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An improved oscillation result for advanced differential equations on time scale

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Abstract: In this paper, we will establish some oscillation criteria for for advanced differential equations on time scale

$$u^{\Delta}(t) - \eta(t)\, u(\lambda(t)) = 0, \quad \text{for } t \in [t_0, \infty) \cap \mathbb{T},$$

on a time scales, where $\sup \mathbb{T} = \infty$ This study aims to present some new sufficient conditions for the oscillatory of solutions to a class of first-order advanced differential equation on time scale.

Keywords: Time scale, Oscillation, Advanced differential equations.

1 Introduction

In this article, we consider the advanced differential equation on time scale of the form

$$u^{\Delta}(t) - \eta(t)\, u(\lambda(t)) = 0, \quad \text{for } t \in [t_0, \infty) \cap \mathbb{T}, \tag{1}$$

on a time scale $\mathbb{T}$, since we are interested in oscillation, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $I_{t_0} = [t_0, \infty) \cap \mathbb{T}$, with $t_0 \in \mathbb{T}$. The functions $\eta \in \mathscr{C}\left(I_{t_0}, [0, \infty)\right)$ and $\lambda \in \mathscr{C}\left(I_{t_0}, I_{t_0}\right)$, such as $\eta \neq 0$ on any interval of the form I_{t_0}, $\lambda(t) > t$, for $t \in I_{t_0}$ and $\lim_{t\to\infty} \lambda(t) = \infty$.

By a solution of (1) we mean a nontrivial real-valued function $u \in \mathscr{C}^1(I_{T_u}, \mathbb{R})$, $T_u \in I_{t_0}$ which satisfies (1) on I_{T_u}. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution u of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory. The theory of time scales was introduced by Hilger [1] in order to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. The books on the subjects of time scale, that is, measure chain, by Bohner and Peterson [2,3], summarize and organize much of time scale calculus. The theory of oscillations is an important branch of the applied theory of dynamic equations related to the study of oscillatory phenomena in technology, natural and social sciences.

In recent years, there has been much research activity concerning the oscillation of solutions of various dynamic equations on time scales.

Today there has been an increasing interest in obtaining sufficient conditions for oscillation and non oscillation of solutions of advanced type differential equations, we refer the reader to the articles [19]-[26] and the references cited therein. So far, there are any results on oscillatory of (1). Hence the aim of this paper is to give some oscillation criteria for this equation.

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2 Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. Then, one defines the graininess function $\mu : \mathbb{T} \to [0, +\infty[$ by $\mu(t) = \sigma(t) - t$. If $\sigma(t) > t$, then we say that t is right-scattered; if $\rho(t) < t$, then t is left-scattered. Moreover, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. If $u : \mathbb{T} \to \mathbb{R}$, then $u^{\sigma} : \mathbb{T} \to \mathbb{R}$ is given by $u^{\sigma}(t) = u(\sigma(t))$ for all $t \in \mathbb{T}$.

Let $u : \mathbb{T} \to \mathbb{R}$ be a real valued function on a time scale $\mathbb{T}$. Then, for $t \in \mathbb{T}^{k}$, we define $u^{\Delta}(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$, there is a neighborhood $\mathscr{V}$ of t such that for all $s \in \mathscr{V}$,

$$\left| u^{\sigma}(t) - u(s) - u^{\Delta}(t)(\sigma(t) - s) \right| \leq \varepsilon \left| \sigma(t) - s \right|.$$

We say that u is delta differentiable on $\mathbb{T}$ provided $u^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{k}$. We will make use of the following product and quotient rules for the derivative of the product uv and the quotient $\frac{u}{v}$ (where $vv^{\sigma} \neq 0$) of two differentiable function u and v

$$(uv)^{\Delta} = u^{\Delta} v^{\sigma} + u v^{\Delta}, \quad \text{and} \quad \left(\frac{u}{v}\right)^{\Delta} = \frac{u^{\Delta} v - u v^{\Delta}}{v v^{\sigma}}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ will be called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write $f \in \mathscr{C}_{rd}(\mathbb{T}) = \mathscr{C}_{rd}(\mathbb{T}, \mathbb{R})$.

For a, $b \in \mathbb{T}$, and for a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\, \Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t) g^{\Delta}(t)\, \Delta t = [f(t) g(t)]_a^b - \int_a^b f^{\Delta}(t) g^{\sigma}(t)\, \Delta t.$$

and the improper integrals are defined in the usual way by

$$\int_a^{\infty} f(t)\, \Delta t = \lim_{b \to \infty} \int_a^b f(t)\, \Delta t.$$

For more on the calculus on time scales, we refer the reader to the books [2,3].

3 Oscillation results

To derive main results in this section, we need the following lemma.

Definition 1. *Let us define a sequence of functions by the recurrence relation*

$$w_{n+1}(t) := \int_t^{\lambda(t)} \eta(s) \exp(w_n(t))\, \Delta s, \quad \text{for } t \in I_{t_0}, \tag{2}$$

with

$$w_0(t) := \int_t^{\lambda(t)} \eta(s)\, \Delta s, \quad \text{for } t \in I_{t_0}. \tag{3}$$

Lemma 1. *If u is an positive solution of (1), then the sequence $\{w_n(t) : n \in \mathbb{N}\}$ converges.*

Proof. Let u be an eventually positive solution of (1). From (1), we have $u^{\Delta}(t) \geq 0$, for $t \in I_{t_0}$, by Pöotzsche's chain rule [2, Theorem 1.90], we see that

$$(\ln(u(t)))^{\Delta} = u^{\Delta}(t) \int_0^1 (hu(t) + (1-h) u^{\sigma}(t))^{-1} dh \geq \frac{u^{\Delta}(t)}{u(t)}, \text{ for } t \in I_{t_0},$$

so, we get

$$\ln\left(\frac{u(\lambda(t))}{u(x)}\right) \geq \int_t^{\lambda(t)} \frac{u^{\Delta}(s)}{u(s)} \Delta s = \int_t^{\lambda(t)} \eta(s) \frac{u(\lambda(s))}{u(s)} \Delta s > \int_t^{\lambda(t)} \eta(s) \Delta s = w_0(t), \quad \text{for } t \in I_{t_0}.$$

This means,

$$\frac{u(\lambda(t))}{u(t)} \geq \exp(w_0(t)), \quad \text{for } t \in I_{t_0}.$$

Multiplying the left-hand side by $\eta(t)$, we get

$$\frac{\eta(t) u(\lambda(t))}{u(t)} \geq \eta(t) \exp(w_0(t)), \quad \text{for } t \in I_{t_1}.$$

It follows from (**??**) and the above inequality, we obtain

$$\ln\left(\frac{u(\lambda(t))}{u(t)}\right) \geq \int_t^{\lambda(t)} \eta(s) \exp(w_0(s)) \Delta s := w_1(t), \quad \text{for } t \in I_{t_0}.$$

By induction, we can see that if

$$\ln\left(\frac{u(\lambda(t))}{u(t)}\right) \geq w_n(t), \quad \text{for } t \in I_{t_0}.$$

In the same way, we find that the inequality is true for $n+1$. By (2) and the above inequality, we conclude that the sequence $\{w_n(t) : n \in \mathbb{N}\}$ is increasing and increased, then $\{w_n(t) : n \in \mathbb{N}\}$ is converges.

Lemma 2.*The sequence $\{w_n(t) : n \in \mathbb{N}\}$ defined by (2), converges if and only if*

$$\int_t^{\lambda(t)} \eta(s) \Delta s \leq \frac{1}{e}, \quad \textit{for all } t \in I_{t_0}. \tag{4}$$

*Proof.*Sufficient: Suppose that (3) is true. Then

$$w_0(t) \leq \frac{1}{e} = a_0, \text{ for all } t \in I_{t_0},$$

Then, we get

$$w_1(t) \leq \int_t^{\lambda(t)} \eta(s) \exp(w_0(t)) \Delta s \leq a_0 \exp(a_0) = a.$$

By induction, we can see that if

$$w_n(t) \leq a_0 \exp(a_n) < 1.$$

In view of Lemma [19, Lemma 2.1], $\{w_n(t) : n \in \mathbb{N}\}$ converges.
Necessary: Suppose that $\{w_n(t) : n \in \mathbb{N}\}$ converges. then there is a positive real function denoted $w(t)$, such as $w(t) = \lim_{n\to\infty} w_n(t)$, by (2), we find that the function w is satisfied

$$w(t) = \int_t^{\lambda(t)} \eta(s) \exp(w(t)) \Delta s, \quad \text{for } t \in I_{t_0}.$$

The above equality, we conclude that the function w is increased on $[t_0,\infty)$. Let $\psi(t)=\exp(w(t))\geq 1$, for $t\in I_{t_0}$, we have

$$\psi(t)=\exp\left(\int_t^{\lambda(t)}\eta(s)\psi(s)\Delta s\right),\quad \text{for } t\in I_{t_0}.$$

It follows from ψ is increased on $[t_0,\infty)$ and the above equality, we obtain

$$\exp\left(\psi(t)\int_t^{\lambda(t)}\eta(s)\Delta s\right)\leq\exp\left(\int_t^{\lambda(t)}\eta(s)\psi(s)\Delta s\right)=\psi(t),\quad \text{for } t\in I_{t_0}.$$

Then,

$$\int_t^{\lambda(t)}\eta(s)\Delta s\leq\frac{\ln(\psi(t))}{\psi(t)},\quad \text{for } t\in I_{t_0}. \tag{5}$$

On the other hand, we have

$$\max\left\{\frac{\ln(x)}{x}:x\geq 1\right\}=\frac{1}{e}.$$

By (5) and the above inequality, we have

$$\int_t^{\lambda(t)}\eta(s)\Delta s\leq\frac{1}{e},\quad \text{for } t\in I_{t_0}.$$

This completes the proof.

*Remark.*If u is an positive solution of (1), then inequality (4) is satisfied.

Now, we establish some sufficient conditions which guarantee that every solution u of (1) oscillates on $[t_0,\infty)$.

Theorem 1.*For all sufficiently large $t_1\in I_{t_0}$, such as*

$$\int_t^{\lambda(t)}\eta(s)\Delta s>\frac{1}{e},\quad \textit{for } t\in I_{t_1}. \tag{6}$$

Then any solution of (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatory solution u on $[t_0,\infty)$. Since $-u$ is also a solution of (1), we can confine our discussion only to the case where the solution u is eventually positive solution of (1). We may assume without loss of generality that there exists $t_1\in I_{t_0}$, such that

$$u(t)>0\quad\text{and}\quad u(\lambda(t))>0, \text{ for all } t\in I_{t_1}.$$

This means the following equation (1) has a positive solution u on I_{t_1}.

$$u^{\Delta}(t)-\eta(t)u(\lambda(t))=0,\quad \text{for } t\in I_{t_1}$$

By Lemma 1 and Lemma 2, we obtain

$$\int_t^{\lambda(t)}\eta(s)\Delta s\leq\frac{1}{e},\quad \text{for } t\in I_{t_1}.$$

which contradicts (6). This completes the proof.

As a Theorem of the previous result, we deduce the following corollaries.

Corollary 1. *If*

$$\liminf_{t\to\infty}\int_t^{\lambda(t)}\eta(s)\Delta s>\frac{1}{e}.$$

Then any solution of (1) *is oscillatory.*

Corollary 2. *If*

$$\limsup_{t\to\infty}\int_{t}^{\lambda(t)}\eta(s)\Delta s>1.$$

Then any solution of (1) *is oscillatory.*

4 Application

In this section, we give applications and examples to illustrate our main result. Next, we consider the advanced differential equation on time scale of the form

$$u^{\Delta}(t)+q(t)u^{\sigma}(t)-\eta(t)u(\lambda(t))=0,\quad \text{for } t\in I_{t_0}, \tag{7}$$

and

$$u^{\Delta}(t)-q(t)u(t)-\eta(t)u(\lambda(t))=0,\quad \text{for } t\in I_{t_0}, \tag{8}$$

with, the functions $q\in\mathscr{C}\left(I_{t_0},[0,\infty)\right)$.

Theorem 2. *For all sufficiently large* $t_1\in I_{t_0}$, *such as*

$$\int_{t}^{\lambda(t)}\eta(s)e_{\ominus q}(s,t_0)e_{\ominus q}(\tau(s),t_0)\Delta s>\frac{1}{e},\quad \textit{for } t\in I_{t_1}, \tag{9}$$

Then any solution of (7) *is oscillatory.*

Proof. By equation (7), we find

$$[u(t)e_q(t,t_0)]^{\Delta}=\eta(t)e_{\ominus q}(t,t_0)u(\lambda(t)),\quad \text{for } t\in I_{t_0}.$$

Let $v(t)=u(t)e_q(t,t_0)$, for $t\in I_{t_0}$, we have

$$v^{\Delta}(t)=\eta(t)e_{\ominus q}(t,t_0)e_{\ominus q}(\tau(t),t_0)v(\lambda(t)),\quad \text{for } t\in I_{t_0},$$

we conclude that the latter's equation is the same as the equation (1). And from it we conclude if it is achieved (9), then any solution of (7) is oscillatory.

Theorem 3. *For all sufficiently large* $t_1\in I_{t_0}$, *such as*

$$\int_{t}^{\lambda(t)}\eta(s)\frac{e_q(\lambda(s),t_0)}{e_q^{\sigma}(s,t_0)}\Delta s>\frac{1}{e},\quad \textit{for } t\in I_{t_1}, \tag{10}$$

Then any solution of (8) *is oscillatory.*

Proof. Let u be an eventually positive solution of (8), then

$$\left[\frac{u(t)}{e_q(t,t_0)}\right]^{\Delta}=\frac{\eta(t)}{e_q(t,t_0)e_q^{\sigma}(t,t_0)}u(\lambda(t)),\quad \text{for } t\in I_{t_0}.$$

Let $v(t)=\dfrac{u(t)}{e_q(t,t_0)}$, for $t\in I_{t_0}$, we have

$$v^{\Delta}(t)=\eta(t)\frac{e_q(\lambda(t),t_0)}{e_q(t,t_0)}u(\lambda(t)),\quad \text{for } t\in I_{t_0},$$

we conclude that the latter's equation is the same as the equation (1). And from it we conclude if it is achieved (10), then any solution of (8) is oscillatory.

Next, we give an example to illustrate our main result.

Example 1. Consider the delay differential equation

$$x^{\Delta}(t) - (t+1)x(t+1) = 0, \quad \text{for all } t \in \mathbb{N}. \tag{11}$$

Here, $\mathbb{T} = \mathbb{N}$, $\eta(t) = t+1$, and $\lambda(t) = t+1 > t$, for all $t \in \mathbb{N}$. On the other hand, we have

$$\int_t^{\lambda(t)} \eta(s)\Delta s = \int_t^{t+1} (s+1)\Delta s = t+1 > \frac{1}{e}, \quad \text{for all } t \in \mathbb{N}.$$

Thus, (6) holds. By Theorem 1, equation (11) is oscillatory.

Example 2. Consider the delay differential equation

$$x^{\Delta}(t) - tx(2t) = 0, \quad \text{for all } t \in \overline{2^{\mathbb{N}}}. \tag{12}$$

Here, $\mathbb{T} = \overline{2^{\mathbb{N}}}$, $\eta(t) = t$, and $\lambda(t) = 2t > t$, for all $t \in \overline{2^{\mathbb{N}}}$. Then

$$\int_t^{\lambda(t)} \eta(s)\Delta s = \int_t^{2t} s\Delta s = 2t^2 \geq 1 = \lambda, \text{ for all } t \in \overline{2^{\mathbb{N}}}.$$

Thus, (6) holds. By Theorem 1, equation (12) is oscillatory.

5 Conclusion

In this paper, we use the recursive sequence we have constructed to establish some new oscillation results of first-order linear dynamic equations with damping. Our results not only unify the oscillation of differential equations but also improve the differential equations established in [19].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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