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# Solving nonlinear fractional PDEs using novel wavelet collocation method

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**Abstract:** In this paper, we consider an effective technique based on wavelet collation method for solving time fractional Chafee-Infante equation. We transform the fractional differential equation to an algebraic equations system by using wavelet operational matrices of fractional integrals. We give an illustrative example to show the applicability and effectiveness of the method. We compare the exact and wavelet solutions in a table and figure. The results show that the method easily applicable to fractional differential equations

**Keywords:** Bernoulli wavelets, fractional differential equations.

## 1 Introduction

Fractional calculus has various application fields in natural science and engineering. Many processes in engineering, physics, mathematical biology, fluid mechanics, electrochemistry, and other fields may be modelled using fractional order differential equations (FODE). Obtaining the analytical solutions of most FODE can not be possible via existing analytical methods. To solve fractional order differential equations, various numerical techniques such as wavelet method [1,2,3], the finite difference method [4], variational iteration method [5], finite element method [6] have been developed.
In this paper, we solve numerically time fractional Chafee-Infante (CI) equation using Bernoulli wavelets. Time fractional CI equation is given as follows:

$$\left(D_t^{\alpha} u\right)(x,t) - u_{xx}(x,t) = \sigma u(x,t)\left(1-u^2(x,t)\right) = 0, \tag{1}$$

where $\sigma$ is a constant that adjusts the relative balance of the nonlinear and diffusion term. When $\alpha = 1$, eq. 1 is transformed to a classical CI equation [7,8,9]. The analytical solution of the classical CI equation was given as follows in the reference [9]

$$u(x,t) = \frac{\lambda}{ce^{\sqrt{\frac{\sigma}{2}}x - \frac{3\sigma t}{2}} + \lambda}. \tag{2}$$

## 2 Preliminaries

An N-set of block pulse functions is given as follows [10]:

$$b_i(x) = \begin{cases} 1, & \frac{i-1}{N}a \le x < \frac{i}{N}a \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

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$\alpha$ order Riemann-Liouville fractional integral is defined [11]:

$$(I^{\alpha}u)(t)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^t (t-j)^{\alpha-1}u(j)dj, & \alpha\in\mathbb{R}^+\\ u(t), & \alpha=0\end{cases}\tag{4}$$

where $\Gamma(.)$ is the gamma function. $\alpha$ order Caputo fractional derivative is given by [12]:

$$(D_t^{\alpha}u)(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{1}{(t-j)^{(\alpha-n+1)}}\frac{d^n u(j)}{dj^n}dj, & n-1<\alpha\leq n,\\ \\ \frac{d^n u(j)}{dj^n}, & \alpha=n,\end{cases}\tag{5}$$

in which $\alpha,t>0$ and $n\in\mathbb{N}$.

**Remark.** The following relation between fractional derivative and fractional integral will be used:

$$(I^{\alpha}D_t^{\alpha}u)(t)=u(t)-\sum_{j=0}^{n-1}u^{(j)}\left(0^+\right)\frac{t^j}{j!},\quad t>0, n-1<\alpha\leqslant n.\tag{6}$$

## *2.1 Bernoulli wavelets*

Bernoulli wavelets $\psi_{n,m}(t)=\psi(k,\hat{n},m,t)$ is defined on the interval $[0,1)$ as follows [13]:

$$\psi_{n,m}(t)=\begin{cases}2^{\frac{k-1}{2}}\tilde{\beta}_m\left(2^{k-1}t-\hat{n}\right), & \frac{\hat{n}}{2^{k-1}}\leqslant t<\frac{\hat{n}+1}{2^{k-1}},\\ 0, & \text{otherwise}\end{cases}\tag{7}$$

where $\hat{n}=n-1$, $(n=1,2,3,\ldots,2^{k-1})$, k is a positive integer, $m$ is the order of Bernoulli polynomials:

$$\tilde{\beta}_m(t)=\begin{cases}1, & m=0\\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m)^2}{(2m)!}\alpha_{2m}}}\beta_m(t), & m>0\end{cases}$$

where $m=0,1,2,\ldots,M-1$ and $n=1,2,\ldots,2^{k-1}$. Here, $\beta_m(t)$ is Bernoulli polynomials that can be given by:

$$\beta_m(t)=\sum_{i=0}^{m}\binom{m}{i}\alpha_{m-i}t^i,$$

where $\alpha_i$, are Bernoulli numbers that are given by:

$$\frac{t}{e^t-1}=\sum_{i=0}^{\infty}\alpha_i\frac{t^i}{i!},$$

A function $g(t)\in L^2[0,1]$ can be given by Bernoulli wavelets as:

$$g(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}g_{nm}\psi_{n,m}(t),\tag{8}$$

where $g_{nm}=\langle g(t),\psi_{n,m}(t)\rangle$ and $\langle\cdot,\cdot\rangle$ symbolizes the inner product. For simplicity, the series can be truncated as follows:

$$g(t)=\sum_{n=1}^{2^{(k-1)}}\sum_{m=0}^{M-1}g_{nm}\psi_{n,m}(t)=\boldsymbol{G}^{\intercal}\boldsymbol{\Psi}(t),\tag{9}$$

where the superscript $^\intercal$ represents the transpose, $\boldsymbol{G}$ and $\boldsymbol{\Psi}(t)$ are $N = 2^{k-1}M$ matrices as:

$$\boldsymbol{G} = \left[g_{10}, g_{11}, \ldots, g_{1M-1}, g_{20}, g_{21}, \ldots, g_{2M-1}, \ldots, g_{2^{k-1}0}, g_{2^{k-1}1}, \ldots, g_{2^{k-1}M-1}\right]^\intercal,$$

$$\boldsymbol{\Psi}(t) = \left[\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \psi_{21}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^{k-1}0}, \psi_{2^{k-1}1}, \ldots, \psi_{2^{k-1}M-1}\right]^\intercal.$$

The wavelet transform of the function $g(t)$ can be given by:

$$g(t) = \sum_{i=1}^{N} g_i \psi_i(x) = \boldsymbol{G}^\intercal \boldsymbol{\Psi}(t). \tag{10}$$

In here, $i = m + M(n-1) + 1$ and $N = 2^{k-1}M, \boldsymbol{G} = [g_1, g_2, \ldots, g_N]^\intercal$, $\boldsymbol{\Psi}(t) = [\psi_1, \psi_2, \ldots, \psi_N]^\intercal$. For $i = 1, 2, \ldots N$, the collocation points $x_i$ and $t_i$ are found by:

$$x_i = \frac{2i-1}{2N}, \quad t_i = \frac{2i-1}{2N}, \quad (i = 1, 2, \ldots, N). \tag{11}$$

Bernoulli wavelet matrix $\boldsymbol{\Phi}_{N\times N}$ is:

$$\boldsymbol{\Phi}_{N\times N} = \left[\boldsymbol{\Psi}\left(\frac{1}{2N}\right), \boldsymbol{\Psi}\left(\frac{3}{2N}\right), \ldots, \boldsymbol{\Psi}\left(\frac{2N-1}{2N}\right)\right]. \tag{12}$$

A function $u(x,t) \in L^2([0,1] \times [0,1])$ can be expanded by Bernoulli wavelets as:

$$u(x,t) = \sum_{i=1}^{N} \sum_{j=1}^{N} u_{ij} \psi_i(x) \psi_j(t) = \boldsymbol{\Psi}^\intercal(x) \boldsymbol{U} \boldsymbol{\Psi}(t), \tag{13}$$

where the elements $u_{ij}$ of the matrix $\boldsymbol{U}$ are:

$$u_{ij} = \left\langle \psi_i(x), \left\langle u(x,t), \psi_j(t) \right\rangle \right\rangle. \tag{14}$$

The $n-$ times operational matrix $P^n$ of integration of $\boldsymbol{\Psi}(t)$ is written by:

$$\underbrace{\int_0^t \ldots \int_0^t \boldsymbol{\Psi}(t) ds \ldots ds}_{n-\text{ times}} \simeq \boldsymbol{P}^n \boldsymbol{\Psi}(t). \tag{15}$$

The fractional integration of the vector $\boldsymbol{\Psi}(t)$ is approximated as [14]:

$$(I^\alpha \boldsymbol{\Psi})(t) \simeq \boldsymbol{P}^\alpha \boldsymbol{\Psi}(t), \tag{16}$$

where $\boldsymbol{P}^\alpha$ is named the Bernoulli wavelet operational matrix of fractional integration. $\boldsymbol{P}^\alpha$ is defined as:

$$\boldsymbol{P}^\alpha \cong \boldsymbol{P}^\alpha_{N\times N} = \boldsymbol{\Phi} \boldsymbol{P}^\alpha_B \boldsymbol{\Phi}^{-1}, \tag{17}$$

where $\boldsymbol{P}^\alpha_B$, the BPFs operational matrix of integration, is given by:

$$\boldsymbol{P}^\alpha_B = \frac{1}{N^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \Upsilon_1 & \Upsilon_2 & \cdots & \Upsilon_{N-1} \\ 0 & 1 & \Upsilon_1 & \ldots & \Upsilon_{N-2} \\ 0 & 0 & 1 & \ldots & \Upsilon_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \tag{18}$$

where $\Upsilon_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$ and $\boldsymbol{\Phi}$ is the wavelet matrix eq. 12.

## 3 Solution algorithm

Let' s suppose that

$$\frac{\partial^{\alpha+2} u(x,t)}{\partial t^\alpha \partial x^2} = \boldsymbol{\Psi}^\top(x)\boldsymbol{U}\boldsymbol{\Psi}(t), \tag{19}$$

where $U = [u_{ij}]_{N\times N}$ is an unknown matrix to be found later. Integrating the eq. 19 $\alpha$ times w.r.t. $t$ and considering the initial condition, we have:

$$u_{xx}(x,t) = \boldsymbol{\Psi}^\top(x)\boldsymbol{U}\boldsymbol{P}^\alpha\boldsymbol{\Psi}(t) + U_0''(x). \tag{20}$$

Integrating the eq. 19 two times w.r.t. $x$, one get:

$$\left(D_t^\alpha u\right)(x,t) = \boldsymbol{\Psi}^\top(x)\left(\boldsymbol{P}^2\right)^\top \boldsymbol{U}\boldsymbol{\Psi}(t) + \left.\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right|_{x=0} + x\frac{\partial}{\partial x}\left.\left(\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right)\right|_{x=0}. \tag{21}$$

Substituting $x = 1$ to the eq. 21, we get:

$$\frac{\partial}{\partial x}\left.\left(D_t^\alpha u\right)(x,t)\right|_{x=0} = \frac{\partial^\alpha U_2(t)}{\partial t^\alpha} - \frac{\partial^\alpha U_1(t)}{\partial t^\alpha} - \boldsymbol{\Psi}^\top(1)\left(\boldsymbol{P}^2\right)^\top \boldsymbol{U}\boldsymbol{\Psi}(t). \tag{22}$$

Substituting the eq. 22 to the eq. 21, we have:

$$\left(D_t^\alpha u\right)(x,t) = \boldsymbol{\Psi}^\top(x)\left(\boldsymbol{P}^2\right)^\top \boldsymbol{U}\boldsymbol{\Psi}(t) - x\boldsymbol{\Psi}^\top(1)\left(\boldsymbol{P}^2\right)^\top \boldsymbol{U}\boldsymbol{\Psi}(t) + \frac{\partial^\alpha U_1(t)}{\partial t^\alpha} + x\frac{\partial^\alpha U_2(t)}{\partial t^\alpha} - x\frac{\partial^\alpha U_1(t)}{\partial t^\alpha}. \tag{23}$$

Integrating the eq. 23 $\alpha$ times w.r.t. $t$, one attain:

$$u(x,t) = \left(\boldsymbol{\Psi}^\top(x) - x\boldsymbol{\Psi}^\top(1)\right)\left(\boldsymbol{P}^2\right)^\top \boldsymbol{U}\boldsymbol{P}^\alpha\boldsymbol{\Psi}(t) + U_0(x) + U_1(t) - U_1(0) + x\big(U_2(t) - U_2(0) - U_1(t) + U_1(0)\big). \tag{24}$$

Substituting the eq. 20, eq. 21, and eq. 24 to the eq. 1, and then substituting $x_i$ and $t_i$ to the new equation, one can obtain a system of algebraic equation. Solving the system, one can determine the unknown matrix $\boldsymbol{U}$. Substituting $\boldsymbol{U}$ to the eq. 24, one get the solution of the main equation.

**Example 1.** Let's deal with the eq. 1 for $\lambda = 1, \sigma = 2, c = 1$. So, we have:

$$\left(D_t^\alpha u\right)(x,t) - u_{xx}(x,t) = 2u(x,t)\left(1 - u^2(x,t)\right) = 0, \tag{25}$$

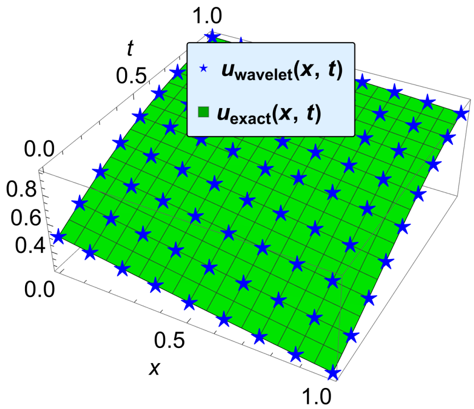
the initial and boundary conditions are given, respectively:

$$u(x,0) = \frac{1}{e^x + 1}, \tag{26}$$

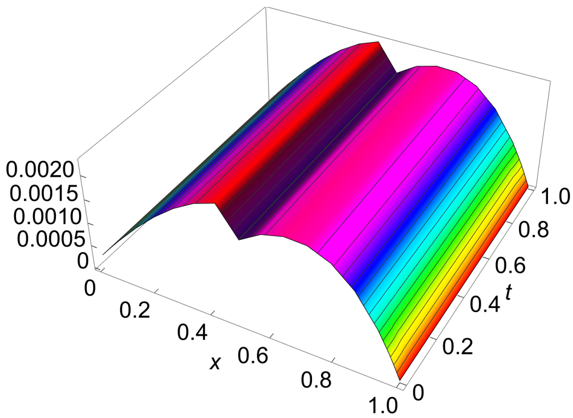
and

$$u(0,t) = \frac{1}{e^{-3t} + 1}, \quad u(1,t) = \frac{1}{e^{1-3t} + 1}. \tag{27}$$

When $\alpha = 1$, the analytic solution is $u(x,t) = \frac{1}{e^{x-3t}+1}$ in the reference [9]. For $k = M = 2$, using the wavelet collocation method, we attain the results in the Table 1 and Fig. 1.




(a) Both wavelet and exact solution.




(b) The absolute error.

**Figure 1:** Comparison between wavelet and exact solution.

| $x$ | $t$ | Wavelet | Exact | Absolute error |
|---|---|---|---|---|
| 0.1125 | 0.1125 | 0.156222 | 0.156218 | $1.06066 \times 10^{-3}$ |
| 0.1125 | 0.1375 | 0.173095 | 0.173106 | $1.06066 \times 10^{-3}$ |
| 0.1125 | 0.1625 | 0.185210 | 0.185195 | $1.06066 \times 10^{-3}$ |
| 0.1125 | 0.1875 | 0.192429 | 0.19241 | $1.06066 \times 10^{-3}$ |
| 0.1375 | 0.1125 | 0.150038 | 0.150000 | $2.1854 \times 10^{-3}$ |
| 0.1375 | 0.1375 | 0.167948 | 0.167918 | $2.1854 \times 10^{-3}$ |
| 0.1375 | 0.1625 | 0.181849 | 0.181757 | $2.1854 \times 10^{-3}$ |
| 0.1375 | 0.1875 | 0.190535 | 0.19047 | $2.1854 \times 10^{-3}$ |
| 0.1625 | 0.1125 | 0.143856 | 0.143782 | $2.08518 \times 10^{-3}$ |
| 0.1625 | 0.1375 | 0.162356 | 0.162246 | $2.08518 \times 10^{-3}$ |
| 0.1625 | 0.1625 | 0.177902 | 0.177730 | $2.08518 \times 10^{-3}$ |
| 0.1625 | 0.1875 | 0.188200 | 0.188080 | $2.08518 \times 10^{-3}$ |
| 0.1875 | 0.1125 | 0.137877 | 0.137754 | $1.3947 \times 10^{-3}$ |
| 0.1875 | 0.1375 | 0.156459 | 0.156218 | $1.3947 \times 10^{-3}$ |
| 0.1875 | 0.1625 | 0.173362 | 0.173106 | $1.3947 \times 10^{-3}$ |
| 0.1875 | 0.1875 | 0.185366 | 0.185195 | $1.3947 \times 10^{-3}$ |

**Table 1:** Comparison between wavelet and exact solutions at the collocation points.

## 4 Conclusion

In this paper, we have successfully applied the Bernoulli wavelet collocation method to the fractional CI equation. We have used Wolfram Mathematica to do the numerical computations and to draw figures. The obtained results imply that the used technique is an effective one for fractional differential equations.

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## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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