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Existence and uniqueness solution for integral boundary value problem of fractional differential equation

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Abstract: In this work, we will study the existence and uniqueness results of a class of nonlinear fractional differential equations with integral boundary value conditions. By using Leray-Schauder nonlinear alternative and the Banach contraction mapping principle. As an application, an example is given to prove our conclusions.

Keywords: Fractional differential equations; Fixed point theorem; Banach contraction theorem; uniqueness, existence solution.

1 Introduction

Fractional calculus describe various phenomena in diverse areas of natural science such as physics, aerodynamics, biology, control theory, and chemistry. Boundary value problems with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems. In recent years, many researchers focused on the solutions for boundary value problems of fractional differential equations for details, see $[1-3,5-7,10,...]$ and references therein.

Motivated by the above work, we investigate the following integral boundary value problems of fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}u(t)+f(t,u(t))=0, \quad t\in(0,1). \\ u(0)=u'(0)=0, \quad u(1)=\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}g(s,u(s))\,ds, \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ is the Riemann-Liouville differential operator, of order $2<\alpha\leq 3$ and $0<\beta<\alpha$. f, $g\in C([0,1]\times(0,+\infty),[0,+\infty)), g(t,u)$ is nondecreasing on u for any $t\in[0,1]$.

The organization of the paper is as follows. In section 2, we introduce some definitions and lemmas needed in our proofs. In section 3, we establish the existence and uniqueness of the solution by using Leray-Schauder nonlinear alternative and the Banach contraction theorem. Last, we give an example illustrating the previous results.

2 Preliminaries

In this section, we present some definitions and lemmas.from fractional calculus theory. Let E be the Banach space of continuous functions $C[0,1]$, endowed with the norm $\|u\|_E=\max\limits_{t\in[0,1]}|u(t)|$.

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Definition 1. [8]*The fractional integral*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where $\alpha > 0$, is called Riemann-Liouville fractional integral of order α of a function $f : (0,+\infty) \to \mathbb{R}$ and $\Gamma(.)$ is the gamma function.

Definition 2. [8] *The Riemann-Liouville fractional derivative of order $\alpha > 0$, of a continuous function $f : (0,+\infty) \to \mathbb{R}$ is given by*

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha+1} f(s)\, ds.$$

$\Gamma(.)$ is the gamma function, provided that the right side is point-wise defined on $(0,+\infty)$ and $n = [\alpha]+1$, $[\alpha]$ stands for the integer less than α.

Lemma 1. [9] *Let $\alpha, \beta > 0$, $f \in L(0,1)$, then*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t).$$

Lemma 2. [4] *Let $\alpha > 0$, $f \in L^1\left([a,b], \mathbb{R}^N\right)$ and $I^{n-\alpha} f \in AC^n\left([a,b], \mathbb{R}^N\right)$, then*

$$I^{\alpha}\left(D^{\alpha} f(t)\right) = f(t) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)} t^{\alpha-j},$$

almost everywhere on $[a,b]$, where n is the smallest integer greater than or equal to α.

Lemma 3. [9] *If $\alpha, \beta > 0$, then*

$$I^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}.$$

Now, we give solution of an auxiliary problem.

Lemma 4. *Let $y \in L^1[0,1]$, the unique solution of the fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1). \\ u(0) = u'(0) = 0, \quad u(1) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s,u(s))\, ds, \end{cases} \tag{2.0}$$

is given by

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 G(t,s) y(t)\, ds + \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 H(t,s) g(s,u(s))\, ds, \tag{2.1}$$

where

$$G(t,s) = \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

$$H(t,s) = t^{\alpha-1}(1-s)^{\alpha-\beta-1}, \quad t,s \in [0,1]$$

Proof. We can see that, $u(t) = -I_{0^+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3}$. From $u(0) = u'(0) = 0$ and, from $u(1) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} g(s,u(s))\, ds$, we deduce that

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t -(t-s)^{\alpha-1} y(t)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} y(t)\, ds$$

$$+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t t^{\alpha-1}(1-s)^{\alpha-\beta-1}g(s,u(s))\,ds.$$

And, that is equivalente to

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)\,y(t)\,ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 H(t,s)\,g(s,u(s))\,ds.$$

Lemma 5. [7] *The function $G(t,s)$ defined by (2.2) satisfies the following properties*

(i) $G(t,s)\geq 0$ *and* $G(t,s)\in C([0,1]\times[0,1],\mathbb{R}_+)$.

(ii) $\max\limits_{t\in[0,1]} G(t,s)=G_1(s)$

(iii) *If* $t,s\in[\tau,1]$, $\tau>0$, *then*

$$\tau^{\alpha-1}G_1(s)\leq G(t,s)\leq\frac{1}{\tau}G_1(s),\ \textit{where}\ G_1(s)=s(1-s)^{\alpha-1}.$$

To use the fixed point theorem, according to *Lemma* 6, we define the operator T as

$$Tu(t)=\frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)\,f(s,u(s))\,ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 H(t,s)\,g(s,u(s))\,ds. \tag{2.3}$$

Lemma 6. [7] *The operator $T:E\longrightarrow E$, is completely continuous.*

3 Existence and uniqueness results

In this section, we prove the existence and uniqueness solution for the boundary value problem (1.1) by Leray-Schauder nonlinear alternative and the Banach contraction mapping principle.

Theorem 1. *Assume that there exists $L_1,L_2>0$ such that*

$$|f(t,u)-f(t,v)|\leq L_1\|u-v\|,\quad |g(t,u)-g(t,v)|\leq L_2\|u-v\|,\ \ \forall u,v\in\mathbb{R}_+,\ t\in[0,1],$$

and if

$$C=\frac{1}{\tau}\left(\frac{L_1}{\Gamma(\alpha)}+\frac{L_2}{\Gamma(\alpha-\beta)}\right)\int_0^1 G_1(s)\,ds<1,\quad \tau\in[0,1].$$

Then the boundary value broblem (1.1), has a unique solution in E

Proof. We have,

$$|Tu(t)-Tv(t)|\leq\frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)\,|f(s,u(s))-f(s,v(s))|\,ds+\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 H(t,s)\,|g(s,u(s))-g(s,v(s))|\,ds.$$

Obviously, we have

$$\|Tu-Tv\|_E\leq C\|u-v\|_E.$$

Then T is a contraction mapping. Therefore, by the Banach contraction mapping principle, it has a unique fixed point which is the unique solution of the boundary value broblem (1.1).

We will employ the following Leray-Schauder nonlinear alternative [7].

Lemma 7. *Let F be Banach space and Ω be a bounded open subset of F, $0\in\Omega$. $T:\overline{\Omega}\to F$ be a completely continuous operator. Then, either there exists $x\in\partial\Omega$, $\lambda>1$ such that $T(x)=\lambda x$, or there exists a fixed point $x^*\in\overline{\Omega}$*

Theorem 2. *We assume that* $f(t,0) \neq 0$, *there exists a constant* M, *such that* $f(t,u)$, $g(t,u) \leq M$, *for some* $u \in \mathbb{R}_+$, $t \in [0,1]$ *and there exists* $m > 0$ *such that,* $\frac{M}{\tau}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta)}\right)\int_0^1 G_1(s)\,ds < m$.

Then the fractional boundary value problem (1.1) *has at least one nontrivial solution* $u^* \in E$.

Proof. Let $\Omega = \{u \in E : \|u\|_E < m\}$. We assume that $u \in \partial\Omega$, $\lambda > 1$ such that $Tu = \lambda u$, then, $\lambda m = \lambda \|u\|_E = \|Tu\|_E = \max_{t\in[0,1]} |Tu(t)|$, we have

$$\|Tu\|_\infty \leq \frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s)\,f(s,u(s))\,ds + \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 H(t,s)\,g(s,u(s))\,ds.$$

So, we can obtain, $\|Tu\|_E \leq \frac{M}{\tau}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta)}\right)\int_0^1 G_1(s)\,ds.$ and we have, $\lambda m = \lambda \|u\|_E = \|Tu\|_E \leq \frac{M}{\tau}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-\beta)}\right)\int_0^1 G_1(s)\,ds \leq m$. Consequently $\lambda < 1$. This contradicts $\lambda > 1$. By applying *Lemma* 10, T has a fixed point $u^* \in \overline{\Omega}$ and then the fractional boundary value broblem (1.1), has a nontrivial solution $u^* \in E$. The proof is complete.

Example 1. Consider the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\frac{7}{2}} u(t) + \frac{t^2}{a} u(t) = 0, \ \ 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u(1) = \frac{1}{\Gamma(2)}\int_0^1 (1-s)\left(1 + \frac{u(s)}{b}\right) ds. \end{cases} \tag{P_1}$$

Let $\alpha = \frac{7}{2}$, $\beta = \frac{3}{2}$,and $f(t,u(t)) = \frac{t^2}{a}u(t)$, $\quad g(t,u(t)) = 1 + \frac{u(t)}{b}$, $\quad a,b > 0$. Then ,$|f(t,u) - f(t,v)| \leq \frac{t^2}{a}\|u-v\|$, $|g(t,u) - g(t,v)| \leq \frac{1}{b}\|u-v\|$, $\forall u,v \in \mathbb{R}_+$, $t \in [0,1]$, and $C = \frac{1}{\tau}\left(\frac{L_1}{\Gamma(\alpha)} + \frac{L_2}{\Gamma(\alpha-\beta)}\right)\int_0^1 G_1(s)\,ds < 1$, $\quad \tau \in [0,1]$. Hence, by *Theorem* 9, the boundary value problem (P_1) has a unique solution in E.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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