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Boundedness of non regular pseudodifferential operators on variable Besov spaces

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Abstract: We study the boundedness of non regular pseudodifferential operators, with symbols belonging to certain vector-valued Besov space, on Besov spaces with variable smoothness and integrabilty. These symbols include the classical Hörmander classes.

Keywords: Variable Besov spaces · Pseudodifferential operators · Non regular symbols.

1 Introduction

Pseudo-differential operators play an import role in Harmonic analysis and in nonlinear partial differential. The boundedness these operators has been extensively addressed in several works. In Lebesgue spaces with symbols in the Hörmander classes can be found in [5-8, 10-11, 20-22, 35, 41] and references therein. In another function spaces, such that Besov spaces, Triebel-Lizorkin spaces, BMO spaces and Hardy spaces, see [23, 28-29, 32, 39-40].

In [9] J. Marschall introduced the class $SB_{\delta}^{m}(r,\mu,v;N,\lambda)$, which is defined by means of vector-valued Besov spaces, and proved the boundedness of the corresponding pseudodifferential operators on Besov spaces and Triebel-Lizorkin spaces.

Boundedness of pseudodifferential operators, with symbols in the Hörmander classes, on weighted variable exponent Lebesgue and Bessel potential spaces was studied by V.S. Rabinovich and S. Samko [30-31] and by A. Yu. Karlovich and I. M. Spitkovsky in [8] (in variable Lebesgue space). Since Besov spaces can be written as a (real) interpolation space between appropriate Bessel potential spaces, Almeida and Hasto [2] extend the results of V.S. Rabinovich and S. Samko to Besov spaces with variable integrability $B_{p(\cdot),q}^{s}$.

Our main result in this paper concerns the boundedness properties of the pseudodifferential operators on Besov spaces with variable smoothness and integrability with symbols in the class $SB_{\delta}^{m}(r,\mu,v;N,\lambda)$.

2 Preliminaries

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f\lesssim g$ means that $f\leq c\,g$ for some independent constant c (and non-negative functions f and g), and $f\approx g$ means $f\lesssim g\lesssim f$. As usual for any $x\in\mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.

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By supp$f$ we denote the support of the function f , i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function.

The Hardy-Littlewood maximal operator $\mathscr{M}$ is defined on L^1_{loc} by

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy$$

and $\mathscr{M}_\tau f = \left(\mathscr{M}\,|f|^\tau\right)^{1/\tau}$, $0 < \tau < \infty$. The symbol $\mathscr{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathscr{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The Fourier transform of a tempered distribution f is denoted by $\mathscr{F} f$ while its inverse transform is denoted by $\mathscr{F}^{-1} f$.

2.1 Variable exponents.

The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $[c,\infty[$ for some $c > 0$. We denote the set of such functions by $\mathscr{P}_0$. The subset of variable exponents with range $[1,\infty[$ is denoted by $\mathscr{P}$. We use the standard notation $p^- := \operatorname*{ess-inf}_{x\in\mathbb{R}^n} p(x)$, $\quad p^+ := \operatorname*{ess-sup}_{x\in\mathbb{R}^n} p(x)$.

The variable exponent Lebesgue space $L^{p(\cdot)}$ is the class of all measurable functions f on $\mathbb{R}^n$ such that the modular

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |\frac{f(x)}{\lambda}|^{p(x)}\,dx$$

is finite for some $\lambda > 0$. This is a quasi-Banach function space equipped with the quasi-norm

$$\|f\|_{p(\cdot)} := \inf\Big\{\mu > 0 : \rho_{p(\cdot)}\Big(\frac{1}{\mu} f\Big) \le 1\Big\}.$$

If $p(x) := p$ is constant, then $L^{p(\cdot)} = L^p$ is the classical Lebesgue space.

Let $p,q \in \mathscr{P}_0$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the modular

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \inf\Big\{\lambda_v > 0 : \rho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\Big) \le 1\Big\}.$$

The (quasi)-norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\Big\{\mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}\{f_v\}_v\Big) \le 1\Big\}. \tag{1}$$

If $q^+ < \infty$, then we can replace (1) by the simpler expression $\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \Big\| |f_v|^{q(\cdot)} \Big\|_{\frac{p(\cdot)}{q(\cdot)}}$. Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. The case $p := \infty$ can be included by replacing the last modular by $\rho_{\ell^{q(\cdot)}(L^\infty)}((f_v)_v) := \sum_v \big\| |f_v|^{q(\cdot)} \big\|_\infty$.

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\text{loc}}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \tag{2}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-*Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log-*Hölder constant* and the log-*Hölder decay constant*, respectively. We note that all functions $g \in C^{\log}_{\text{loc}}$ always belong to L^∞.

We say that g is *globally*-log-*Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. We define the following class of variable exponents

$$\mathscr{P}^{\log} := \left\{ p \in \mathscr{P} : \frac{1}{p} \in C^{\log} \right\},$$

were introduced in [7, Section 2]. We define $1/p_\infty := \lim_{|x|\to\infty} 1/p(x)$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded. It was shown in [6], Theorem 4.3.8 that $\mathscr{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathscr{P}^{\log}$ and $p^- > 1$, see also [7], Theorem 1.2. Also if $p \in \mathscr{P}^{\log}$, then the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$: $\|\varphi * f\|_{p(\cdot)} \leq c\|\varphi\|_1 \|f\|_{p(\cdot)}$. We also refer to the papers [3] and [4], where various results on maximal function in variable Lebesgue spaces were obtained. Recall that $\eta_{v,m}(x) := 2^{nv}(1 + 2^v |x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\|\eta_{v,m}\|_1 = c_m$ is independent of v.

2.2 Variable Besov spaces

In this subsection we present the Fourier analytical definition of the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity. Let Ψ be a function in $\mathscr{S}'(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq 2$. We put $\mathscr{F}\varphi_0(x) = \Psi(x)$, $\mathscr{F}\varphi_1(x) = \Psi(x) - \Psi(2x)$ and $\mathscr{F}\varphi_v(x) = \mathscr{F}\varphi_1(2^{-v}x) \quad \text{for} \quad v = 2, 3, \ldots$. Then $\{\mathscr{F}\varphi_v\}_{v\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity, $\sum_{v=0}^{\infty} \mathscr{F}\varphi_v(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{v=0}^{\infty} \varphi_v * f$$

of all $f \in \mathscr{S}'(\mathbb{R}^n)$ (convergence in $\mathscr{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Definition 1.*Let $\{\mathscr{F}\varphi_v\}_{v\in\mathbb{N}_0}$ be as resolution of unity. For $s : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathscr{P}_0$, the Besov space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\| \left(2^{vs(\cdot)}\varphi_v * f\right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \tag{3}$$

For any $p, q \in \mathscr{P}^{\log}_0$ and $s \in C^{\log}_{\text{loc}}$, the space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathscr{F}\varphi_v\}_{v\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms) and

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathscr{S}'(\mathbb{R}^n).$$

3 Boundedness of pseudodifferential operators

For a function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, we write

$$a_j(x, \xi) = \mathscr{F}^{-1}_{y\to x}(\varphi_j(y)\mathscr{F}a(y, \xi)).$$

Let $0<\mu\leq\infty$, $1\leq\lambda\leq\infty$, $r\geq\frac{n}{\mu}$ and $N>\frac{n}{\lambda}$. The space $B^{r}_{\mu,\nu}(B^{N}_{\lambda,\infty})$ consists of all distributions $a\in\mathscr{S}'(\mathbb{R}^n\times\mathbb{R}^n)$ such that

$$\|a\|_{B^{r}_{\mu,\nu}(B^{N}_{\lambda,\infty})}=\left\|\left\{2^{jr}\left\|a_j(x,\cdot)\right\|_{B^{N}_{\lambda,\infty}}\right\}_j\right\|_{\ell^{\nu}(L^{\mu})}<\infty.$$

Notice that these spaces are just the spaces $SB^{\bar{r}}_{\bar{p},\bar{q}}$ with $\bar{r}=(N,r)$, $\bar{p}=(\lambda,\mu)$ and $\bar{q}=(\infty,\nu)$, see [?] for further properties of these function spaces. Let $m,r,N\in\mathbb{R}$, $0\leq\delta\leq1$, $0<\mu\leq\infty$, $r>\frac{n}{\mu}$ and $N>\frac{n}{\lambda}$. We say that a symbol a belongs to $SB^{m}_{\delta}(r,\mu,\nu;N,\lambda)$ if

$$\sup_k 2^{-km}\left\|\left\|a\left(x,2^k\cdot\right)\varphi_k\left(2^k\cdot\right)\right\|_{B^{N}_{\lambda,\infty}}\right\|_{L^{\infty}(dx)}<\infty$$
$$\sup_k 2^{-k(m+\delta r)}\left\|a\left(x,2^k\cdot\right)\varphi_k\left(2^k\cdot\right)\right\|_{B^{r}_{\mu,\nu}(B^{N}_{\lambda,\infty})}<\infty,$$

which, introduced by J. Marschall [9] and [10]. Choosing $\mu=\nu=N=\lambda=\infty$, these symbols include the classical Hörmander classes $S^{m}_{1,\delta}$. Moreover the class $SB^{m}_{0}(r,\mu,\nu;\infty,1)$ equal the class $S'(B^{(1,\dots,1),r}_{\mu,\nu})^m$ of M. Yamazaki [11]. Notice that

$$SB^{m}_{\delta}(r,\mu,\nu;N,\lambda)\hookrightarrow SB^{m}_{\delta_1}(r_1,\mu_1,\nu;N,\lambda),\qquad(4)$$

if $0<\mu<\mu_1\leq\infty$, $0<\nu\leq\infty$, $r-\frac{n}{\mu}=r_1-\frac{n}{\mu_1}$ and $\delta r=\delta_1 r_1$, see [10, Lemma 10].

A pseudo-differential operator with symbol $a\in SB^{m}_{\delta}(r,\mu;N,\lambda)$ is defined by

$$a(x,D)f(x)=\frac{1}{(2\pi)^n}\int e^{ix\xi}a(x,\xi)\mathscr{F}f(\xi)d\xi,$$

where $f\in\mathscr{S}(\mathbb{R}^n)$. Besov estimates, with fixed exponents, for such operators were given by J. Marschall [10].
The following lemma is from A. Almeida and P. Hästö [1, Lemma 4.7] (we use it, since the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$, see [1, Example 4.1]).

Lemma 1. *Let $p\in\mathscr{P}^{\log}$, $q\in\mathscr{P}^{\log}_0$ with $0<q^-\leq q^+<\infty$ and $p^->1$. For $m>n+c_{\log}(1/q)$, there exists $c>0$ such that*

$$\|(\eta_{\nu,m}*f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}\leq c\|(f_\nu)_\nu\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

The the next three lemmas are used in the proof of our result, see [10] for the constant exponents.

Lemma 2. *Let $A,B>0$, p, $q\in\mathscr{P}^{\log}_0$ and $s\in C^{\log}_{\text{loc}}$ such that $q^+<\infty$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that*

$$\text{supp}\mathscr{F}f_0\subseteq\{\xi\in\mathbb{R}^n:|\xi|\leq A\}$$

and

$$\text{supp}\mathscr{F}f_k\subseteq\left\{\xi\in\mathbb{R}^n:B2^{k+1}\leq|\xi|\leq A2^{k+1}\right\}.$$

Then it holds that:

$$\left\|\sum_{k=0}^{\infty}f_k\right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim\left\|(2^{ks(\cdot)}f_k)_k\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Lemma 3. *Let $A>0,p,q\in\mathscr{P}^{\log}_0$ and $s\in C^{\log}_{\text{loc}}$ such that $0<q^+<\infty$. Let $s^->n(\max\{1,1/p^-\}-1)$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that $\text{supp}\mathscr{F}f_k\subseteq\left\{\xi\in\mathbb{R}^n:|\xi|\leq A2^{k+1}\right\}$. Then it holds that*

$$\left\|\sum_{k=0}^{\infty}f_k\right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim\left\|(2^{ks(\cdot)}f_k)_k\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Lemma 4. *Let $A > 0, p, q \in \mathscr{P}_0^{\log}$ such that $0 < q^+ < \infty$. Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions such that*

$$\operatorname{supp}\mathscr{F} f_k \subseteq \left\{\xi \in \mathbb{R}^n : |\xi| \leq A2^{k+1}\right\}.$$

Let $\alpha = n(\max\{1, 1/p^-\} - 1)$. Then it holds that, for some constant $c > 0$,

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{B^{\alpha}_{p(\cdot),\infty}} \lesssim \left\| (2^{k\alpha} f_k)_k \right\|_{\ell^{\min(1,p^-)}(L^{p(\cdot)})}. \tag{5}$$

Moreover if the right-hand side inequality in (5) is finite, then $\left\{\sum_{k=0}^{N} f_k\right\}_N$ converges in $\mathscr{S}'(\mathbb{R}^n)$ to a distribution $\sum_{k=0}^{\infty} f_k$ satisfying this inequality.

The following proposition plays a fundamental role.

Proposition 1. *Let $s \in C^{\log}_{\text{loc}}$, p_1, p_2, $q \in \mathscr{P}_0^{\log}$, $0 < \mu \leq \infty$ and $1 \leq \lambda \leq \infty$ with $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{\mu}$. Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a bounded and measurable symbol such that*

$$\operatorname{supp} a(x, \cdot) \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq c2^k\}.$$

If $p_1^- \geq 1$ or if $0 < p_1^- < 1$ and

$$\operatorname{supp}\mathscr{F} f \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq c2^k\},$$

and if $N > n\max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p_2^-}\right\} + c_{\log}(s) + c_{\log}(\frac{1}{q})$, then

$$\left\| 2^{ks(\cdot)} \delta^{-\frac{1}{q(\cdot)}} a(x, D) f \right\|_{p_1(\cdot)} \lesssim \left\| \left\| a\left(\cdot, 2^k \cdot\right) \right\|_{B^N_{\lambda,\infty}} \right\|_{\mu} \left\| 2^{ks(\cdot)} \delta^{-\frac{1}{q(\cdot)}} f \right\|_{p_2(\cdot)}$$

for any $k \in \mathbb{N}_0$ and any $\delta \in \left[2^{-k}, 1 + 2^{-k}\right]$, with the implicit constant not depending on k.

Now we are ready to formulate our main result.

Theorem 1. *Let $s \in C^{\log}_{\text{loc}}$, p, $q \in \mathscr{P}_0^{\log}$ with $0 < q^+ < \infty$. Let $a \in SB^m_\delta(r, \mu, v; N, \lambda)$ be such that $0 < \mu, v \leq \infty$, $r > 0$, $(1 - \delta) r \geq \frac{n}{\mu}$ and $1 \leq \lambda \leq \infty$. Let $N > n\max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p^-}\right\} + c_{\log}(s) + c_{\log}(\frac{1}{q})$.*

(i) *If*

$$n\max\left\{1, \frac{1}{\mu} + \frac{1}{p^-}\right\} - n - (1 - \delta) r < s^-$$

and

$$s^+ < r - n\max\left\{\frac{1}{\mu} - \frac{1}{p^+}, 0\right\},$$

then $a(x, D)$ is a continuous linear mapping from $B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$ to $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

(ii) *If $(1 - \delta) r > \frac{n}{\mu}$, $v \leq q^- < \infty$. and*

$$s := r - n\max\left\{\frac{1}{\mu^-} - \frac{1}{p^+}, 0\right\},$$

then $a(x, D)$ is a continuous linear mapping from $B^{s+m}_{p(\cdot),q(\cdot)}$ to $B^{s}_{p(\cdot),q(\cdot)}$.

(iii) *We suppose that $\frac{1}{\mu} + \frac{1}{p^-} \leq 1$ or $0 < p^+ \leq 1$ and $\frac{1}{\mu} + \frac{1}{p^-} > 1$. If $(1 - \delta) r > \frac{n}{\mu}$, $0 < q^+ \leq \min\{1, p^-\}$ and*

$$s := n\max\left\{1, \frac{1}{\mu} + \frac{1}{p^-}\right\} - n - (1 - \delta) r,$$

then $a(x,D)$ *is a continuous linear mapping from* $B^{s(\cdot)+m}_{p(\cdot),q(\cdot)}$ *to* $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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