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On some fixed point results for multivalued contractions with an application

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Abstract: In this paper, we prove the existence of multivalued fixed point, by combining the contractions of Geraghty type with θ-contraction and α,η-admissible concepts. Some consequences are given in metric spaces endowed with partial order or with graph, also we provide an example and an application to the existence of solutions of a boundary value problemma for fractional differential inclusions to demonstrate the usability of our outcomes.

Keywords: θ-contraction, α,η-admissible, fractional differential inclusion.

1 Introduction

The idea of the metric fixed point theory for multivalued mappings has been started by Nadler [14], where he gave an important generalization to Banach principle in the setting of multivalued mappings. Afterward, several results were given in this way for instance, see [5,7,11,17,16].

Samet et al. [15] introduced a new notion called as α-admissibility, they obtained some results for $\alpha-\psi$-contractive mappings. Later, several authors investigated such concepts to establish some results, for example see [2,7,13]. Recently, a new contractions type is given by Jleli et al. [12], called θ-contraction to show the existence of fixed points by using such concept. It is worth mentioning here, that a contraction in the sense of Banach is θ-contraction, while the converse may be true. After that, several authors have studied on different variations of θ-contraction for single-valued and multivalued mappings, for example, see [1,3,8,18].

In this study, we combine the (α,η)-admissibility concept with θ- contraction and Geraghty contraction type in the multivalued mappings context, some existence results of a fixed point in complete metric spaces were furnished. We present some results in metric spaces equipped with a partially order relationship and in metric spaces with graph by using obtained main results. Finally, an example is provided and we suggest a study of an existence problemma of the solutions for a fractional differential inclusion to demonstrate the validity of obtained results. Let (X,d) be a metric space, and let $CB(X)$ be a set of closed, bounded and nonempty subsets of X, the Pompeiu-Hausdorff distance is defined as:

$$H(A,B)=\begin{cases}\max\left\{\sup\limits_{x\in A} d(x,B),\ \sup\limits_{y\in B} d(y,A)\right\} & \text{if the maximum exists;}\\ \infty, & \text{otherwise,}\end{cases}$$

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for all $A,B \in CB(X)$, where $d(x,B) = \inf\{d(x,y) : y \in B\}$. For more details we refer the readers to [4,10].
Note that, if $A = \{x\}$ (singleton) and $B = \{y\}$, then $H(A,B) = d(x,y)$. Denote by $CL(X)$ the family of closed and nonempty subsets of X and let $K(X)$ be the set of compact subsets of X.

Lemma 1. *[14] Let (X,d) be a metric space and let A,B be elemmaents of $CL(X)$ such that $H(A,B) > 0$. Then, for an arbitrary $h > 1$ and for each $a \in A$, there exists $b \in B$ such that $d(a,b) < hH(A,B)$.*

Firstly, Samet et al. [15] introduced the concept of α-admissible single valued mapping, afterwards Asl et al. [2] extended this notion to multivalued mappings by introducing the notion of α_*-admissible mappings. Later, Mohammadi et al. [13] introduced the concept of α-admissible for multivalued mappings.

Definition 1. *Let (X,d)be a metric space and $\alpha : X \times X \to \mathbb{R}_+$ be a given function. $T : X \to CL(X)$ is*

(1) *α_*-admissible if for $\alpha(x,y) \geq 1$, $x,y \in X$ then $\alpha_*(Tx,Ty) \geq 1$, whenever $\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$;*
(2) *α-admissible, if for each $z \in X$ and $x \in Tz$ with $\alpha(z,x) \geq 1$, then $\alpha(x,y) \geq 1$ for all $y \in Tx$.*

A mapping T is α_*-admissible, then it is also α-admissible, while there are some α-admissible mappings are never α_*-admissible.

Definition 2. *[6] Let (X,d) be a metric space and $\eta, \alpha : X \times X \to \mathbb{R}_+$ be given functions with η bounded. A mapping $T : X \to CL(X)$ is α_*-admissible w.r.t η, if $\alpha(x,y) \geq \eta(x,y)$ then $\alpha_*(Tx,Ty) \geq \eta_*(Tx,Ty)$, where*

$$\alpha_*(Tx,Ty) = \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$$

and

$$\eta_*(Tx,Ty) = \sup\{\eta(a,b) : a \in Tx, b \in Ty\}.$$

Definition 3. *[9] Let (X,d) be a metric space, multivalued map $T : X \to Cl(X)$ and a function $\alpha : X \times X \to [0,\infty)$. T is called to be an α,η- lower semi continuous map if for $x \in X$ and a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} d(x_n,Tx_n) = 0$ and $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \geq 0$ implies*

$$\lim_{n\to\infty} \inf d(x_n,Tx_n) \geq d(x,Tx).$$

Definition 4. *[12] Let Θ be the set of all functions $\theta : (0,+\infty) \to (1,+\infty)$ be a function satisfying:*

(θ_1) *:θ is non decreasing,*
(θ_2) *:for each sequence $\{t_n\}$ in $(0,+\infty)$, $\lim_{n\to\infty} t_n = 1$ if and only if $\lim_{n\to\infty} t_n = 0$,*
(θ_3) *:there exist $\sigma \in (0,1)$ and $\rho \in [0,\infty)$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^\sigma} = \rho$.*

Let Ω be the set of the functions $\beta : [0,\infty) \to [0,1)$ satisfying $\lim_{n\to+\infty} \beta(t_n) = 1$ implies $\lim_{n\to+\infty} t_n = 0$.

2 Main results

Definition 5.*Let (X,d) be a metric space and $\alpha,\eta : X \times X \to (0,\infty)$. $T : X \to CL(X)$ is a multivalued (α_*,η_*,θ) contraction of Geraghty type, if there exist a function $\theta \in \Theta$, $\beta \in \Omega$ and $\kappa : (0,\infty) \to [0,1)$ satisfying $\lim_{t\to s^+} \sup \kappa(t) < 1$ for all $s \in (0,\infty)$ such that*

$$\theta(H(Tx,Ty)) \leq \Big[\theta(\beta(d(x,y))d(x,y)\Big]^{\kappa(d(x,y))}, \tag{1}$$

for all $x,y \in X$ with $\alpha_*(x,y) \geq \eta_*(x,y)$ and $H(Tx,Ty) > 0$.

Theorem 1.*Let (X,d) be a complete metric space and $T:X\to K(X)$ be a multivalued (α_*,η_*,θ) contraction of Geraghty type. Assume that the following conditions are satisfied:*

(H_1) :*T is α_*-admissible w.r.t η.*
(H_2) :*There exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq\eta(x_0,x_1)$.*
(H_3) :*X is $\alpha-\eta$ regular, that's, for every sequence $\{x_n\}$ in X such that $x_n\to x$ and $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$, for all $n\in\mathbb{N}$, we have $\alpha(x_n,x)\geq\eta(x_n,x)$, for all $n\in\mathbb{N}$.*

Then T admits a fixed point in X.

Remark. We can replace the hypothesis (H_4) by α,η- lower semi continuity of T, that is if T is α,η-lower semi continuous, then for some $x\in X$ and a sequence $\{x_n\}\subset X$ such $\lim_{n\to+\infty} d(x_n,x)=0$ and $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}$, we have

$$0\leq d(x,Tx)\leq\lim_{n\to+\infty}\inf d(x_n,Tx_n)=0.$$

Hence, $x\in Tx$.

Theorem 2. *Let (X,d) be a complete metric space and $T:X\to CB(X)$ be a multivalued (α_*,η_*,θ) Geraghty contraction, with θ is right continuous. Assume that the following conditions are hold:*

(H_1) :*T is α_*-admissible w.r.t η.*
(H_2) :*There exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq\eta(x_0,x_1)$.*
(H_3) :*X is $\alpha-\eta$ regular, that's for every sequence $\{x_n\}$ in X such that $x_n\to x$ and $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$, for all $n\in\mathbb{N}$, we have $\alpha(x_n,x)\geq\eta(x_n,x)$, for all $n\in\mathbb{N}$.*

Then T admits a fixed point.

3 Fixed point on a metric space equipped with a partial ordering/graph

Firstly, we give an existence theorem of fixed point in a partially order metric space, by using the results provided in previous section. Define $\alpha: X\times X\to[0,+\infty)$, $\eta: XtimesX\to[0,+\infty)$ as follows:

$$\alpha(x,y)=\begin{cases}2, \text{ if } x\preceq y,\\ 0, \text{ otherwise.}\end{cases}$$

$$\eta(x,y)=\begin{cases}1,\ x\preceq y,\\ 0, \text{ otherwise.}\end{cases}$$

Theorem 3. *Let $(X,\preceq,d)$ be a complete ordered metric space and $T: X\to CB(X)$ be a multivalued mapping. Assume that the following assertions hold:*

(1) *For each $x\in X$ and $y\in Tx$ with $x\preceq y$, we have $y\preceq z$ for all $z\in Ty$.*
(2) *There exist $x_0\in X$ and $x_1\in Tx_0$ such that $x_0\preceq x_1$.*
(3) *For every nondecreasing sequence $\{x_n\}$ in X with $x_n\to x\in X$, then $x_n\preceq x$, for all $n\in\mathbb{N}$;*
(4) *There exist a right continuous function $\theta\in\widehat{\Theta}$, $\beta\in\Omega$ and $\kappa:(0,\infty)\to[0,1)$ satisfying $\lim_{t\to s^+}\sup\kappa(t)<1$ for all $s\in(0,\infty)$ such that*

$$\theta(H(Tx,Ty))\leq\left[\theta(\beta(d(x,y))d(x,y))\right]^{\kappa(d(x,y))}, \tag{2}$$

for all $x,y\in\mathscr{Y}$ with $x\preceq y$ and$H(Tx,Ty)>0$.

Then T has a fixed point.

Now, we give an existence theorem of a fixed point for multivalued mappings from a metric space X, equipped with a graph, into the space of nonempty closed and bounded subsets of the metric space. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops; that is, $E(G) \supseteq \Delta$, where $\Delta = \{(x,x) : x \in \mathscr{X}\}$. Assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Define $\alpha, \eta : X \times X \to [0, +\infty)$ as follows:

$$\alpha(x,y) = \begin{cases} 2, \ (x,y) \in E(G), \\ 0, \ \text{otherwise}, \end{cases}$$

$$\eta(x,y) = \begin{cases} 1, \ \text{if } (x,y) \in E(G), \\ 0, \ \text{otherwise}. \end{cases}$$

Theorem 4. *Let (X,d) be a complete metric space equipped with a graph G and $T : X \to CB(X)$ be a multivalued mapping. Assume that the following conditions are satisfied:*

(1) *For each $x \in X$ and $y \in Tx$ with $(x,y) \in E(G)$, we have $(y,z) \in E(G)$ for all $z \in Ty$;*
(2) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*
(3) *For every sequence $\{x_n\}$ in X such that $x_n \to x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $\{x_n, x) \in E(G)$.*
(4) *There exist a right continuous function $\theta \in \Theta$, $\beta \in \Omega$ and $\kappa : (0,\infty) \to [0,1)$ such that $\lim\sup_{t \to s^+} \kappa(t) < 1$ for all $s \in (0,\infty)$*

$$\theta(H(Tx,Ty)) \leq \left[\theta(\beta(d(x,y))d(x,y)\right]^{\kappa(d(x,y))}, \tag{3}$$

for all $x, y \in X$ with $(x,y) \in E(G)$ and $H(Tx,Ty) > 0$.

Then there exists a point x satisfying $x \in Tx$.

4 Existence of the solution for a fractional differential inclusion

Let the boundary value problems with fractional order differential inclusion and boundary integral conditions:

$$\begin{cases} {}^cD^\alpha x(t) \in F(t, x(t)), \ t \in I = [0,1], \ 1 < \alpha \leq 2 \\ x(0) - x'(0) = 0 \\ x(1) = \int_0^1 g(s, x(s))ds \end{cases} \tag{4}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of α order, F and g are given continuous functions
$F : I \times \mathbb{R} \times \mathbb{R} \longrightarrow K(\mathbb{R})$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$.
Denote by $X = C(I,\mathbb{R})$ the Banach space of continuous functions $x : I \longrightarrow \mathbb{R}$, with the usual supermum norm

$$\| x \|_\infty = \sup\{|x(t)|, \ t \in I\}.$$

X can be endowed with the partial order relationship $\preceq$, that is, for all $x, y \in X$ $x \preceq y$ if and only if $x(s) \leq y(s)$, so $(X, d_\infty, \preceq)$ is a complete ordered metric space. x is a solution of problem (4) if there exists $v \in F(s, x(t))$ such that

$$\begin{cases} {}^cD^\alpha x(t) = v(t), \ 0 \leq t \leq 1, \ 1 < \alpha \leq 2 \\ ax(0) - x'(0) = 0 \\ x(1) = \int_0^1 g(s, x(s))ds \end{cases} \tag{5}$$

Lemma 2. *A function x is a solution of (5) if and only if it is a solution of the fractional integral equation:*

$$x(s) = \int_0^1 G(t,s)v(s)ds + \frac{t+1}{2}\int_0^1 g(s,x(s))ds,$$

where G is the Green function given by

$$G(t,s)=\begin{cases}\frac{(t+1)(1-s)^{\alpha-1}}{2\Gamma(\alpha)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\leq s\leq t\\ \frac{(t+1)(1-s)^{\alpha-1}}{2\Gamma(\alpha)}, & t\leq s\leq 0.\end{cases} \tag{6}$$

G is continuous on I^2, so let $G_0=\sup_{0\leq t,s\leq 1}\int_0^1|G(t,s)|ds$. For all $x\in X$, define a set valued mapping:

$$Tx(t)=\{z\in X, z(t)=\int_0^1 G(t,s)v(s)ds+\frac{t+1}{2}\int_0^1 g(s,x(s))ds, v\in F(t,x(t))\}.$$

The problem (4) has a solution if and only if T has a fixed point.
Firstly, we show that T is well defined, in fact since F is continuous, then from Michael selection theorem there exists a continuous function in $v\in F(t,x)$, then Tx is non empty. Moreover, since F has compact values, then for all $x\in X$ the set Tx is compact. Assume that:

(A_1) : For each $x\in X$ and $y\in Tx$ with $x\preceq y$, we have $y\preceq z$ for all $z\in Ty$.
(A_2) : There exist $x_0\in X$ and $x_1\in Tx_0$ such that $x_0\preceq x_1$.
(A_3) : There exists $\varphi\in C(I,\mathbb{R}_+)$ such that for all $x_1,x_2\in\mathbb{R}$, we have

$$H(F(s,x_1(s)),F(s,x_2(t)))\leq\varphi(s)\ln(1+|x_1(t)-x_2(t)|),$$

with $\varphi_0=\sup_{t\in[0,1]}|\varphi(t)|$.
(A_4) : There exists $\psi\in C(I,\mathbb{R}_+)$ such that for all $x_1,x_2\in\mathbb{R}$, we have

$$\|g(s,x_1(s))-g(s,x_2(s))\|\leq\psi(t)\ln(1+|x_1(t)-x_2(t)|),$$

with $\psi_0=\sup_{t\in[0,1]}|\psi(t)|$ and $\kappa_0=G_0\varphi_0+\psi_0<1$.

Theorem 5. *Under the assumptions $(A_1)-(A_4)$ the problem (4) possesses a solution.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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