New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2022.472

On Some New Types of Separation Axioms Via δ^*-semiopen Sets

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Received: 11 February 2022, Accepted: 7 March 2022
Published online: 25 March 2022.

Abstract: In this paper, we study different properties of δ^*-semiopen set. We define the concept of δ^*-semi generalized closed sets and present some characteristics. In addition, as applications to δ^*-semi generalized closed set, we introduce δ^*-semi $T_{\frac{1}{2}}$ space and obtain some of their basic properties. Moreover, we defined the notions of δ^*-semi symmetric space, δ^*-semi difference sets and δ^*-semi kernel of sets, and investigate some of their fundamental properties. At the latest, some new types of spaces are introduced and the relationships of these spaces are studied.

Keywords: δ^*-semiopen set, δ^*-semiclosed set, δ^*-semi generalized closed.

1 Introduction

The study of semiopen sets and their properties was initiated by N. Levine [4] which is one of the well-known notion of generalized open sets. After the work of Levine on semiopen sets, various mathematicians turned their attention to the generalizations of various concepts of topology by considering semiopen sets instead of open sets. While open sets are replaced by semiopen sets, new results are obtained in some occasions and in other occasions substantial generalizations are exhibited. Njastad [7] defined the class of α-open sets. Mashhour et al [5] introduced the concept of preopen sets in topological spaces.

In this direction, Maheshwari and Prasad [6] used semiopen sets to define and investigate three new separation axioms, called semi-T_0, semi-T_1 and semi-T_2. Later, Bhattacharyya and Lahiri [1] generalized the concept of closed sets to semi-generalized closed sets with the help of semi-openness. Also, they defined the concept of a new class of topological spaces called semi-$T_{\frac{1}{2}}$ and further investigated the separation axioms semi-T_0, semi-T_1 and semi-T_2. Tong [8] introduced the nation of D-sets and used these sets to introduce a separation axiom D_1 which is strictly between T_0 and T_1. Miguel Caldas [2] introduce a new separation axiom semi-D_1 which is strictly between semi-T_0 and semi-T_1 and discuss its relations with the axioms mentioned above. In 2015, Ibrahim [3] introduced some new classes of sets used the open sets and functions in topological spaces. He defined δ^*-open set, δ^*-α-open set, δ^*-preopen set and δ^*-semiopen set, and investigated the relationships between them. Let (Y,δ) be a topological spaces and let f be a function from X into Y, then a subset G in δ is called δ^*-open if $f^{-1}(G) = X$, or $f^{-1}(G) = \phi$, that is $\delta^* = \{G \in \delta : f^{-1}(G) = X, or f^{-1}(G) = \phi\}$. The family of all δ^*-open sets in Y is denoted by δ^*. A subset F of Y is called δ^*-closed if $Y \setminus F$ is δ^*-open. The family of all δ^*-closed sets in Y is denoted by δ^*c.

We recall the following results from [3].

Definition 1. Let $f : X \to Y$ be any function and A be any subset of a topological space (Y,δ). Then,

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(1) the union of all δ^*-open sets contained in A is called the δ^*-interior of A and denoted by δ^*-int(A).
(2) the intersection of all δ^*-closed sets containing A is called the δ^*-closure of A and denoted by δ^*-cl(A).

Definition 2. Let $f: X \to Y$ be any function. A subset A of a space Y is called:

(1) δ^*-α-open if $A \subset \delta^*$-int(δ^*-cl(δ^*-int(A))).
(2) δ^*-preopen if $A \subset \delta^*$-int(δ^*-cl((A)).
(3) δ^*-semiopen if $A \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(A))$.

The family of all δ^*-semiopen sets in Y denoted by δ^*SO.

Proposition 1. Let $f: X \to Y$ be a function. Then,

(1) for every δ^*-open set G and for every subset $A \subseteq Y$ we have $\delta^*\text{-}cl(A) \cap G \subseteq \delta^*\text{-}cl(A \cap G)$.
(2) every δ^*-open set is δ^*-α-open.
(3) the concepts of δ^*-semiopen and semiopen are independent.

2 δ^*-semiopen Sets

In this section, we discuss some the properties of δ^*-semiopen sets.

Theorem 1. An arbitrary union of δ^*-semiopen sets is δ^*-semiopen.

Proof. Let $\{A_i : i \in I\}$ be a family of δ^*-semiopen sets. Then for each i, $A_i \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(A_i))$ and so

$$\begin{aligned}\cup_{i\in I} A_i &\subseteq \cup_{i\in I}[\delta^*\text{-}cl(\delta^*\text{-}int(A_i))]\\ &\subseteq [\cup_{i\in I}\delta^*\text{-}cl(\delta^*\text{-}int(A_i))]\\ &\subseteq [\delta^*\text{-}cl(\cup_{i\in I}\delta^*\text{-}int(A_i))]\\ &\subseteq [\delta^*\text{-}cl(\delta^*\text{-}int(\cup_{i\in I}A_i))].\end{aligned}$$

Thus, $\cup_{i\in I} A_i$ is a δ^*-semiopen set.

Theorem 2. Let $f: X \to Y$ be a any function. If A is δ^*-α-open in Y and B is δ^*-semiopen in Y, then $A \cap B$ is δ^*-semiopen in Y.

Proof. By assumption, $A \subseteq \delta^*\text{-}int(\delta^*\text{-}cl(\delta^*\text{-}int(A)))$ and $B \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(B))$, then by Proposition 1, we have that

$$\begin{aligned}A \cap B &\subseteq \delta^*\text{-}int(\delta^*\text{-}cl(\delta^*\text{-}int(A))) \cap \delta^*\text{-}cl(\delta^*\text{-}int(B))\\ &\subseteq \delta^*\text{-}cl[\delta^*\text{-}int(\delta^*\text{-}cl(\delta^*\text{-}int(A))) \cap \delta^*\text{-}int(B)]\\ &\subseteq \delta^*\text{-}cl[\delta^*\text{-}cl(\delta^*\text{-}int(A)) \cap \delta^*\text{-}int(B)]\\ &\subseteq \delta^*\text{-}cl[\delta^*\text{-}cl[\delta^*\text{-}int(A) \cap \delta^*\text{-}int(B)]]\\ &= \delta^*\text{-}cl(\delta^*\text{-}int(A \cap B)).\end{aligned}$$

Thus, $A \cap B$ is δ^*-semiopen.

Remark. If A is δ^*-open in Y and B is δ^*-semiopen in Y, then $A \cap B$ is δ^*-semiopen in Y.

Remark. We note that the intersection of two δ^*-semiopen sets need not be δ^*-semiopen as can be seen from the following example.

Example 1. Consider $X = \{a,b,c\}$ and $Y = \{1,2,3\}$ with $\delta = \{\phi, Y, \{2\}, \{3\}, \{2,3\}\}$. Let $f : X \to Y$ be a function such that $f(a) = f(b) = f(c) = 2$, then $\delta^* = \{\phi, Y, \{2\}, \{3\}, \{2,3\}\}$. Let $A = \{1,2\}$ and $B = \{1,3\}$, then A and B are δ^*-semiopen but $A \cap B = \{1\}$ is not δ^*-semiopen.

Theorem 3. Let $f : X \to Y$ be any function and $V \subseteq Y$. Then, V is δ^*-semiopen if and only if for each $s \in V$, there exists a δ^*-semiopen set K such that $s \in K \subseteq V$.

Proof. It is obvious.

Theorem 4. Let $f : X \to Y$ be a any function and $A, B \subseteq Y$. Then,

(1) A is δ^*-semiopen if and only if there exists a δ^*-open set U such that $U \subseteq A \subseteq \delta^*\text{-}cl(U)$.
(2) B is δ^*-semiopen if A is δ^*-semiopen and $A \subseteq B \subseteq \delta^*\text{-}cl(A)$.
(3) A is δ^*-semiopen if and only if $\delta^*\text{-}cl(A) = \delta^*\text{-}cl(\delta^*\text{-}int(A))$.

Proof. (1) Let A be δ^*-semiopen, then $A \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(A))$. Take $U = \delta^*\text{-}int(A)$, then U is δ^*-open such that $U = \delta^*\text{-}int(A) \subseteq A \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(A)) = \delta^*\text{-}cl(U)$.

Conversely, since $U \subseteq A$ implies that $U = \delta^*\text{-}int(U) \subseteq \delta^*\text{-}int(A)$ and so $A \subseteq \delta^*\text{-}cl(U) = \delta^*\text{-}cl(\delta^*\text{-}int(U)) \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(A))$. Thus, A is δ^*-semiopen.

(2) Since A is δ^*-semiopen, then by (1) there exists a δ^*-open set U such that $U \subseteq A \subseteq \delta^*\text{-}cl(U)$. Since $A \subseteq B$, so $U \subseteq B$. But $\delta^*\text{-}cl(A) \subseteq \delta^*\text{-}cl(U)$, then $B \subseteq \delta^*\text{-}cl(U)$. Hence, $U \subseteq B \subseteq \delta^*\text{-}cl(U)$. Thus, B is δ^*-semiopen.

(3) Let A be δ^*-semiopen, then $A \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(A))$ which implies that $\delta^*\text{-}cl(A) \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(A)) \subseteq \delta^*\text{-}cl(A)$ and hence $\delta^*\text{-}cl(A) = \delta^*\text{-}cl(\delta^*\text{-}int(A))$.

Conversely, since $\delta^*\text{-}int(A)$ is δ^*-semiopen such that $\delta^*\text{-}int(A) \subseteq A \subseteq \delta^*\text{-}cl(A) = \delta^*\text{-}cl(\delta^*\text{-}int(A))$ and therefore A is δ^*-semiopen.

Definition 3. A subset F of Y is called δ^*-semiclosed if $Y \setminus F$ is δ^*-semiopen.

Theorem 5. Let $f : X \to Y$ be a any function and $A, B \subseteq Y$. If A is δ^*-semiclosed and $\delta^*\text{-}int(A) \subseteq B \subseteq A$, then B is δ^*-semiclosed.

Proof. Since A is δ^*-semiclosed, then $Y \setminus A$ is δ^*-semiopen. By hypothesis $\delta^*\text{-}int(A) \subseteq B \subseteq A$, so $Y \setminus A \subset Y \setminus B \subset Y \setminus \delta^*\text{-}int(A)$. But $Y \setminus \delta^*\text{-}int(A) = \delta^*\text{-}cl(Y \setminus A)$. Therefore $Y \setminus A \subset Y \setminus B \subset \delta^*\text{-}cl(Y \setminus A)$, and hence by Theorem 4 (2), $Y \setminus B$ is δ^*-semiopen. Thus, B is δ^*-semiclosed.

Remark. (1) An arbitrary intersection of δ^*-semiclosed sets is δ^*-semiclosed.
(2) The union of two δ^*-semiclosed sets may not be δ^*-semiclosed.

Definition 4. Let $f : X \to Y$ be any function and $A \subseteq Y$. Then,

(1) the union of all δ^*-semiopen sets contained in A is called the δ^*-semi-interior of A and denoted by $\delta^*\text{-}sint(A)$.
(2) the intersection of all δ^*-semiclosed sets containing A is called the δ^*-semi-closure of A and denoted by $\delta^*\text{-}scl(A)$.

Now, we state the following theorem without proof.

Theorem 6. Let $f : X \to Y$ be any function. For any subsets A and B of Y, we have the following:

(1) A is δ^*-semiopen if and only if $A = \delta^*\text{-}sint(A)$.
(2) A is δ^*-semiclosed if and only if $A = \delta^*\text{-}scl(A)$.
(3) If $A \subseteq B$, then $\delta^*\text{-}sint(A) \subseteq \delta^*\text{-}sint(B)$ and $\delta^*\text{-}scl(A) \subseteq \delta^*\text{-}scl(B)$.

(4) $\delta^*\text{-}sint(A) \cup \delta^*\text{-}sint(B) \subseteq \delta^*\text{-}sint(A \cup B)$.
(5) $\delta^*\text{-}sint(A \cap B) \subseteq \delta^*\text{-}sint(A) \cap \delta^*\text{-}sint(B)$.
(6) $\delta^*\text{-}scl(A) \cup \delta^*\text{-}scl(B) \subseteq \delta^*\text{-}scl(A \cup B)$.
(7) $\delta^*\text{-}scl(A \cap B) \subseteq \delta^*\text{-}scl(A) \cap \delta^*\text{-}scl(B)$.
(8) $\delta^*\text{-}sint(X \setminus A) = X \setminus \delta^*\text{-}scl(A)$.
(9) $\delta^*\text{-}scl(X \setminus A) = X \setminus \delta^*\text{-}sint(A)$.
(10) $X \setminus \delta^*\text{-}scl(X \setminus A) = \delta^*\text{-}sint(A)$.
(11) $X \setminus \delta^*\text{-}sint(X \setminus A) = \delta^*\text{-}scl(A)$.
(12) $x \in \delta^*\text{-}sint(A)$ if and only if there exists a δ^*-semiopen set L such that $x \in L \subseteq A$.

Theorem 7. Let $f : X \to Y$ be any function and A be a subset of Y. Then, $y \in \delta^*\text{-}scl(A)$ if and only if for every δ^*-semiopen subset L of Y containing $y \in Y$, $A \cap L \neq \phi$.

Proof. Let $y \in \delta^*\text{-}scl(A)$ and suppose that $L \cap A = \phi$ for some δ^*-semiopen set L which contains y. Then, $(Y \setminus L)$ is δ^*-semiclosed and $A \subset (Y \setminus L)$, thus $\delta^*\text{-}scl(A) \subset (Y \setminus L)$. But this implies that $y \in Y \setminus L$, a contradiction. Thus, $L \cap A \neq \phi$. Conversely, let $A \subseteq Y$ and $y \in Y$ such that for each δ^*-semiopen set L_1 which contains y, $L_1 \cap A \neq \phi$. If $y \notin \delta^*\text{-}scl(A)$, there is a δ^*-semiclosed set F such that $A \subseteq F$ and $y \notin F$. Then, $(Y \setminus F)$ is a δ^*-semiopen set with $y \in (Y \setminus F)$, and thus $(Y \setminus F) \cap A \neq \phi$, which is a contradiction.

Remark. Let A be any subset of Y. Then the following relation holds, $\delta^*\text{-}int(A) \subseteq \delta^*\text{-}sint(A) \subseteq A \subseteq \delta^*\text{-}scl(A) \subseteq \delta^*\text{-}cl(A)$

Let H be any subset of Y. Then, the collection $\tau_H = \{U \cap H : U \in \delta^*\}$ is called a subspace topology on H. The pair (H, τ_H) is called a subspace, and each member of τ_H is called a δ^*-open set in H. For any $B \subseteq Y$, $\delta^*\text{-}int_{\tau_H}(B)$ is called δ^*-interior of B in H and $\delta^*\text{-}cl_{\tau_H}(B)$ is called δ^*-closure of B in H.

Remark. Let $f : X \to Y$ be any function and H be a subset of Y. Then, $\delta^*\text{-}cl_{\tau_H}(B) = \delta^*\text{-}cl(B) \cap H$, for any $B \subseteq H$.

Theorem 8. Let $f : X \to Y$ be any function and $A, B \subseteq Y$. If A is δ^*-preopen in Y and B is δ^*-semiopen in Y, then

(1) $A \cap B$ is δ^*-semiopen in A.
(2) $A \cap B$ is δ^*-preopen in B.

Proof. By assumption, $A \subseteq \delta^*\text{-}int(\delta^*\text{-}cl(A))$ and $B \subseteq \delta^*\text{-}cl(\delta^*\text{-}int(B))$.

(1) Then,

$$\begin{aligned} A \cap B &\subseteq \delta^*\text{-}int(\delta^*\text{-}cl(A)) \cap \delta^*\text{-}cl(\delta^*\text{-}int(B)) \\ &\subseteq \delta^*\text{-}cl[\delta^*\text{-}int(\delta^*\text{-}cl(A)) \cap \delta^*\text{-}int(B)] \\ &\subseteq \delta^*\text{-}cl[\delta^*\text{-}cl(A) \cap \delta^*\text{-}int(B)] \\ &\subseteq \delta^*\text{-}cl[\delta^*\text{-}cl[A \cap \delta^*\text{-}int(B)]] \\ &= \delta^*\text{-}cl[A \cap \delta^*\text{-}int(B)]. \end{aligned}$$

Hence, $A \cap B \subseteq \delta^*\text{-}cl(A \cap \delta^*\text{-}int(B))$ and so $A \cap B \subseteq \delta^*\text{-}cl(A \cap \delta^*\text{-}int(B)) \cap A = \delta^*\text{-}cl_{\tau_A}(A \cap \delta^*\text{-}int(B))$. Since $A \cap \delta^*\text{-}int(B)$ is a δ^*-open set in A, so $A \cap B \subseteq \delta^*\text{-}cl_{\tau_A}(A \cap \delta^*\text{-}int(B)) = \delta^*\text{-}cl_{\tau_A}(\delta^*\text{-}int_{\tau_A}(A \cap \delta^*\text{-}int(B))) \subseteq \delta^*\text{-}cl_{\tau_A}(\delta^*\text{-}int_{\tau_A}(A \cap B))$. Therefore, $A \cap B$ is δ^*-semiopen in A.

(2) Now,

$$\begin{aligned} A \cap B \subseteq \delta^*\text{-}int(\delta^*\text{-}cl(A)) \cap B &= \delta^*\text{-}int_{\tau_B}[\delta^*\text{-}int(\delta^*\text{-}cl(A)) \cap B] \\ &\subseteq \delta^*\text{-}int_{\tau_B}[\delta^*\text{-}int(\delta^*\text{-}cl(A)) \cap \delta^*\text{-}cl(\delta^*\text{-}int(B))] \end{aligned}$$

$$\subseteq \delta^*\text{-}int_{\tau_B}[\delta^*\text{-}cl[\delta^*\text{-}int(\delta^*\text{-}cl(A))\cap\delta^*\text{-}int(B)]]$$
$$\subseteq \delta^*\text{-}int_{\tau_B}[\delta^*\text{-}cl[\delta^*\text{-}cl(A)\cap\delta^*\text{-}int(B)]]$$
$$\subseteq \delta^*\text{-}int_{\tau_B}[\delta^*\text{-}cl[\delta^*\text{-}cl[A\cap\delta^*\text{-}int(B)]]]$$
$$\subseteq \delta^*\text{-}int_{\tau_B}[\delta^*\text{-}cl[\delta^*\text{-}cl[A\cap B]]]$$
$$= \delta^*\text{-}int_{\tau_B}(\delta^*\text{-}cl(A\cap B)).$$

Since δ^*-$int_{\tau_B}(\delta^*$-$cl(A\cap B))$ is δ^*-open in B, then δ^*-$int_{\tau_B}(\delta^*$-$cl(A\cap B))\cap B = \delta^*$-$int_{\tau_B}(\delta^*$-$cl(A\cap B)\cap B)$, and hence $A\cap B\subseteq\delta^*$-$int_{\tau_B}(\delta^*$-$cl(A\cap B)\cap B) = \delta^*$-$int_{\tau_B}(\delta^*$-$cl_{\tau_B}(A\cap B))$.

Therefore, $A\cap B$ is δ^*-preopen in B.

Theorem 9. Let $f : X \to Y$ be any function, $A\subseteq B\subseteq Y$ and B be δ^*-semiopen in Y. Then, A is δ^*-semiopen in Y if and only if A is δ^*-semiopen in B.

Proof. Let A be δ^*-semiopen in Y, then there is a δ^*-open set U such that $U\subseteq A\subseteq\delta^*$-$cl(U)$ implies that $U\subseteq A\subseteq B$. Hence, $U\subseteq A\subseteq\delta^*$-$cl(U)\cap B = cl_{\tau_B}(U)$. Since $U\cap B = U$ is also δ^*-open in B, then A is δ^*-semiopen in B.

Conversely, let A be δ^*-semiopen in B. Then there is a δ^*-open set U in B such that $U\subseteq A\subseteq cl_{\tau_B}(U)$. Since U is δ^*-open in B, there exists a δ^*-open set V such that $U = V\cap B$. Then, $V\cap B = U\subseteq A\subseteq cl_{\tau_B}(U) = cl_{\tau_B}(V\cap B)\subseteq\delta^*$-$cl(V\cap B)$. By Rematrk 2, $V\cap B$ is δ^*-semiopen, then by Theorem 4 (2), A is δ^*-semiopen in Y.

3 δ^*-semi $T_{\frac{1}{2}}$ Space

In this section, we defne and study some properties of δ^*-semi generalized closed set and δ^*-semi $T_{\frac{1}{2}}$ space.

Definition 5. *Let $f : X \to Y$ be any function. A subset A of Y is said to be a δ^*-semi generalized closed (briefly, δ^*-sgc) if δ^*-$scl(A)\subseteq U$ whenever $A\subseteq U$ and U is a δ^*-semiopen set in Y.*

Remark. It is clear that every δ^*-semiclosed subset of Y is also a δ^*-sgc set. The following example shows that a δ^*-sgc set need not be δ^*-semiclosed.

Example 2. Consider $X = \{1,2,3\}$ and $Y = \{a,b,c\}$ with $\delta = \{Y,\phi,\{a\},\{b,c\}\}$. If $f : X\to Y$ is a function such that $f(1) = f(2) = f(3) = a$, then $\delta^*SO = \{Y,\phi,\{a\},\{b,c\}\}$. Thus, $\{b\}$ is δ^*-sgc but it is not δ^*-semiclosed.

Theorem 10. Let $f : X\to Y$ be any function. A subset A of Y is δ^*-sgc if and only if δ^*-$scl(\{x\})\cap A\neq\phi$ holds for every $x\in\delta^*$-$scl(A)$.

Proof. Let L be a δ^*-semiopen set such that $A\subseteq L$. Let $x\in\delta^*$-$scl(A)$, then there exists an element $z\in\delta^*$-$scl(\{x\})$ and $z\in A\subseteq L$. It follows from Theorem 7, that $L\cap\{x\}\neq\phi$ and hence $x\in L$. This implies δ^*-$scl(A)\subseteq L$. Thus, A is δ^*-sgc set in Y.
Conversely, let A be a δ^*-sgc subset of Y and $x\in\delta^*$-$scl(A)$ such that δ^*-$scl(\{x\})\cap A = \phi$. Since, δ^*-$scl(\{x\})$ is δ^*-semiclosed set in Y. Thus, by Definition 3, $Y\setminus(\delta^*$-$scl(\{x\}))$ is a δ^*-semiopen set. Since $A\subseteq Y\setminus(\delta^*$-$scl(\{x\}))$ and A is δ^*-sgc implies that δ^*-$scl(A)\subseteq Y\setminus(\delta^*$-$scl(\{x\}))$ holds, and hence $x\notin\delta^*$-$scl(A)$. This is a contradiction. Thus, δ^*-$scl(\{x\})\cap A\neq\phi$.

Theorem 11. If δ^*-$scl(\{x\})\cap A\neq\phi$ holds for every $x\in\delta^*$-$scl(A)$, then δ^*-$scl(A)\setminus A$ does not contain a non empty δ^*-semiclosed set.

Proof. Suppose there exists a non empty δ^*-semiclosed set F such that $F \subseteq \delta^*\text{-}scl(A) \setminus A$. Let $x \in F$, then $x \in \delta^*\text{-}scl(A)$. It follows that $F \cap A = \delta^*\text{-}scl(F) \cap A \supseteq \delta^*\text{-}scl(\{x\}) \cap A \neq \phi$. Hence, $F \cap A \neq \phi$. This is a contradiction. Thus, $F = \phi$.

Theorem 12. If a subset A of Y is δ^*-sgc and $A \subseteq B \subseteq \delta^*\text{-}scl(A)$, then B is a δ^*-sgc set in Y.

Proof. Let A be a δ^*-sgc set such that $A \subseteq B \subseteq \delta^*\text{-}scl(A)$. Let L be a δ^*-semiopen subset of Y such that $B \subseteq L$. Since A is δ^*-sgc, then $\delta^*\text{-}scl(A) \subseteq L$. Now, $\delta^*\text{-}scl(A) \subseteq \delta^*\text{-}scl(B) \subseteq \delta^*\text{-}scl(\delta^*\text{-}scl(A)) = \delta^*\text{-}scl(A) \subseteq L$, that is $\delta^*\text{-}scl(B) \subseteq L$. Thus, B is a δ^*-sgc set in Y.

Theorem 13. Let $f : X \to Y$ be any function. Then, for each $x \in Y$, either $\{x\}$ is δ^*-semiclosed or $Y \setminus \{x\}$ is δ^*-sgc.

Proof. Suppose that $\{x\}$ is not δ^*-semiclosed, then by Definition 3, $Y \setminus \{x\}$ is not δ^*-semiopen. Let L be any δ^*-semiopen set such that $Y \setminus \{x\} \subseteq L$, so $L = Y$. Hence, $\delta^*\text{-}scl(Y \setminus \{x\}) \subseteq L$. Thus, $Y \setminus \{x\}$ is δ^*-sgc.

Definition 6. *Let $f : X \to Y$ be any function. Then, Y is called δ^*-semi symmetric if for $x, y \in Y$ such that $x \in \delta^*\text{-}scl(\{y\})$ implies $y \in \delta^*\text{-}scl(\{x\})$.*

Example 3. Consider $X = \{1,2,3\}$ and $Y = \{a,b,c\}$ with
$\delta = \{Y, \phi, \{a\}, \{b,c\}\}$. If $f : X \to Y$ is a function such that $f(1) = f(2) = f(3) = a$ then $\delta^*SO = \{Y, \phi, \{a\}, \{b,c\}\}$. Thus, Y is δ^*-semi symmetric.

Theorem 14. Let $f : X \to Y$ be any function. Then, the following statements are equivalent:

(1) Y is a δ^*-semi symmetric.
(2) $\{x\}$ is δ^*-sgc, for each $x \in Y$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\{x\} \subseteq L \in \delta^*SO$, but $\delta^*\text{-}scl(\{x\}) \nsubseteq L$. Then, $\delta^*\text{-}scl(\{x\}) \cap Y \setminus L \neq \phi$. Now, we take $y \in \delta^*\text{-}scl(\{x\}) \cap Y \setminus L$, then by hypothesis $x \in \delta^*\text{-}scl(\{y\}) \subseteq Y \setminus L$ and $x \notin L$, which is a contradiction. Therefore, $\{x\}$ is δ^*-sgc for each $x \in Y$.
(2) $\Rightarrow$ (1): Suppose that $x \in \delta^*\text{-}scl(\{y\})$, but $y \notin \delta^*\text{-}scl(\{x\})$. Then, $\{y\} \subseteq Y \setminus \delta^*\text{-}scl(\{x\})$ and hence $\delta^*\text{-}scl(\{y\}) \subseteq Y \setminus \delta^*\text{-}scl(\{x\})$. Therefore, $x \in Y \setminus \delta^*\text{-}scl(\{x\})$ which is a contradiction and so $y \in \delta^*\text{-}scl(\{x\})$.

Definition 7. *Let $f : X \to Y$ be any function. Then, Y is said to be δ^*-semi $T_{\frac{1}{2}}$ if every δ^*-sgc in Y is δ^*-semiclosed.*

Example 4. Consider $X = \{1,2,3\}$ and $Y = \{a,b,c\}$ with
$\delta = \{Y, \phi, \{a\}, \{a,c\}\}$. If $f : X \to Y$ is a function such that $f(1) = f(2) = c$ and $f(3) = b$, then $\delta^*SO = \{Y, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Then, Y is δ^*-semi $T_{\frac{1}{2}}$.

Theorem 15. Let $f : X \to Y$ be any function. Then, Y is a δ^*-semi $T_{\frac{1}{2}}$ if and only if $\{x\}$ is either δ^*-semiclosed or δ^*-semiopen for each $x \in Y$.

Proof. Suppose $\{x\}$ is not δ^*-semiclosed. Then it follows from assumption and Theorem 13, that $\{x\}$ is δ^*-semiopen. Conversely, let F be δ^*-sgc set in Y and x be any point in $\delta^*\text{-}scl(F)$, then $\{x\}$ is δ^*-semiopen or δ^*-semiclosed.

(1) Suppose $\{x\}$ is δ^*-semiopen. Then by Theorem 7, we have $\{x\} \cap F \neq \phi$ and hence $x \in F$. This implies $\delta^*\text{-}scl(F) \subseteq F$, therefore F is δ^*-semiclosed.
(2) Suppose $\{x\}$ is δ^*-semiclosed. Assume $x \notin F$, then $x \in \delta^*\text{-}scl(F) \setminus F$. This is not possible by Theorem 11. Thus, we have $x \in F$. Therefore, $\delta^*\text{-}scl(F) = F$ and hence F is δ^*-semiclosed.

Definition 8. *Let $f : X \to Y$ be any function. Then, a subset A of Y is called a δ^*-semi Difference set (briefly, δ^*sD-set) if there are $L, K \in \delta^*SO$ such that $L \neq Y$ and $A = L \setminus K$.*

It is true that every δ^*-semiopen set L different from Y is a δ^*sD-set if $A = L$ and $K = \phi$. So, we can observe the following.

Remark. Every proper δ^*-semiopen set is a $\delta^* sD$-set. But, the converse is not true in general as the next example shows.

Example 5. Consider $X = \{1,2,3,4,5\}$ and $Y = \{a,b,c,d,e\}$ with $\delta = \{Y,\phi,\{a\},\{a,b,c,d\},\{b,c,d\}\}$. If $f : X \to Y$ is a function such that $f(1) = f(2) = f(3) = c$ and $f(4) = f(5) = d$, then
$\delta^* SO = \{Y,\phi,\{a\},\{a,e\},\{b,c,d\},\{a,b,c,d\},\{b,c,d,e\}\}$. If we take $L = \{b,c,d,e\} \neq Y$ and $K = \{b,c,d\}$, then $L \setminus K = \{b,c,d,e\} \setminus \{b,c,d\} = \{e\}$. Thus, $\{e\}$ is a $\delta^* sD$-set but it is not δ^*-semiopen.

Definition 9. *Let $f : X \to Y$ be any function and A be a subset of Y. Then, the δ^*-semi kernel of A denoted by $\delta^* sker(A)$ is defined to be the set*

$$\delta^* sker(A) = \cap\{L \in \delta^* SO\text{: } A \subseteq L\}.$$

Example 6. From example 5, let $A = \{b,c\}$, then $\delta^* sker(\{b,c\}) = \{b,c,d\}$.

Theorem 16. Let $f : X \to Y$ be any function and $x \in Y$. Then, $y \in \delta^* sker(\{x\})$ if and only if $x \in \delta^*\text{-}scl(\{y\})$.

Proof. Suppose that $y \notin \delta^* sker(\{x\})$. Then, there exists a δ^*-semiopen set K containing x such that $y \notin K$. Therefore, we have $x \notin \delta^*\text{-}scl(\{y\})$. The proof of the converse case can be done similarly.

Theorem 17. Let $f : X \to Y$ be any function and A be a subset of Y. Then, $\delta^* sker(A) = \{x \in Y\text{: } \delta^*\text{-}scl(\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in \delta^* sker(A)$ and suppose $\delta^*\text{-}scl(\{x\}) \cap A = \phi$. Hence, $x \notin Y \setminus \delta^*\text{-}scl(\{x\})$ which is a δ^*-semiopen set containing A. This is impossible, since $x \in \delta^* sker(A)$. Consequently, $\delta^*\text{-}scl(\{x\}) \cap A \neq \phi$. Next, let $x \in Y$ such that $\delta^*\text{-}scl(\{x\}) \cap A \neq \phi$ and suppose that $x \notin \delta^* sker(A)$. Then, there exists a δ^*-semiopen set K containing A and $x \notin K$. Let $y \in \delta^*\text{-}scl(\{x\}) \cap A$. Hence, K is a δ^*-semiopen set containing y which does not contain x. By this contradiction $x \in \delta^* sker(A)$.

Theorem 18. Let $f : X \to Y$ be any function. Then, the following properties hold for the subsets A,B of Y:

(1) $A \subseteq \delta^* sker(A)$.
(2) $A \subseteq B$ implies that $\delta^* sker(A) \subseteq \delta^* sker(B)$.
(3) If A is δ^*-semiopen in Y, then $A = \delta^* sker(A)$.
(4) $\delta^* sker(\delta^* sker(A)) = \delta^* sker(A)$.

Proof. (1), (2) and (3) are immediate consequences of Definition 9. To prove (4), first observe that by (1) and (2), we have $\delta^* sker(A) \subseteq \delta^* sker(\delta^* sker(A))$. If $x \notin \delta^* sker(A)$, then there exists $L \in \delta^* SO$ such that $A \subseteq L$ and $x \notin L$. Hence, $\delta^* sker(A) \subseteq L$, and so we have $x \notin \delta^* sker(\delta^* sker(A))$. Thus $\delta^* sker(\delta^* sker(A)) = \delta^* sker(A)$.

Theorem 19. Let $f : X \to Y$ be any function. Then, for any points x and y in Y the following statements are equivalent:

(1) $\delta^* sker(\{x\}) \neq \delta^* sker(\{y\})$.
(2) $\delta^*\text{-}scl(\{x\}) \neq \delta^*\text{-}scl(\{y\})$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\delta^* sker(\{x\}) \neq \delta^* sker(\{y\})$, then there exists a point z in Y such that $z \in \delta^* sker(\{x\})$ and $z \notin \delta^* sker(\{y\})$. From $z \in \delta^* sker(\{x\})$ it follows that $\{x\} \cap \delta^*\text{-}scl(\{z\}) \neq \phi$ which implies $x \in \delta^*\text{-}scl(\{z\})$. By $z \notin \delta^* sker(\{y\})$, we have $\{y\} \cap \delta^*\text{-}scl(\{z\}) = \phi$. Since $x \in \delta^*\text{-}scl(\{z\})$, then $\delta^*\text{-}scl(\{x\}) \subseteq \delta^*\text{-}scl(\{z\})$ and $\{y\} \cap \delta^*\text{-}scl(\{x\}) = \phi$. Therefore, it follows that $\delta^*\text{-}scl(\{x\}) \neq \delta^*\text{-}scl(\{y\})$. Thus, $\delta^* sker(\{x\}) \neq \delta^* sker(\{y\})$ implies that $\delta^*\text{-}scl(\{x\}) \neq \delta^*\text{-}scl(\{y\})$.
(2) $\Rightarrow$ (1): Suppose that $\delta^*\text{-}scl(\{x\}) \neq \delta^*\text{-}scl(\{y\})$. Then, there exists a point z in Y such that $z \in \delta^*\text{-}scl(\{x\})$ and $z \notin \delta^*\text{-}scl(\{y\})$. Then, there exists a δ^*-semiopen set containing z and therefore x but not y. Hence, $y \notin \delta^* sker(\{x\})$ and thus $\delta^* sker(\{x\}) \neq \delta^* sker(\{y\})$.

Theorem 20. Let $f : X \to Y$ be any function. Then, $\cap\{\delta^*\text{-}scl(\{x\}) : x \in Y\} = \phi$ if and only if $\delta^* sker(\{x\}) \neq Y$ for every $x \in Y$.

Proof. Necessity. Suppose that $\cap\{\delta^*\text{-}scl(\{x\}) : x \in Y\} = \phi$. Assume that there is a point y in Y such that $\delta^*sker(\{y\}) = Y$. Let x be any point of Y. Then, $x \in K$ for every δ^*-semiopen set K containing y and hence $y \in \delta^*\text{-}scl(\{x\})$ for any $x \in Y$. This implies that $y \in \cap\{\delta^*\text{-}scl(\{x\}) : x \in Y\}$. But this is a contradiction.
Sufficiency. Assume that $\delta^*sker(\{x\}) \neq Y$ for every $x \in Y$. If there exists a point y in Y such that $y \in \cap\{\delta^*\text{-}scl(\{x\}) : x \in Y\}$, then every δ^*-semiopen set containing y must contain every point of Y. This implies that the space Y is the unique δ^*-semiopen set containing y. Hence, $\delta^*sker(\{y\}) = Y$ which is a contradiction. Therefore, $\cap\{\delta^*\text{-}scl(\{x\}) : x \in Y\} = \phi$.

Theorem 21. Let $f : X \to Y$ be any function and Y be δ^*-semi $T_{\frac{1}{2}}$. If $\delta^*sker(\{x\}) \neq Y$ for a point $x \in Y$, then $\{x\}$ is a δ^*sD-set in Y.

Proof. Let $\delta^*sker(\{x\}) \neq Y$ for a point $x \in Y$, then there exists a subset $L \in \delta^*SO$ such that $\{x\} \subseteq L$ and $L \neq Y$. Using Theorem 15, for the point x, we have $\{x\}$ is δ^*-semiopen or δ^*-semiclosed in Y. When the singleton $\{x\}$ is δ^*-semiopen, then $\{x\}$ is a δ^*sD-set in Y. When the singleton $\{x\}$ is δ^*-semiclosed, then $Y \setminus \{x\}$ is δ^*-semiopen in Y. Put $L_1 = L$ and $L_2 = Y \cap (Y \setminus \{x\})$. Then, $\{x\} = L_1 \setminus L_2$ and $L_1 \neq Y$. Thus, $\{x\}$ is a δ^*sD-set.

Theorem 22. Let $f : X \to Y$ be any function. If a singleton $\{x\}$ is a δ^*sD-set in Y, then $\delta^*sker(\{x\}) \neq Y$.

Proof. Let $\{x\}$ be a δ^*sD-set in Y, then there exist two subsets $L_1, L_2 \in \delta^*SO$ such that $\{x\} = L_1 \setminus L_2$, $\{x\} \subseteq L_1$ and $L_1 \neq Y$. Thus, we have that $\delta^*sker(\{x\}) \subseteq L_1 \neq Y$ and so $\delta^*sker(\{x\}) \neq Y$.

Definition 10. Let $f : X \to Y$ be any function. Then, Y is said to be:

(1) δ^*-semi D_0 if for any pair of distinct points x and y of Y there exists a δ^*sD-set of Y containing x but not y or a δ^*sD-set of Y containing y but not x.
(2) δ^*-semi D_1 if for any pair of distinct points x and y of Y there exists a δ^*sD-set of Y containing x but not y and a δ^*sD-set of Y containing y but not x.

Remark. If Y is δ^*-semi D_1, then it is δ^*-semi D_0.

Example 7. Consider $X = \{1,2\}$ and $Y = \{a,b\}$ with $\delta = \{Y, \phi, \{a\}\}$. If $f : X \to Y$ is a function such that $f(1) = f(2) = a$ then $\delta^*SO = \{Y, \phi, \{a\}\}$. Then, Y is δ^*-semi D_0 but not δ^*-semi D_1 because there is no δ^*sD-set containing b but not a.

Theorem 23. Let $f : X \to Y$ be any function and Y be a δ^*-semi $T_{\frac{1}{2}}$ with at least two points. Then, Y is δ^*-semi D_1 if and only if $\delta^*sker(\{x\}) \neq Y$ holds for every point $x \in Y$.

Proof. Necessity. Let $x \in Y$. For a point $y \neq x$, there exists a δ^*sD-set L such that $x \in L$ and $y \notin L$. Say $L = L_1 \setminus L_2$, where $L_i \in \delta^*SO$ for each $i \in \{1,2\}$ and $L_1 \neq Y$. Thus, for the point x, we have a δ^*-semiopen set L_1 such that $\{x\} \subseteq L_1$ and $L_1 \neq Y$. Hence, $\delta^*sker(\{x\}) \neq Y$.
Sufficiency. Let x and y be a pair of distinct points of Y. We prove that there exist δ^*sD-sets A and B containing x and y, respectively, such that $y \notin A$ and $x \notin B$. Using Theorem 15, we can take the subsets A and B for the following four cases for two points x and y.
Case1. $\{x\}$ is δ^*-semiopen and $\{y\}$ is δ^*-semiclosed in Y. Since $\delta^*sker(\{y\}) \neq Y$, then there exists a δ^*-semiopen set K such that $y \in K$ and $K \neq Y$. Put $A = \{x\}$ and $B = \{y\}$. Since $B = K \setminus (Y \setminus \{y\})$, then K is a δ^*-semiopen set with $K \neq Y$ and $Y \setminus \{y\}$ is δ^*-semiopen, and B is a required δ^*sD-set containing y such that $x \notin B$. Obviously, A is a required δ^*sD-set containing x such that $y \notin A$.
Case 2. $\{x\}$ is δ^*-semiclosed and $\{y\}$ is δ^*-semiopen in Y. The proof is similar to Case 1.
Case 3. $\{x\}$ and $\{y\}$ are δ^*-semiopen in Y. Put $A = \{x\}$ and $B = \{y\}$.
Case 4. $\{x\}$ and $\{y\}$ are δ^*-semiclosed in Y. Put $A = Y \setminus \{y\}$ and $B = Y \setminus \{x\}$.
For each case of the above, the subsets A and B are the required δ^*sD-sets. Therefore, Y is δ^*-semi D_1.

4 δ^*-semi T_k $(k = 0,1,2)$

In this section, we introduce some new separation axioms using the notion of δ^*-semiopen set. Moreover, we give some characterization of these types of spaces and study the relationships between them and other spaces defined in section three.

Definition 11. *Let $f : X \to Y$ be any function. Then, Y is said to be:*

(1) *δ^*-semi T_0 if for each pair of distinct points x, y in Y, there exists a δ^*-semiopen set L such that either $x \in L$ and $y \notin L$ or $x \notin L$ and $y \in L$.*
(2) *δ^*-semi T_1 if for each pair of distinct points x, y in Y, there exist two δ^*-semiopen sets L and K such that $x \in L$ but $y \notin L$ and $y \in K$ but $x \notin K$.*
(3) *δ^*-semi T_2 if for each distinct points x, y in Y, there exist two disjoint δ^*-semiopen sets L and K containing x and y respectively.*

Theorem 24. Let $f : X \to Y$ be any function. Then, Y is δ^*-semi T_0 if and only if for each pair of distinct points x, y of Y, δ^*-$scl(\{x\}) \neq \delta^*$-$scl(\{y\})$.

Proof. Necessity. Let Y be δ^*-semi T_0 and x, y be any two distinct points of Y. Then, there exists a δ^*-semiopen set L containing x or y, say x but not y. Then, $Y \setminus L$ is a δ^*-semiclosed set which does not contain x but contains y. Since δ^*-$scl(\{y\})$ is the smallest δ^*-semiclosed set containing y, then δ^*-$scl(\{y\}) \subseteq Y \setminus L$ and therefore $x \notin \delta^*$-$scl(\{y\})$. Consequently δ^*-$scl(\{x\}) \neq \delta^*$-$scl(\{y\})$.
Sufficiency. Suppose that $x, y \in Y$, $x \neq y$ and δ^*-$scl(\{x\}) \neq \delta^*$-$scl(\{y\})$. Let z be a point of Y such that $z \in \delta^*$-$scl(\{x\})$ but $z \notin \delta^*$-$scl(\{y\})$. We claim that $x \notin \delta^*$-$scl(\{y\})$. For, if $x \in \delta^*$-$scl(\{y\})$ then δ^*-$scl(\{x\}) \subseteq \delta^*$-$scl(\{y\})$. This contradicts the fact that $z \notin \delta^*$-$scl(\{y\})$. Consequently x belongs to the δ^*-semiopen set $Y \setminus \delta^*$-$scl(\{y\})$ to which y does not belong.

Theorem 25. Let $f : X \to Y$ be any function. Then, Y is δ^*-semi T_1 if and only if the singletons are δ^*-semiclosed sets.

Proof. Let Y be δ^*-semi T_1 and x any point of Y. Suppose $y \in Y \setminus \{x\}$, then $x \neq y$ and so there exists a δ^*-semiopen set L such that $y \in L$ but $x \notin L$. Consequently $y \in L \subseteq Y \setminus \{x\}$, that is $Y \setminus \{x\} = \cup\{L : y \in Y \setminus \{x\}\}$ which is δ^*-semiopen.
Conversely, suppose $\{p\}$ is δ^*-semiclosed for every $p \in Y$. Let $x, y \in Y$ with $x \neq y$. Now, $x \neq y$ implies $y \in Y \setminus \{x\}$. Hence, $Y \setminus \{x\}$ is a δ^*-semiopen set contains y but not x. Similarly $Y \setminus \{y\}$ is a δ^*-semiopen set contains x but not y. Accordingly Y is δ^*-semi T_1.

Theorem 26. Let $f : X \to Y$ be any function. Then, the following statements are equivalent:

(1) Y is δ^*-semi T_2.
(2) Let $x \in Y$. For each $y \neq x$, there exists a δ^*-semiopen set L containing x such that $y \notin \delta^*$-$scl(L)$.
(3) For each $x \in Y$, $\cap\{\delta^*$-$scl(L) : L \in \delta^*SO$ and $x \in L\} = \{x\}$.

Proof. $(1) \Rightarrow (2)$: Since Y is δ^*-semi T_2, then there exist disjoint δ^*-semiopen sets L and K containing x and y respectively. So, $L \subseteq Y \setminus K$. Therefore, δ^*-$scl(L) \subseteq Y \setminus K$. So, $y \notin \delta^*$-$scl(L)$.
$(2) \Rightarrow (3)$: If possible for some $y \neq x$, we have $y \in \delta^*$-$scl(L)$ for every δ^*-semiopen set L containing x, which then contradicts (2).
$(3) \Rightarrow (1)$: Let $x, y \in Y$ and $x \neq y$. Then, there exists a δ^*-semiopen set L containing x such that $y \notin \delta^*$-$scl(L)$. Let $K = Y \setminus \delta^*$-$scl(L)$, then $y \in K$, $x \in L$ and $L \cap K = \phi$. Thus, Y is δ^*-semi T_2.

Theorem 27. Let $f : X \to Y$ be any function. Then, then the following statements are hold:

(1) Every δ^*-semi T_2 space is δ^*-semi T_1.
(2) Every δ^*-semi T_1 space is δ^*-semi $T_{\frac{1}{2}}$.

(3) Every δ^*-semi $T_{\frac{1}{2}}$ space is δ^*-semi T_0.

Proof. (1) The proof is straightforward from the definitions.
(2) The proof is obvious by Theorem 25.
(3) Let x and y be any two distinct points of Y. By Theorem 15, the singleton set $\{x\}$ is δ^*-semiclosed or δ^*-semiopen.
 (1) If $\{x\}$ is δ^*-semiclosed, then $Y \setminus \{x\}$ is δ^*-semiopen. So $y \in Y \setminus \{x\}$ and $x \notin Y \setminus \{x\}$. Therefore, we have Y is δ^*-semi T_0.
 (2) If $\{x\}$ is δ^*-semiopen, then $x \in \{x\}$ and $y \notin \{x\}$. Therefore, we have Y is δ^*-semi T_0.

Example 8. Consider $X = \{1,2,3,4\}$ and $Y = \{a,b,c,d\}$ with
$\delta = \{Y, \phi, \{d\}, \{a,d\}, \{a,c,d\}, \{a,b,d\}\}$. If $f : X \to Y$ is a function such that $f(1) = f(2) = f(3) = f(4) = d$, then $\delta^*SO = \{Y, \phi, \{d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,c,d\}, \{a,b,d\}\}$. Then, Y is δ^*-semi T_0 but not δ^*-semi $T_{\frac{1}{2}}$.

Example 9. Consider $X = \{1,2,3\}$ and $Y = \{a,b,c\}$ with $\delta = \{Y, \phi, \{a\}\}$. If $f : X \to Y$ is a function such that $f(1) = f(2) = f(3) = a$ then $\delta^*SO = \{Y, \phi, \{a\}, \{a,b\}, \{a,c\}\}$. Then, Y is δ^*-semi $T_{\frac{1}{2}}$ but not δ^*-semi T_1.

Remark. Let $f : X \to Y$ be any function. If Y is δ^*-semi T_k, then it is δ^*-semi D_k, for $k = 0,1$.

*Proof.*Obvious.

Theorem 28. Let $f : X \to Y$ be any function. Then, Y is δ^*-semi D_0 if and only if it is δ^*-semi T_0.

Proof. Suppose that Y is δ^*-semi D_0. Then, for each distinct pair $x,y \in Y$, at least one of x,y, say x, belongs to a δ^*sD-set G but $y \notin G$. Let $G = L_1 \setminus L_2$ where $L_1 \neq Y$ and $L_1, L_2 \in \delta^*SO$. Then, $x \in L_1$, and for $y \notin G$ we have two cases: (a) $y \notin L_1$ (b) $y \in L_1$ and $y \in L_2$.
In case (a), $x \in L_1$ but $y \notin L_1$.
In case (b), $y \in L_2$ but $x \notin L_2$.
Thus in both the cases, we obtain that Y is δ^*-semi T_0.
Conversely, if Y is δ^*-semi T_0, by Remark 4, Y is δ^*-semi D_0.

Corollary 1. If Y is δ^*-semi D_1, then it is δ^*-semi T_0.

Proof. Follows from Remark 3 and Theorem 28.

Here is an example which shows that the converse of Corollary 1 is not true in general.

Example 10. From Example 7, it is clear that Y is δ^*-semi T_0 but not δ^*-semi D_1.

Corollary 2. Let $f : X \to Y$ be any function. If Y is δ^*-semi T_1, then it is δ^*-semi symmetric.

Proof. In δ^*-semi T_1, every singleton is δ^*-semiclosed and therefore is δ^*-sgc. Then, by Theorem 14, Y is δ^*-semi symmetric.

Corollary 3. Let $f : X \to Y$ be any function. Then, the following statements are equivalent:

(1) Y is δ^*-semi symmetric and δ^*-semi T_0.
(2) Y is δ^*-semi T_1.

Proof. By Corollary 2 and Theorem 27, it suffices to prove only $(1) \Rightarrow (2)$.
Let $x \neq y$ and as Y is δ^*-semi T_0, we may assume that $x \in L \subseteq Y \setminus \{y\}$ for some $L \in \delta^*SO$. Then, $x \notin \delta^*\text{-}scl(\{y\})$ and hence $y \notin \delta^*\text{-}scl(\{x\})$. There exists a δ^*-semiopen set K such that $y \in K \subseteq Y \setminus \{x\}$ and thus Y is a δ^*-semi T_1 space.

Theorem 29. Let $f : X \to Y$ be any function. If Y is δ^*-semi symmetric, then the following statements are equivalent:

(1) Y is δ^*-semi T_0.
(2) Y is δ^*-semi $T_{\frac{1}{2}}$.
(3) Y is δ^*-semi T_1.

Proof. $(1) \Leftrightarrow (3)$: Obvious from Corollary 3.
$(3) \Rightarrow (2)$ and $(2) \Rightarrow (1)$: Directly from Theorem 27.

Corollary 4. Let $f : X \to Y$ be any function. If Y is δ^*-semi symmetric, then the following statements are equivalent:

(1) Y is δ^*-semi T_0.
(2) Y is δ^*-semi D_1.
(3) Y is δ^*-semi T_1.

Proof.$(1) \Rightarrow (3)$. Follows from Corollary 3.
$(3) \Rightarrow (2) \Rightarrow (1)$. Follows from Remark 4 and Corollary 1.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

References

[1] P. Bhattacharyya and B. K. Lahiri, Semi-generalized closed set in topology, *lnd. Jr. Math*.29 (1987). 375-382.
[2] M. Caldas, A separation axiom between semi-T_0 and semi-T_1, *Mem. Fac. Sci. Kochi Univ. Ser. Math.*, 18 (1997) 37-42.
[3] H. Z. Ibrahim, δ^*-open sets and relation between some weak and strong forms of δ^*-open sets in topological spaces, *Journal of Advanced Studies in Topology*, 6 (1) (2015), 20-27.
[4] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly,* 70 (1) (1963), 36-41.
[5] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous functions, *Proc. Math. Phys. Soc. Egypt*, 51 (1982), 47-53.
[6] S. N. Maheshwari and R. Prasad, Some new separation axioms, *Ann. Soc. Sci. Bruxelles*, 89 (1975), 395-402.
[7] O. Njstad, On some classes of nearly open sets, *Pacific J. Math.*, 15 (1965), 961-970.
[8] J. Tong, A separation. axiom between T_0 and T_1, *Ann. Soc. Sci. Bruxelles*, 96 (1982), 85-90.