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A Numerical Comparison of Solutions of Non-Linear Initial Value Problems of First Order

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Abstract: The approximate solutions of first order non-linear initial value problems are obtained using both the perturbation method and the Adomian's decomposition method. The results are then compared with the exact solution, and they are presented in a number of figures for the various values of parameter ε.

Keywords: Perturbation Method, Adomian's Decomposition Method, Initial Value Problems, exact solution, approximate solution.

1 Introduction

Over the last ten years or so many mathematical methods that are aimed at solving nonlinear partial and ordinary differential equations have appeared in the research literature [1,2,9,10,11,12,13,14]. However, most of them require a tedious analysis or a large computer memory to handle these problems. In the beginning of the 1980s, a so-called Adomian Decomposition Method (ADM) was introduced by Adomian [3] for solving the nonlinear problems. It is well known that this method avoids linearization and unrealistic assumptions, and provides an efficient numerical solution with high accuracy [3,7,8,9,10,11].

Also, another method called Perturbation Series Method (PSM) has its roots in 17th century. It is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε. These methods are so powerful that sometimes it is actually advisable to introduce a parameter ε temporarily into a difficult problem having no small parameter, and then finally to set $\varepsilon = 1$ to recover the original problem [5]. The main idea of perturbation theory can be thought of decomposing a difficult problem into an infinite number of relatively easy ones. Hence, perturbation theory is most useful when the first few steps reveal the important features of the solution and the remaining ones give small corrections [5]. In Ref. [4], the authors have made a comparison of the ADM and a regular perturbation technics applied to the solution of nonlinear vector random differential equations. They have observed that the ADM is superior for their problem. Also, in Refs. [1,2] the authors have obtained the numerical solutions of Blasius equation and integral equations by using "homotopy perturbation method" and ADM. From the comparison they have made, one observes that the accuracy and the effectiveness of the method change according to the problem studied.

In this paper we consider the non-linear initial value problem

$$\frac{dy(x)}{dx} = f(x, y(x), \varepsilon), \tag{1}$$

$$y(x_0) = y_0, \tag{2}$$

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where x is independent variable, $y = y(x)$ is dependent variable, and ε is some positive parameter. Here $f : \mathbf{R}\times\mathbf{R}\times\mathbf{R}^{+} \to \mathbf{R}$ together with a point $(x_0, y_0) \in \mathbf{R}\times\mathbf{R}$.

2 Adomian's decomposition method

In this section we consider the model equation of the form

$$\frac{dy(x)}{dx} = f(x, y(x), \varepsilon), \quad y(x_0) = y_0, \tag{3}$$

where f is a nonlinear function in y. The decomposition method consists of approximating the solution of Eq.(3) as an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{4}$$

and decomposing f as

$$f(x, y(x), \varepsilon) = \sum_{n=0}^{\infty} A_n, \tag{5}$$

where A_n's are the Adomian polynomials given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(x, \sum_{n=0}^{\infty} \lambda^n y_n(x), \varepsilon \right) \right]_{\lambda=0}, \quad n = 0, 1, 2\cdots. \tag{6}$$

The convergence of the decomposition series (6) is studied in [6]. Applying the decomposition method [3], Eq.(3) can be written as

$$Ly(x) = f(x, y(x), \varepsilon) \tag{7}$$

where the notation $L = \frac{d}{dx}$ symbolizes the linear differential operator. We assume the integration inverse operators L^{-1} exists, and it is defined as $L^{-1} = \int_0^x (.)d\tau$. Therefore, applying on both sides of Eq.(7) with L^{-1} yields

$$y(x) = y_0 + L^{-1} f(x, y(x), \varepsilon). \tag{8}$$

Using Eqs.(4) and (5) it follows that

$$\sum_{n=0}^{\infty} y_n(x) = y_0 + L^{-1} \sum_{n=0}^{\infty} A_n. \tag{9}$$

Therefore, one determines the iterates in the following recursive way:

$$y_0(x) \equiv y(0),$$
$$y_{n+1}(x) = L^{-1} A_n, \quad n = 0, 1, 2\cdots.$$

We then define the solution $y(x)$ as

$$y(x) = \lim_{n\to\infty} \sum_{k=0}^{n} y_k(x). \tag{10}$$

3 Perturbation series method

The general procedure of the perturbation series method is to introduce a small parameter ε, such that when $\varepsilon = 0$ the problem becomes soluble. The global solution to the given problem can then be studied by a local analysis about $\varepsilon = 0$ [5]. A perturbative solution is constructed by local analysis about $\varepsilon = 0$ as a series of powers of ε:

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots . \tag{11}$$

This series is called a perturbation series. It has the attractive feature that $y_n(x)$ can be computed in terms of $y_0(x), y_1(x), ..., y_{n-1}(x)$ as long as the problem obtained by setting $\varepsilon = 0$ is soluble [5]. One expects that $y(x)$ is well approximated by only a few terms of the perturbation series if ε is very small.

In the following section we compare the exact solution of an initial-value problem with the approximate solutions obtained by (ADM) and (PSM). We then present some figures that show how close the approximate solutions are to the exact solution.

4 Application and results

Example. We consider the initial-value problem

$$y'(x) = -\frac{y(x)}{x+\varepsilon y(x)}, \quad y(1) = 1. \tag{12}$$

Firstly, we find the exact solution to (12). Therefore, let us first write the differential equation as $(x+\varepsilon y)y' + y = 0$, and let $x + \varepsilon y = t$. Differentiating both sides with respect to x, we then have the equation $tdt - xdx = 0$. From here one gets $t - x = c/(t+x)$, where c is an arbitrary constant. Therefore,

$$y = \frac{1}{\varepsilon}(t-x) = \frac{c}{\varepsilon(t+x)} = \frac{c}{\varepsilon(\varepsilon y + 2x)}$$

from which it follows that $y = \frac{-x}{\varepsilon} \mp \frac{1}{\varepsilon}\sqrt{x^2+c}$. Since y(1)=1, we get the exact solution to (12)

$$y_{\text{exact}}(x) = \frac{-x}{\varepsilon} + \sqrt{x^2/\varepsilon^2 + 2/\varepsilon + 1}. \tag{13}$$

From (13) one obtains the second order expansion of the exact solution with respect to ε

$$y_{\text{exact}}(x) = \frac{-x}{\varepsilon} + \frac{x}{\varepsilon}\sqrt{1 + \frac{\varepsilon^2 + 2\varepsilon}{x^2}} \tag{14}$$

$$= \frac{1}{x} + \left(\frac{1}{2x} - \frac{1}{2x^3}\right)\varepsilon + \left(\frac{1}{2x^5} - \frac{1}{2x^3}\right)\varepsilon^2 + O(\varepsilon^3) \quad (\varepsilon \to 0). \tag{15}$$

Secondly, we obtain the Adomian decomposition solution to (12). We proceed as in section 2, and take $f(x,y,\varepsilon) = -\frac{y}{x+\varepsilon y}$. Therefore, we have

$$f(x,y,\varepsilon) = -\frac{y}{x}\left(\frac{1}{1+\frac{\varepsilon y}{x}}\right) = -\frac{y}{x} + \frac{y^2}{x^2}\varepsilon - \frac{y^3}{x^3}\varepsilon^2 + \cdots ,$$

and the Adomian polynomials can be derived as follows [3,10] :

$$\begin{aligned}
f(x,y,\varepsilon) &= -\frac{1}{x}(y_0+y_1+y_2+\cdots)+\frac{\varepsilon}{x^2}(y_0+y_1+y_2+\cdots)^2-\frac{\varepsilon^2}{x^3}(y_0+y_1+y_2+\cdots)^3+\cdots \\
&= -\frac{1}{x}(y_0+y_1+y_2+\cdots)+\frac{\varepsilon}{x^2}\left(y_0^2+(2y_0y_1)+(2y_0y_2+y_1^2)+(2y_0y_3+2y_1y_2)+\cdots\right) \\
&\quad -\frac{\varepsilon^2}{x^3}\left(y_0^3+(3y_0^2y_1)+(3y_0^2y_2+3y_0y_1^2)+(y_1^3+3y_0^2y_3+6y_0y_1y_2)+\cdots\right)
\end{aligned}$$

Therefore, we get the following Adomian polynomials:

$$\begin{aligned}
A_0 &= -\frac{1}{x}y_0+\frac{\varepsilon}{x^2}y_0^2-\frac{\varepsilon^2}{x^3}y_0^3, \\
A_1 &= -\frac{1}{x}y_1+\frac{2\varepsilon}{x^2}y_0y_1-\frac{3\varepsilon^2}{x^3}y_0^2y_1, \\
A_2 &= -\frac{1}{x}y_2+\frac{2\varepsilon}{x^2}y_0y_3+\frac{2\varepsilon}{x^2}y_1y_2-\frac{3\varepsilon^2}{x^3}y_0^2y_2-\frac{3\varepsilon^2}{x^3}y_1^2y_0, \\
&\vdots
\end{aligned}$$

Since $y(1)=1$ we take $y_0(x)\equiv 1$. Therefore, we have

$$\begin{aligned}
A_0 &= -\frac{1}{x}+\frac{\varepsilon}{x^2}-\frac{\varepsilon^2}{x^3}, \\
y_1(x) = L^{-1}A_0 &= -\ln x-\frac{\varepsilon}{x}+\frac{\varepsilon^2}{2x^2}, \\
A_1 &= \frac{\ln x}{x}+\left(\frac{1}{x^2}-2\frac{\ln x}{x^2}\right)\varepsilon+\left(-\frac{5}{2x^3}+3\frac{\ln x}{x^3}\right)\varepsilon^2+O(\varepsilon^3) \quad (\varepsilon\to 0), \\
y_2(x) = L^{-1}A_1 &= \frac{(\ln x)^2}{2}+\left(\frac{1}{x}+2\frac{\ln x}{x}\right)\varepsilon+\left(\frac{1}{2x^2}-3\frac{\ln x}{2x^2}\right)\varepsilon^2+O(\varepsilon^3) \quad (\varepsilon\to 0), \\
&\vdots
\end{aligned}$$

and so on. Substituting these terms into (10), we obtain the three-term decomposition series solution

$$y_{\text{decomp}}(x) = 1-\ln x+\frac{(\ln x)^2}{2}+\left(2\frac{\ln x}{x}\right)\varepsilon+\left(\frac{1}{x^2}-3\frac{\ln x}{2x^2}\right)\varepsilon^2+O(\varepsilon^3) \quad (\varepsilon\to 0). \tag{16}$$

Lastly, we obtain the perturbation series solution to (12), by assuming that (12) has a solution, $y(x)$ of the form given by (11), namely

$$y(x)=\sum_{n=0}^{\infty}\varepsilon^n y_n(x), \quad \varepsilon \ll 1. \tag{17}$$

To do this, let us write Eq.(12) in the form

$$(x+\varepsilon y(x))y(x)'+y(x)=0, \quad y(1)=1. \tag{18}$$

Now, let us plug Eq.(17) in Eq.(18):

$$\left[x+\varepsilon\sum_{n=0}^{\infty}\varepsilon^n y_n(x)\right]\sum_{n=0}^{\infty}\varepsilon^n y_n'(x)+\sum_{n=0}^{\infty}\varepsilon^n y_n(x)=0$$

$$x\sum_{n=0}^{\infty}\varepsilon^n y_n'(x)+\sum_{n=0}^{\infty}\varepsilon^{n+1}[y_0(x)y_n'(x)+...+y_n(x)y_0'(x)]+\sum_{n=0}^{\infty}\varepsilon^n y_n(x)=0$$

$$xy_0'(x)+y_0(x)+\sum_{n=1}^{\infty}\varepsilon^n[xy_n'(x)+y_0(x)y_{n-1}'(x)+...+y_{n-1}(x)y_0'(x)+y_n(x)]=0.$$

Therefore, the zeroth-order problem $xy_0'(x)+y_0(x)=0$ is obtained by setting $\varepsilon=0$, and the solution which satisfies the initial condition $y_0(1)=1$ is

$$y_0(x)=\frac{1}{x}.$$

The nth-order problem ($n\geq 1$) is obtained by setting the coefficient of ε^n ($n\geq 1$) equal to 0. The result is

$$xy_n'(x)+y_0(x)y_{n-1}'(x)+...+y_{n-1}(x)y_0'(x)+y_n(x)=0,\ \ y_n(1)=0\ \ (n\geq 1). \tag{19}$$

It is clear from Eq.(19) that the first-order problem $xy_1'(x)+y_0(x)y_0'(x)+y_1(x)=0$, and the solution which satisfies the initial condition $y_1(1)=0$ is

$$y_1(x)=\frac{1}{2x}-\frac{1}{2x^3}.$$

Similarly, we obtain the second-order problem from Eq.(19) that $xy_2'(x)+y_0(x)y_1'(x)+y_1(x)y_0'(x)+y_2(x)=0$ with the initial condition $y_2(1)=0$. The solution to this initial value problem is easy to obtain

$$y_2(x)=\frac{1}{2x^5}-\frac{1}{2x^3}.$$

Let us now put $y_0(x),y_1(x),y_2(x)$ in Eq.(17) to obtain the three-term perturbation series approximation to $y(x)$:

$$y_{\text{perturb}}(x)=\frac{1}{x}+\left(\frac{1}{2x}-\frac{1}{2x^3}\right)\varepsilon+\left(\frac{1}{2x^5}-\frac{1}{2x^3}\right)\varepsilon^2+O(\varepsilon^3)\quad(\varepsilon\to 0). \tag{20}$$

As seen from the Eq.s (15) and (20), the exact solution and the perturbation series approximation to Eq.(12) are the same up to the second order of ε $(\varepsilon\to 0)$. This shows that PSM is the best approximation between PSM and ADM to the exact solution for this example. Therefore, it suffices to compare the exact solution with the Adomian decomposition series solution. It is clear that the perturbation series approximation obtained in (20) is non-uniform for small x. For this reason we take $0.8\leq x\leq 1.5$ to make a numerical comparison of the solutions. In figures 1, 2 and 3 we take $\varepsilon=0.03$, $\varepsilon=0.01$ and $\varepsilon=0.001$, respectively. As seen from the three figures we achieve a very good approximation to the exact solution as ε gets closer to 0. By using only 3 terms of the decomposition series (10), we have approximated to the exact solution as desired, which shows that the speed of convergence of the ADM is very fast. In conclusion, the ADM is easier to compute and supplies quantitatively reliable results, and the overall errors in the application of the ADM can be made very small by adding new terms to the series (10).

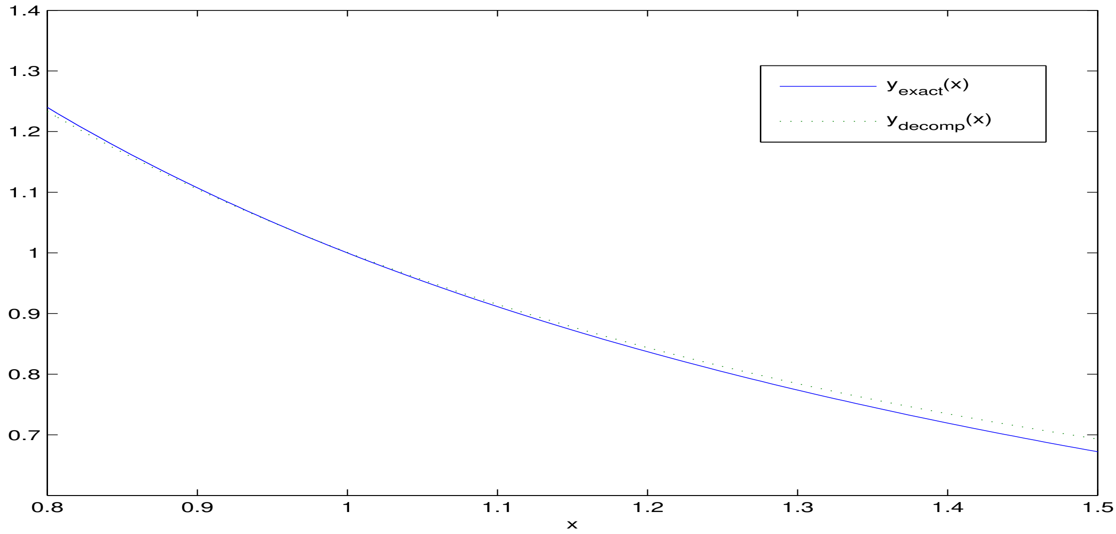

Fig. 1: Comparison of the Exact Sol. with the Decomposition Series Sol. ($\varepsilon = 0.03$).

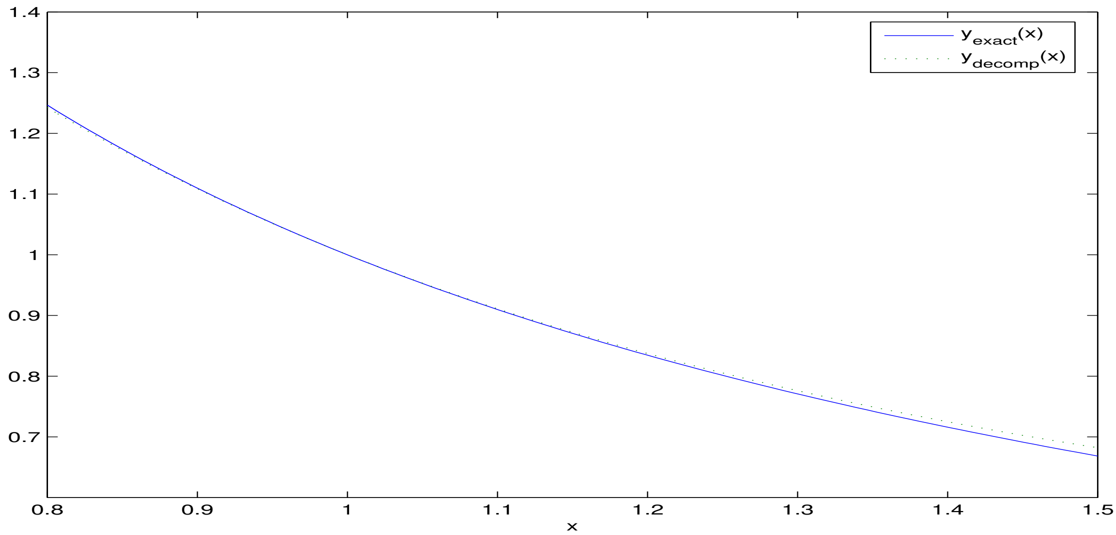

Fig. 2: Comparison of the Exact Sol. with the Decomposition Series Sol. ($\varepsilon = 0.01$).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

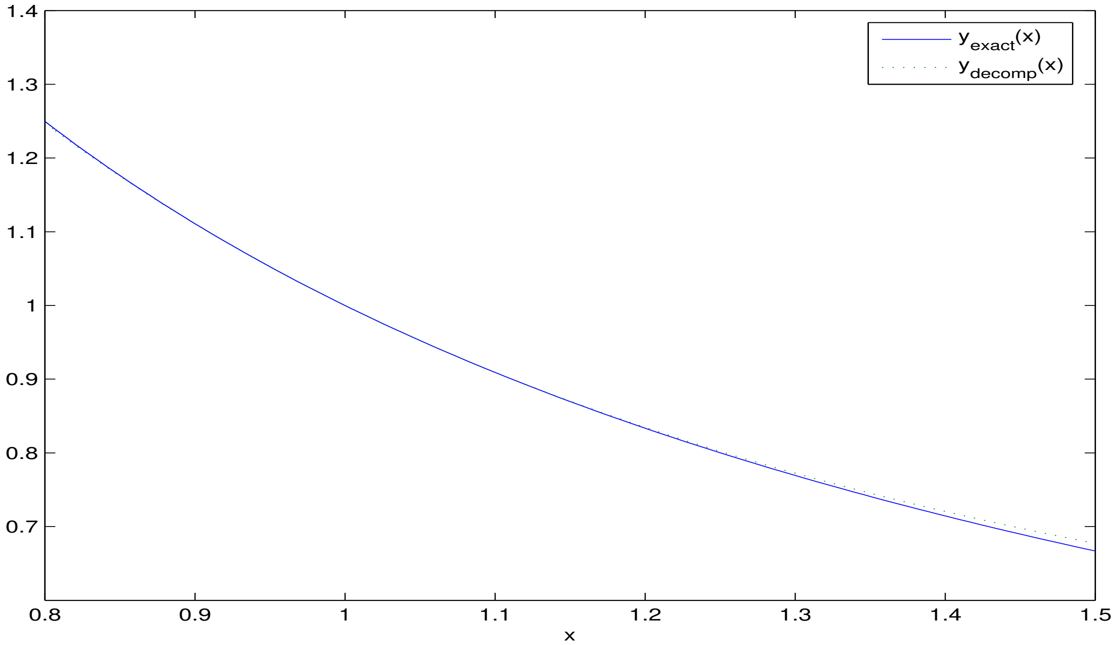

Fig. 3: Comparison of the Exact Sol. with the Decomposition Series Sol. ($\varepsilon = 0.001$).

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