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Generalized Derivations on a Prime Rings

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Abstract: In this research, firstly, we have extended Ashraf's results in [4,5] for (θ,θ)-derivation that acting as a homomorphism (resp. an anti-homomorphism) on a Jordan ideal and a subring of a prime ring R with characteristic non equal two. Secondly, we have expanded Zaidi's results in [12] for a generalized (θ,θ)-derivations. Lastly, we have found the relationship between the commutativity of a prime ring and the existence of certain specific types of generalized derivations on R.

Keywords: Generalized derivation, semiprime rings, Jordan Ideal.

1 Introduction

Let R be an associative ring with identity, $Z(R)$ is the center of R. A ring R is prime if $sRt = 0$, then either $s = 0$ or $t = 0$ and R is semiprime if the identity $sRs = 0$ gives $s = 0$. The $charR \neq 2$ of a ring R if whenever $2s = 0$, $s \in R$, then $s = 0$. The derivation is an additive map $\delta : R \to R$ satisfies

$$\delta(st) = \delta(s)t + s\delta(t)\ \forall s,t \in R.$$

The additive map δ is said to be (θ,φ)-derivation if

$$\delta(st) = \delta(s)\theta(t) + \varphi(s)\delta(t)\ \forall s,t \in R,$$

where, $\theta,\varphi : R \to R$ are maps on R.

An additive map $F : R \to R$ is called a generalized derivation associated with δ if there exists a derivation $\delta : R \to R$ such that

$$F(st) = F(s)t + s\delta(t)\ \forall s,t \in R.$$

An additive map $F : R \to R$ is called a generalized (θ,φ)-derivation associated with δ where θ,φ are maps on R , if there exists a (θ,φ)-derivation δ satisfies

$$F(st) = F(s)\theta(t) + \varphi(s)\delta(t)\ \forall s,t \in R.$$

All other definitions are standard and they can be found in [1,2,3,4,6,7,8,9,10] and [11].

2 Preliminaries

We will state some lemmas, which helps us to prove the main results,

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Lemma 1.*[12, Lemma 2-5] Let $V \neq \{0\}$ be a Jordan ideal of a prime ring R. If*

$$rV = \{0\} \text{ or } Vr = \{0\}, r \in R, \text{ then } r = 0.$$

Lemma 2.*[12, Lemma 2-6] Let $V \neq \{0\}$ be a Jordan ideal of a prime ring R of $charR \neq 2$. If $sVt = \{0\}$, then $s = 0$ or $t = 0$.*

Lemma 3.*[12, Lemma 2-7] Let $V \neq \{0\}$ be a Jordan ideal of a prime ring R of $charR \neq 2$. Then the commutativity of V gives that $V \subseteq Z(R)$.*

3 Generalized (θ, θ)-derivation

Now we will generalize Zaidi's theorem [12] to left (θ, θ)-derivations that acting as a homomorphism (resp. an anti-homomorphism) on a Jordan ideal $V \neq \{0\}$ and subring of a prime ring R of $charR \neq 2$.

Theorem 1.*If $V \neq \{0\}$ is a Jordan ideal and subring of a prime ring R of a $charR \neq 2$ and θ an automorphisms on R and δ is a left (θ, θ)-derivation of R which is acting as a homomorphism (resp. an anti-homomorphism) on V. Then $\delta = 0$ or $V \subseteq Z(R)$.*

*Proof.*Assume that δ acting as a homomorphism on V, where V is not contained in the center of R. Thus

$$\delta(st) = \delta(s)\delta(t) = \delta(s)\theta(t) + \theta(s)\delta(t) \ \forall s,t \in V. \tag{1}$$

Now substituting in the identity (1) t by $tr, r \in V$, then

$$\delta(str) = \delta(s)\theta(t)\theta(r) + \theta(s)(\delta(t)\theta(r) + \theta(t)\delta(r)) = \delta(s)(\delta(t)\theta(r) + \theta(t)\delta(r)).$$

From (1) we get $(\delta(s) - \theta(s))\theta(t)\delta(r) = 0$. Thus $\theta^{-1}(\delta(s) - \theta(s))t\theta^{-1}\delta(r) = 0$. Hence $\theta^{-1}(\delta(s) - \theta(s))V\theta^{-1}\delta(r) = \{0\}$. From lemma (2-2), we have $\delta(s) - \theta(s)$ or $\delta(r) = 0$. Let $\delta(r) = 0$ and using lemma (2-3), we conclude that $\delta = 0$. Now let $\delta(s) - \theta(s) = 0$, then from the identity (1)

$$\theta(s)\delta(t) = 0. \tag{2}$$

Substituting s in the identity (2) by sr, we get $\theta(s)\theta(r)\delta(t) = 0$. Hence $sr\theta^{-1}(\delta(t)) = 0$, then $sV\theta^{-1}(\delta(t)) = \{0\}$. Using lemma (2) we have $s = 0$ or $\delta(t) = 0$, since $V \neq \{0\}$, then $\delta(t) = 0$. Thus by lemma (3) $V \subseteq Z(R)$. Assume that δ is acting as an anti-homomorphism on a Jordan ideal $V \neq \{0\}$ of R where V is not contained in the center of R. Hence

$$\delta(st) = \delta(ts) = \delta(t)\delta(s) = \delta(s)\theta(t) + \theta(s)\delta(t) \ \forall s,t \in V. \tag{3}$$

Substituting s by st in (3), then

$$(\delta(s)\theta(t) + \theta(s)\delta(t))\theta(t) + \theta(s)\theta(t)\delta(t) = \delta(t)(\delta(s)\theta(t) + \theta(s)\delta(t)).$$

Then from (3) we get

$$\theta(s)\theta(t)\delta(s) = \delta(t)\theta(s)\delta(t). \tag{4}$$

Now, replace s by cs in identity (4), then

$$\theta(c)\theta(s)\theta(t)\delta(s) = \delta(t)\theta(c)\theta(s)\delta(t) \ \forall c,s,t \in V. \tag{5}$$

Concerning (4), then (5) gives that $[\delta(t),\theta(c)]\theta(s)\delta(t)=0$. Thus

$$\theta^{-1}[\delta(t),\theta(c)]s\theta^{-1}(\delta(t))=0.$$

Equivalently, $\theta^{-1}[\delta(t),\theta(c)]V\theta^{-1}(\delta(t))=0$. From lemma(2) conclude that $[\delta(t),\theta(c)]=0$or $\delta(t)=0$. Let $\delta(t)=0$ and using lemma (3), we conclude that $\delta=0$. Now let

$$[\delta(t),\theta(c)]=0, \tag{6}$$

then replace t by tc in identity (6) we have

$$\begin{aligned}0=[\delta(tc),\theta(c)]&=[\delta(t)\theta(c)+\theta(t)\delta(c),\theta(c)]\\&=[\delta(t)\theta(c),\theta(c)]+[\theta(t)\delta(c),\theta(c)]\\&=\theta(t)[\delta(c)\theta(c)]+[\theta(t),\theta(c)]\delta(c).\end{aligned}$$

This means

$$\theta(t)[\delta(c)\theta(c)]+[\theta(t),\theta(c)]\delta(c)=0. \tag{7}$$

then replace t by rt in identity (6) we have

$$[\theta(r),\theta(c)]\theta(t)\delta(c)=0,$$

Thus $[r,c]t\theta^{-1}(\delta(c))=0$, equivalently, $[r,c]V\theta^{-1}(\delta(c))=\{0\}$. From Lemma(2) conclude that $[r,c]=0$ or $\delta(t)=0$. Assume that

$$U=\{c\in V:[r,c]=0\,\forall r\in V\}\text{and } W=\{c\in V:\delta(c)=0\}.$$

Then $U\subset V$ and $W\subset V$ as a proper subgroups and $V=U\bigcup W$, hence $V=U$or $V=W$. Now, if $V=U$, then $[r,c]=0$, implies V is commutative, then by Lemma (3) V is contained in the center of R, which is contradict with assumption. Hence $V\subseteq Z(R)$. Now we will extend theorem (1) to generalized (θ,θ)-derivation on R. □

Theorem 2. *Let $V\neq\{0\}$ be a Jordan ideal and subring of a prime ring R of a char$R\neq 2$. Now if θ is an automorphisms on R and $F:R\to R$ is a generalized (θ,θ)-derivation on R which is acting as a homomorphism (resp. an anti-homomorphism) on V and associated with δ . Then $\delta=0$ or $V\subseteq Z(R)$.*

Proof. Assume that δ acting as a homomorphism on V and $V\nsubseteq Z(R)$. Thus

$$F(st)=F(s)F(t)=F(s)\theta(t)+\theta(s)\delta(t)\ \forall s,t\in V. \tag{8}$$

Now substituting in the identity (8) t by $tr,r\in V$, then

$$\begin{aligned}F(str)=F(s)\theta(t)\theta(r)+\theta(s)(\delta(t)\theta(r)+\theta(t)\delta(r))=\\F(s)(F(t)\theta(r)+\theta(t)\delta(r)).\end{aligned}$$

From (1) we get $(F(s)-\theta(s))\theta(t)\delta(r)=0$. Thus $\theta^{-1}(F(s)-\theta(s))t\theta^{-1}\delta(r)=0$. Hence $\theta^{-1}(F(s)-\theta(s))V\theta^{-1}\delta(r)=\{0\}$. From lemma (2), we have $F(s)-\theta(s)$ or $\delta(r)=0$. Let $\delta(r)=0$ and using lemma (3), we conclude that $\delta=0$. Now let $F(s)-\theta(s)=0$, then

$$\theta(s)\delta(t)=0. \tag{9}$$

substituting s in the identity (2) by sr, we get $\theta(s)\theta(r)\delta(t)=0$. Hence $sr\theta^{-1}(\delta(t))=0$, then $sV\theta^{-1}(\delta(t))=\{0\}$. Using lemma (2-2) we have $s=0$ or $\delta(t)=0$, since $V\neq\{0\}$, then $\delta(t)=0$. Thus by lemma (2-3) $V\subseteq Z(R)$. Now assume that δ

is acting as an anti-homomorphism on a Jordan ideal $V \neq \{0\}$ of R such that V is not contained in the center of R. Hence

$$F(st) = F(ts) = F(t)F(s) = F(s)\theta(t) + \theta(s)\delta(t). \tag{10}$$

Substituting s by st in (10), then

$$(F(s)\theta(t) + \theta(s)\delta(t))\theta(t) + \theta(s)\theta(t)\delta(t)) = F(t)(F(s)\theta(t) + \theta(s)\delta(t)).$$

Then from (10) we get

$$\theta(s)\theta(t)\delta(t))F(t)\theta(s)\delta(t). \tag{11}$$

Now, replace s by cs in identity (11), then

$$\theta(c)\theta(s)\theta(t)\delta(s) = F(t)\theta(c)\theta(s)\delta(t) \; \forall c,s,t \in V. \tag{12}$$

Concerning (11), then (12) gives that

$$[F(t), \theta(c)]\theta(s)\delta(t) = 0.$$

Thus $\theta^{-1}[F(t), \theta(c)]s\theta^{-1}(\delta(t)) = 0$. Equivalently, $\theta^{-1}[F(t), \theta(c)]V\theta^{-1}(\delta(t)) = 0$. From lemma (2-2) conclude that $[F(t), \theta(c)] = 0 \; or \; \delta(t) = 0$. Let $\delta(t) = 0$ and using lemma (2-3), we conclude that $\delta = 0$. Now let

$$[F(t), \theta(c)] = 0, \tag{13}$$

then replace t by tc in identity (13) we have

$$\begin{aligned} 0 = [F(tc), \theta(c)] &= [F(t)\theta(c) + \theta(t)\delta(c), \theta(c)] \\ &= [F(t)\theta(c), \theta(c)] + [\theta(t)\delta(c), \theta(c)] \\ &= \theta(t)[\delta(c)\theta(c)] + [\theta(t), \theta(c)]\delta(c). \end{aligned}$$

This means

$$\theta(t)[\delta(c)\theta(c)] + [\theta(t), \theta(c)]\delta(c) = 0. \tag{14}$$

then replace t by rt in identity (14) we have $[\theta(r), \theta(c)]\theta(t)\delta(c) = 0$, Thus $[r,c]t\theta^{-1}(\delta(c)) = 0$, equivalently, $[r,c]V\theta^{-1}(\delta(c)) = \{0\}$, From Lemma(2) conclude that $[r,c] = 0 \; or \; \delta(t) = 0$. Assume that

$$U = \{c \in V : [r,c] = 0 \, \forall r \in V\} \; and \; W = \{c \in V : \delta(c) = 0\}.$$

Then $U \subset V$ and $W \subset V$ as a proper subgrops and $V = U \cup W$, hence $V = U \; or \; V = W$. Now, if $V = U$, then $[r,c] = 0$, implies V is commutative, then by Lemma (3) V is contained in the center of R, which is contradict with assumption. Hence $V \subseteq Z(R)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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