall quasifield (also called a Hall system) [9; 10]. To build a Hall quasifield, start with a Galois field $F=GF(p^n)$, for p a prime and a quadratic irreducible polynomial $f(t)=t^2-rt-s$ over F. We consider an algebraic system $(S,\oplus,\odot)$ over the Galois field $(F_3,+,\cdot)$ of order 3. The nine elements of S are $a+\lambda b,\ a,\ b\in F_3,\ \lambda\notin F_3$. Addition in S is defined by the rule

$$(a+\lambda b)\oplus(c+\lambda d) = (a+c)+\lambda\,(b+d)$$

and multiplication by

$$(a+\lambda b)\odot(c+\lambda d)=\begin{cases} ca+\lambda\,(da) & \text{if } b=0\\ ca-db(a^2+1)+\lambda\,(cb-da) & \text{if } b\neq 0\end{cases}$$

where $a,\ b,\ c,\ d\in F_3,\ \lambda\notin F_3$ and $f(t)=t^2+1$ is an irreducible polynom on F_3. For the sake of the shortness if we use ab instead of $a+\lambda b$ in addition and multiplication equations, then addition and multiplication tables are obtained as follows:

$\oplus$	00	01	02	10	11	12	20	21	22
00	00	01	02	10	11	12	20	21	22
01	01	02	00	11	12	10	21	22	20
02	02	00	01	12	10	11	22	20	21
10	10	11	12	20	21	22	00	01	02
11	11	12	10	21	22	20	01	02	00
12	12	10	11	22	20	21	02	00	01
20	20	21	22	00	01	02	10	11	12
21	21	22	20	01	02	00	11	12	10
22	22	20	21	02	00	01	12	10	11

$\odot$	00	01	02	10	11	12	20	21	22
00	00	00	00	00	00	00	00	00	00
01	00	20	10	01	21	11	02	22	12
02	00	10	20	02	12	22	01	11	21
10	00	01	02	10	11	12	20	21	22
11	00	12	21	11	20	02	22	01	10
12	00	22	11	12	01	20	21	10	02
20	00	02	01	20	22	21	10	12	11
21	00	11	22	21	02	10	12	20	01
22	00	21	12	22	10	01	11	02	20

If we use the following equalities

$$\begin{aligned} 00&\to 0\\ 10&\to 1\\ 20&\to 2\\ 01&\to 3\\ 11&\to 4\\ 21&\to 5\\ 02&\to 6\\ 12&\to 7\\ 22&\to 8 \end{aligned}$$

the addition and multiplication tables in $(S,\oplus,\odot)$ can be arranged as follows:

$\oplus$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

$\odot$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	1	6	8	7	3	5	4
3	0	3	6	2	5	8	1	4	7
4	0	4	8	7	2	3	5	6	1
5	0	5	7	4	6	2	8	1	3
6	0	6	3	1	7	4	2	8	5
7	0	7	5	8	3	1	4	2	6
8	0	8	4	5	1	6	7	3	2

The projective plane whose point and the lines are coordinated by elements of $(S,\oplus,\odot)$. The 91 points of P_2S are the elements of the set $N=\{(x,y):x,y\in S\}\cup\{(m):m\in S\}\cup\{(\infty):\infty\notin S\}$. The points of the form (x,y) are called *proper* points, and the unique point (∞) and the points of the form (m) are called *ideal* points. The 91 lines of P_2S are defined as follows:

$$\mathscr{D} = \{[m,k] : m,k \in S\} \cup \{[a] : a \in S\} \cup \{[\infty] : \infty \notin S\}$$

We have 81 lines of the form $[m,k]$, 9 lines of the form $[a]$ are called *proper* lines and the unique line $[\infty]$ is called *ideal* line. And incidence relation

$$(x,y) \circ [m,k] \Leftrightarrow y = m \odot x \oplus k, \forall m,k \in S$$
$$(x,y) \circ [a] \Leftrightarrow x = a$$
$$(m) \circ [\infty]$$
$$(\infty) \circ [\infty]$$

We convert a point expressed in Cartesian coordinates to homogeneous coordinates in left nearfield plane of order 9. We have seen a point (x,y) in the P_2S has homogeneous coordinates $\lambda \odot (x,y,1) = (\lambda \odot x, \lambda \odot y, \lambda \odot 1)$, $\lambda \in S$, $\lambda \neq 0$. Homogeneous coordinates of the form $\lambda \odot (m,1,0)$ do correspond to the unique point at infinity in the P_2S. We have seen that a line $[m,k]$ in the P_2S has homogeneous coordinates $\mu \odot [m,-1,k] = [\mu \odot m, \mu \odot (-1), \mu \odot k]$, $\mu \in S$, $\mu \neq 0$. Homogeneous coordinates of the form $\mu \odot [x,0,1]$ do correspond to all lines $[a]$, $a \neq 0$, $a \in S$ in the P_2S. Homogeneous coordinates of the form $[0,0,\mu]$ do correspond to the unique line $[\infty]$ at infinity in the P_2S. A line in the P_2S has general equation $y = m \odot x \oplus k$. Suppose (x_1, x_2, x_3) , $x_3 \neq 0$ are the homogeneous coordinates of a point (x,y) on the line; hence $x_3^{-1} \odot x_1 = x$ and $x_3^{-1} \odot x_2 = y$. Substituting for x and y in the line equation and multiplying through by x_3 , yields the conditions for (x_1, x_2, x_3) to be the homogeneous coordinates of a point on the line :

$$m \odot x_1 \oplus (-1) \odot x_2 \oplus k \odot x_3 = 0$$

The table of all homogeneous coordinates of the 91 points and lines in the projective plane P_2S defined in terms of a coordinate system over the Hall system $(S, \oplus, \odot)$ is given, see [3].

3 Some properties of arcs

A $k-arc$ in a finite projective or affine plane is a set of k points no three of which are collinear. A $k-arc$ is complete if it is not contained in a $(k+1)-arc$. A line L is secant, tangent or passant to an arc if they have $2,1$ or 0 in common, respectively. In a plane of order q, a $(q+1)-arc$ is called an oval and if q is even, a $(q+2)-arc$ is called a hyperoval. When q is odd, it can be proved that (q+2)-arc do not exist. However, every conic is a (q+1)-arc, and a well-known theorem of Segre proves that also the converse is true when q is odd.

Fano subplanes of projective plane of order 9 were found in [2]. There are 18 Fano subplanes containing $I = (1,1,1), X = (1,0,0), O = (0,0,1), P = (a,b,1)$ with a,b$\in F_3$. We give an algorithm for finding complete $(k,2)-arcs$ with $6 \leq k \leq 10$ of this projective plane and apply the algorithm (implemented C#) to determine complete $(k,2)-arcs$ in [3,7].

For this projective plane containing Fano subplane, in [3] we found that 1 class of the complete $(6,2)-$ arcs, 108 classes of the complete $(7,2)-$ arcs, 319 classes of the complete $(8,2)-$arcs, 11 classes of the complete $(9,2)-$ arcs. For this projective plane not containing Fano subplane, in [7] we found that 168 classes of the complete $(7,2)-$– arcs, 276 classes of the complete $(8,2)-$ arcs, 6 classes of the complete $(9,2)-$ arcs, 3 classes of the complete $(10,2)-$ arcs.

A *conic C* is a set of q+1 points of PG(2,*q)* whoose coordinates (x,y,z) are the zeros of absolutely irreducible quadratic form

$$Q(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fxz$$

with $a,b,c,d,e \in \mathbf{F}_{q.}$ Every 5 points of an arc lie on an unique conic. An arc is called Veronesean if no 6 points of lie on the same conic. If we take $(6,2)$-arc which was found in our previous study [3;7] for projective plane of order 9 containing Fano plane, there is 1 class of complete $(6,2)$-arc $\{1,6,11,21,38,58\}$. We have $1=(1,0,0),\ 6=(4,1,0),\ 11=(0,0,1),\ 21=(1,1,1),\ 38=(0,3,1),\ 58=(2,5,1)$ in homogenous coordinates. Firstly,

$$Q(x,y,z)=ax^2+by^2+cz^2+dxy+eyz+fxz$$

conic equation must be obtained and then it must be shown that whether $(6,2)$-arc satisfies the conic equation found or not.

• If we start with first point $1=(1,0,0)$, put this on $Q(x,y,z)=ax^2+by^2+cz^2+dxy+eyz+fxz$
$Q(1,0,0)=a.1^2+b.0^2+c0^2+d.1.0+e.0.0+f.1.0=0 \Rightarrow Q(1,0,0)=a.1=0 \Rightarrow a=0$
• $6=(4,1,0)$
$Q(4,1,0)=a.4^2+b.1^2+c0^2+d.4.1+e.1.0+f.4.0=0 \Rightarrow Q(4,1,0)=a.2+b.1+d.4=0$
We found previous equation $a=0$, putting this value,
$Q(4,1,0)=0.2+b.1+d.4=0 \Rightarrow b.1+d.4=0 \Rightarrow b+d.4+d.8=d.8 \Rightarrow b+d.(4+8)=d.8 \Rightarrow b+d.0=d.8 \Rightarrow b=d.8 \Rightarrow b.4=d.8.4 \Rightarrow b.4=d.1 \Rightarrow b.4=d$
• $11=(0,0,1)$
$Q(0,0,1)=a.0^2+b.0^2+c.1^2+d.0.0+e.0.1+f.0.1=0 \Rightarrow Q(0,0,1)=c.1=0 \Rightarrow c=0$ •$21=(1,1,1)$
$Q(1,1,1)=a.1^2+b.1^2+c.1^2+d.1.1+e.1.1+f.1.1=0 \Rightarrow Q(1,1,1)=a.1+b.1+c.1+d.1+e.1+f.1=0 \Rightarrow Q(1,1,1)=a+b+c+d+e+f=0$ We found previous equations $a=0, c=0$, $b.4=d$, putting this value,
$b+b.4+e+f=0 \Rightarrow b.(1+4)+e+f=0 \Rightarrow b.5+e+f=0$
• $38=(0,3,1)$
$Q(0,3,1)=a.0^2+b.3^2+c.1^2+d.0.3+e.3.1+f.0.1=0 \Rightarrow Q(0,3,1)=b.3^2+c.1^2+e.3.1=0 \Rightarrow$
$Q(0,3,1)=b.2+c.1^2+e.3=0$ We found previous equation $c=0$, putting this value,
$b.2+e.3=0 \Rightarrow b.2+e.3+e.6=e.6 \Rightarrow b.2+e.(3+6)=e.6 \Rightarrow b.2+e.0=e.6 \Rightarrow b.2.3=e.6.3 \Rightarrow b.6=e$
If we put on this equation $b.6=e$ on $b.5+e+f=0$,
$b.5+b.6+f=0 \Rightarrow b.(5+6)+f=0 \Rightarrow b.2+f=0 \Rightarrow b.2+f+f.2=f.2 \Rightarrow b.2+f.(1+2)=f.2 \Rightarrow b.2+f.0=f.2 \Rightarrow b.2.2=f.2.2 \Rightarrow b=f$
We found that $a=0,\ b.4=d,\ c=0,\ b.6=e,\ b=f$
If we write these equations on $Q(x,y,z)=ax^2+by^2+cz^2+dxy+eyz+fxz$, we get $Q(x,y,z)=by^2+b.4.xy+b.6.yz+b.xz$
• Last point $58=(2,5,1)$
$Q(2,5,1)=b5^2+b.4.2.5+b.6.5.1+b.1.5 \Rightarrow Q(2,5,1)=b.2+b.6+b.4+b.5$
$b.2+b.6+b.4+b.5=b.(2+6+4+5)=b.5 \Rightarrow b.5 \neq 0$
So our last point $58=(2,5,1)$ does not satisfy our conic equation, we can say that this point not lie on $Q(x,y,z)=by^2+b.4.xy+b.6.yz+b.xz$, $\{1,6,11,21,38,58\}$ complete $(6,2)-arc$ is Veronesean arc. We examined complete (k,2)-arcs which were found in our previous study [3;7], we determined that some arcs are Veronesean but some of them are non-Veronesean.
We want to give an example for Non-Veronesean arc:
We take $\{1,4,11,21,35,38,88\}$ $(7,2)-arc$
We have $1=(1,0,0),\ 4=(2,1,0),\ 11=(0,0,1),\ 21=(1,1,1),\ 35=(6,2,1),\ 38=(0,3,1),\ 88=(5,8,1)$ in homogenous coordinates. Firstly we must find what is our

$$Q(x,y,z)=ax^2+by^2+cz^2+dxy+eyz+fxz$$

conic equation must be obtained and then it must be shown that whether $(7,2)$-arc satisfies the conic equation found or not.

• If we start with first point $1=(1,0,0)$, put this on $Q(x,y,z)=ax^2+by^2+cz^2+dxy+eyz+fxz$,
$Q(1,0,0)=a.1^2+b.0^2+c.0^2+d.1.0+e.0.0+f.1.0=0 \Rightarrow Q(1,0,0)=a.1=0 \Rightarrow a=0$
• $4=(2,1,0)$
$Q(2,1,0)=a.2^2+b.1^2+c.0^2+d.2.1+e.1.0+f.2.0=0 \Rightarrow Q(2,1,0)=a.1+b.1+d.2=0$
We found previous equation $a=0$, putting this value,
$Q(2,1,0)=b+d.2=0 \Rightarrow b+d.2=0 \Rightarrow b+d.2+d.1=d.1 \Rightarrow b+d.(2+1)=d \Rightarrow b+d.0=d \Rightarrow b=d$
• $11=(0,0,1)$
$Q(0,0,1)=a.0^2+b.0^2+c.1^2+d.0.0+e.0.1+f.0.1=0 \Rightarrow Q(0,0,1)=c.1=0 \Rightarrow c=0$
• $21=(1,1,1)$
$Q(1,1,1)=a.1^2+b.1^2+c.1^2+d.1.1+e.1.1+f.1.1=0 \Rightarrow Q(1,1,1)=a+b+c+d+e+f=0$
We found previous equations $a=0$, $b=d$, $c=0$, putting this values $b.2+e+f=0$
• $35=(6,2,1)$
$Q(6,2,1)=a.6^2+b.2^2+c.1^2+d.6.2+e.2.1+f.6.1=0 \Rightarrow Q(6,2,1)=a.2+b.1+c.1+d.3+e.2+f.6=0$ We found previous equations $a=0$, $b=d$, $c=0$, putting this values $b.4+e.2+f.6=0$
We found previous equation $b.2+e+f=0$, if we solve these equation system:
$b.4+e.2+f.6=0 \Rightarrow b.2+e+f=0 \Rightarrow b.(4+2)+e.(2+1)+f.(6+1)=0 \Rightarrow b.3+e.0+f.7=0 \Rightarrow b.3+f.7=0 \Rightarrow b.3+f.7+f.5=f.5 \Rightarrow b.3=f.5 \Rightarrow b.3.7=f \Rightarrow b.4=f$
If we put this value on $b.2+e+f=0$, we get $b.2+e+b.4=0$
$b.(2+4)+e=0 \Rightarrow b.3+e=0 \Rightarrow b.3+e+e.2=e.2 \Rightarrow b.3=e.2 \Rightarrow b.3.2=e.2.2 \Rightarrow b.6=e$
We know that every 5 points of an arc lie on an unique conic, so we can determine this conic equation. We have these equations:
$a=0,\ b=d,\ c=0,\ b.6=e,\ b.4=f$ and we obtained $Q(x,y,z)=by^2+b.xy+b.6yz+b.4xz$
• Another point of this $(7,2)-arc$ is $38=(0,3,1)$
$Q(0,3,1)=b.3^2+b.0.3+b.6.3.1+b.4.0.1=0 \Rightarrow b.2+b.1=0 \Rightarrow b.(2+1)=0 \Rightarrow b.0=0$
This $38=(0,3,1)$ point satisfies $Q(x,y,z)=by^2+b.xy+b.6yz+b.4xz$ conic equation, we can say that 6 points of $(7,2)-arc$ lie on the same conic, $(7,2)$-arc is non-Veronesean arc.
Now we examine that whether this complete $(7,2)$-Non-Veronesean satisfy Pascal's Theorem: 6 of its all points which are $\{1,4,11,21,35,38\}$ points line on same conic. But,

$P=[1,4]\wedge[21,35]=10$
$Q=[4,11]\wedge[35,38]=22$
$R=[11,21]\wedge[1,38]=41$
$P=10$, $Q=22$, $R=41$ are not collinear.
We can say that complete $(7,2)-arc$ does not satisfy Pascal's Theorem.

4 Conclusion

We take the set $A=\{O,I,X,P\}$ such that $O=(0,0,1), I=(1,1,1), X=(1,0,0), P=(a,b,1)$ with $a,b \in F_3$. We described procedure for searching all complete $(k,2)-Veronesean$ and $Non-Veronesean$ arcs with $6<k\leq 10$ containing the quadrangles $A=\{O,I,X,P\}$ such that $O=(0,0,1), I=(1,1,1), X=(1,0,0), P=(a,b,1)$ with $a \in F_3$, $b \in F_3-S$ constructing and not constructing the Fano subplanes in the left nearfield plane of order 9. For the projective plane of order 9 containing Fano plane, there is 1 class of complete $(6,2)$-arc-$\{1,6,11,21,38,58\}$. We have

$1 = (1,0,0),\ 6 = (4,1,0),\ 11 = (0,0,1),\ 21 = (1,1,1),\ 38 = (0,3,1),\ 58 = (2,5,1)$ in homogenous coordinates. By applying the conic equation

$$Q(x,y,z) = ax^2 + by^2 + cz^2 + dxy + eyz + fxz$$

we can see that the point $58 = (2,5,1)$ does not satisfy our conic equation and this point not lie on $Q(x,y,z) = by^2 + b.4.xy + b.6.yz + b.xz$. So it is obtained that $\{1,6,11,21,38,58\}$ complete $(6,2)-arc$ is Veronesean arc. And also, some of these complete $(k,2)-arcs$ are Veronesean but some of them are non-Veronesean as $\{1,4,11,21,35,38,88\}$-$(7,2)-arc$ satisfying the conic equation. By applying the method the complete $(k,2)$-Veronesean and Non- Veronesean arcs are determined. Lastly, it is examined that this $(7,2)$-Non-Veronesean arc $\{1,4,11,21,35,38,88\}$ whether satisfy Pascal's Theorem or not. As a result this $\{1,4,11,21,35,38,88\}$ Non-Veronesean does not satisfy Pascal's Theorem.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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