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Note on the projectable linear connection in the semi-tangent bundle

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Abstract: The present paper is devoted to some results concerning with the projectable linear connection in the semi-tangent (pull-back) bundle tM. In this study, horizontal lift problems of projectable linear connection, which are preliminary to the subject of covarient derivates of almost contact structure and almost paracontact structure on semi-tangent bundle, are discussed.

Keywords: Horizontal lift, Projectable linear connection, Pull-back bundle, Semi-tangent bundle, Vector field.

1 Introduction

Let M_n be a differentiable manifold of class C^∞ and finite dimension n, and let (M_n, π_1, B_m) be a differentiable bundle over B_m. We use the notation $(x^i) = (x^a, x^\alpha)$, where the indices $i, j, \ldots$ run from 1 to n, the indices $a, b, \ldots$ from 1 to $n-m$ and the indices $\alpha, \beta, \ldots$ from $n-m+1$ to n, x^α are coordinates in B_m , x^a are fibre coordinates of the bundle

$$\pi_1 : M_n \to B_m.$$

Let now $(T(B_m), \widetilde{\pi}, B_m)$ be a tangent bundle [13] over base space B_m, and let M_n be differentiable bundle determined by a natural projection (submersion) $\pi_1 : M_n \to B_m$. The semi-tangent bundle (pull-back [[2],[3],[9], [10],[14],[15]]) of the tangent bundle $(T(B_m), \widetilde{\pi}, B_m)$ is the bundle $(t(B_m), \pi_2, M_n)$ over differentiable bundle M_n with a total space

$$t(B_m) = \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}} \right) \in M_n \times T_x(B_m) : \pi_1(x^a, x^\alpha) = \widetilde{\pi}\left(x^\alpha, x^{\overline{\alpha}}\right) = (x^\alpha) \right\} \subset M_n \times T_x(B_m)$$

and with the projection map $\pi_2 : t(B_m) \to M_n$ defined by $\pi_2(x^a, x^\alpha, x^{\overline{\alpha}}) = (x^a, x^\alpha)$, where $T_x(B_m)\,(x = \pi_1(\widetilde{x}), \widetilde{x} = (x^a, x^\alpha) \in M_n)$ is the tangent space at a point x of B_m, where $x^{\overline{\alpha}} = y^\alpha \left(\overline{\alpha}, \overline{\beta}, \ldots = n+1, \ldots, 2n\right)$ are fibre coordinates of the tangent bundle $T(B_m)$.

Where the pull-back (Pontryagin [7]) bundle $t(B_m)$ of the differentiable bundle M_n also has the natural bundle structure over B_m, its bundle projection $\pi : t(B_m) \to B_m$ being defined by $\pi : (x^a, x^\alpha, x^{\overline{\alpha}}) \to (x^\alpha)$, and hence $\pi = \pi_1 \circ \pi_2$. Thus $(t(B_m), \pi_1 \circ \pi_2)$ is the composite bundle [[8], p.9] or step-like bundle [6]. Consequently, we notice the semi-tangent bundle $(t(B_m), \pi_2\,)$ is a pull-back bundle of the tangent bundle over B_m by π_1 [9].

If $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another local adapted coordinates in differentiable bundle M_n, then we have

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right) \quad . \end{cases} \tag{1}$$

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The Jacobian of (1) has the components

$$\left(A_j^{i'}\right) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

where $A_b^{a'} = \frac{\partial x^{a'}}{\partial x^b}$, $A_\beta^{a'} = \frac{\partial x^{a'}}{\partial x^\beta}$, $A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}$ [9].

To a transformation (1) of local coordinates of M_n, there corresponds on $t(B_m)$ the change of coordinate

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}(x^\beta), \\ x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta. \end{cases} \tag{2}$$

The Jacobian of (2) is:

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon & A_\beta^{\alpha'} \end{pmatrix}, \tag{3}$$

where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $I, J, ... = 1, ..., 2n$; $A_{\beta\varepsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon}$ [9].

The purpose of this paper is to study the horizontal lifts of projectable linear connection to semi-tangent (pull-back) bundle $(t(B_m)\ \ , \pi_2)$ and their properties.

We denote by $\Im_q^p(M_n)$ the set of all tensor fields of class C^∞ and of type (p,q) on M_n, i.e., contravariant degree p and covariant degree q. We now put $\Im(M_n) = \sum_{p,q=0}^{\infty} \Im_q^p(M_n)$, which is the set of all tensor fields on M_n. Smilarly, we denote by $\Im_q^p(B_m)$ and $\Im(B_m)$ respectively the corresponding sets of tensor fields in the base space B_m.

2 Some lifts of vector and covector fields

If f is a function on B_m, we write ${}^{vv}f$ for the function on $t(B_m)$ obtained by forming the composition of $\pi : t(B_m) \to B_m$ and ${}^{v}f = f \circ \pi_1$, so that

$${}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi.$$

Thus, the vertical lift ${}^{vv}f$ of the function f to $t(B_m)$ satisfies

$${}^{vv}f(x^a, x^\alpha, x^{\overline{\alpha}}) = f(x^\alpha). \tag{4}$$

We note here that value ${}^{vv}f$ is constant along each fibre of $\pi : t(B_m) \to B_m$. Let $X \in \Im_0^1(B_m)$, i.e. $X = X^\alpha \partial_\alpha$. On putting

$${}^{vv}X = ({}^{vv}X^\alpha) = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{5}$$

from (3), we easily see that ${}^{vv}X' = \bar{A}({}^{vv}X)$. The vector field ${}^{vv}X$ is called the vertical lift of X to $t(B_m)$.

Let $\omega \in \Im_1^0(B_m)$, i.e. $\omega = \omega_\alpha dx^\alpha$. On putting

$$^{vv}\omega = \left(^{vv}\omega\right)_\alpha = (0, \omega_\alpha, 0), \tag{6}$$

from (3), we easily see that $^{vv}\omega = \bar{A}\,^{vv}\omega'$. The covector field $^{vv}\omega$ is called the vertical lift of ω to $t(B_m)$.

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [11] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. Now, consider $\widetilde{X} \in \Im_0^1(M_n)$, then $^{cc}\widetilde{X}$ (complete lift) has the components on the semi-tangent bundle $t(B_m)$ [9]

$$^{cc}\widetilde{X} = \left(^{cc}\widetilde{X}^\alpha\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{7}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

For any $F \in \Im_1^1(B_m)$, if we take account of (3), we can prove that $(\gamma F)' = \bar{A}(\gamma F)$, where γF is a vector field defined by

$$\gamma F = (\gamma F^I) = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} \tag{8}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.

Let now $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field on M_n with projection $X \in \Im_0^1(B_m)$ [11]. Then we define the horizontal lift $^{HH}\widetilde{X}$ of $\widetilde{X}$ by

$$^{HH}\widetilde{X} = {}^{cc}\widetilde{X} - \gamma(\nabla\widetilde{X})$$

on $t(M_n)$. Where ∇ is a projectable symmetric linear connection in a differentiable manifold B_m. Then, remembering that $^{cc}\widetilde{X}$ and $\gamma(\nabla\widetilde{X})$ have, respectively, local componenets

$$^{cc}\widetilde{X} = \left(^{cc}\widetilde{X}^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix}, \gamma(\nabla\widetilde{X}) = \left(\gamma(\nabla\widetilde{X})^I\right) = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of X^ε, i.e.,

$$(\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma_\beta^\varepsilon{}_\alpha.$$

We find that the horizontal lift $^{HH}\widetilde{X}$ of $\widetilde{X}$ has the components

$$^{HH}X = \left(^{HH}X^I\right) = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -\Gamma_\beta^\alpha X^\beta \end{pmatrix} \tag{9}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. Where

$$\Gamma_\beta^\alpha = y^\varepsilon \Gamma_\varepsilon^\alpha{}_\beta. \tag{10}$$

3 Complete lifts of projectable linear connection

Let $\Gamma_{\alpha\ \gamma}^{\beta}$ be components of projectable linear connection [[1], [4], [5], [11], [12]] ∇ with respect to local coordinates (x^{α}) in B_m and ${}^{cc}\Gamma_{I\ K}^{J}$ components of ${}^{cc}\nabla$ with respect to the induced coordinates $(x^{a}, x^{\alpha}, x^{\overline{\alpha}})$ in $t(B_m)$. We recall from [11] that components ${}^{cc}\Gamma_{I\ K}^{J}$ of complete lift ${}^{cc}\nabla$ of projectable linear connection ∇ can be calculated from base manifold B_m to semi-tangent bundle $t(B_m)$ also as:

$$\begin{cases} {}^{cc}\Gamma_{a\ c}^{b} = {}^{cc}\Gamma_{a\ \gamma}^{b} = {}^{cc}\Gamma_{a\ \overline{\gamma}}^{b} = {}^{cc}\Gamma_{\alpha\ c}^{b} = {}^{cc}\Gamma_{\alpha\ \overline{\gamma}}^{b} = {}^{cc}\Gamma_{\overline{\alpha}\ c}^{b} = {}^{cc}\Gamma_{\overline{\alpha}\ \gamma}^{b} = {}^{cc}\Gamma_{\overline{\alpha}\ \overline{\gamma}}^{b} = 0, \\ {}^{cc}\Gamma_{\alpha\ \gamma}^{b} = \Gamma_{\alpha\ \gamma}^{b}, \\ {}^{cc}\Gamma_{a\ c}^{\beta} = {}^{cc}\Gamma_{a\ \gamma}^{\beta} = {}^{cc}\Gamma_{a\ \overline{\gamma}}^{\beta} = {}^{cc}\Gamma_{\alpha\ c}^{\beta} = {}^{cc}\Gamma_{\alpha\ \overline{\gamma}}^{\beta} = {}^{cc}\Gamma_{\overline{\alpha}\ c}^{\beta} = {}^{cc}\Gamma_{\overline{\alpha}\ \gamma}^{\beta} = {}^{cc}\Gamma_{\overline{\alpha}\ \overline{\gamma}}^{\beta} = 0, \\ {}^{cc}\Gamma_{\alpha\ \gamma}^{\beta} = \Gamma_{\alpha\ \gamma}^{\beta}, \\ {}^{cc}\Gamma_{a\ c}^{\overline{\beta}} = {}^{cc}\Gamma_{a\ \gamma}^{\overline{\beta}} = {}^{cc}\Gamma_{a\ \overline{\gamma}}^{\overline{\beta}} = {}^{cc}\Gamma_{\alpha\ c}^{\overline{\beta}} = {}^{cc}\Gamma_{\overline{\alpha}\ c}^{\overline{\beta}} = {}^{cc}\Gamma_{\overline{\alpha}\ \overline{\gamma}}^{\overline{\beta}} = 0, \\ {}^{cc}\Gamma_{\alpha\ \overline{\gamma}}^{\overline{\beta}} = \Gamma_{\alpha\ \gamma}^{\beta}, \\ {}^{cc}\Gamma_{\alpha\ \gamma}^{\overline{\beta}} = y^{\varepsilon}\partial_{\varepsilon}\Gamma_{\alpha\ \gamma}^{\beta}, \\ {}^{cc}\Gamma_{\overline{\alpha}\ \gamma}^{\overline{\beta}} = \Gamma_{\alpha\ \gamma}^{\beta}. \end{cases} \tag{11}$$

where $I = (a, \alpha, \overline{\alpha})$, $J = (b, \beta, \overline{\beta})$, $K = (c, \gamma, \overline{\gamma})$. On the other hand, from (11) we obtain:

Theorem 1. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projection $X \in \Im_0^1(B_m)$ and $Y \in \Im_0^1(B_m)$, respectively. We have:*

(i) ${}^{cc}\nabla_{{}^{vv}X}({}^{vv}Y) = 0$,
(ii) ${}^{cc}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y}) = 0$,
(iii) ${}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y) = {}^{vv}(\nabla_X Y)$,
(iv) ${}^{cc}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) = {}^{HH}(\nabla_X Y) + \gamma(R(\ ,X)Y)$,
(v) $[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}] = {}^{cc}[\widetilde{X}, \widetilde{Y}] (i.e. L_{{}^{cc}\widetilde{X}}({}^{cc}\widetilde{Y}) = {}^{cc}\left(L_{\widetilde{X}}\widetilde{Y}\right))$,
(vi) $[{}^{cc}\widetilde{X}, \gamma F] = \gamma(L_X F)\ (F \in \Im_1^1(B_m))$,

where $R(\ ,X)Y \in \Im_1^1(B_m)$ is a tensor field of type of $(1,1)$ defined by $F(Z) = R(Z,X)Y$ for any $Z \in \Im_0^1(B_m)$ and L_X is the operator of Lie derivation with respect to X.

4 Horizontal lifts of projectable linear connection

Let there be given a projectable linear connection ∇ in B_m. We shall define the horizontal lift ${}^{HH}\nabla$ of a projectable linear connection ∇ in B_m to $t(B_m)$ by the conditions:

(i) ${}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y) = 0$,
(ii) ${}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y}) = 0$
(iii) ${}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y) = {}^{vv}(\nabla_X Y)$,

$$(iv)\ {}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{HH}\widetilde{Y}) = {}^{HH}(\nabla_X Y), \tag{12}$$

for any $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$. Thus, if we put

$$\widetilde{S}(\widetilde{X}, \widetilde{Y}) = {}^{HH}\nabla_{\widetilde{X}}\widetilde{Y} - {}^{cc}\nabla_{\widetilde{X}}\widetilde{Y} \tag{13}$$

for any $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$. Then, from (13) and Theorem 1, the tensor $\widetilde{S}$ of type $(1,2)$ in $t(B_m)$ satisfies the conditions

(i) $\widetilde{S}({}^{vv}X, {}^{vv}Y) = 0$,
(ii) $\widetilde{S}({}^{vv}X, {}^{HH}\widetilde{Y}) = 0$,
(iii) $\widetilde{S}({}^{HH}\widetilde{X}, {}^{vv}Y) = 0$,

$$(iv)\widetilde{S}(^{HH}\widetilde{X},^{HH}\widetilde{Y}) = -\gamma(R(\ ,X)Y), \tag{14}$$

for any $\widetilde{X},\widetilde{Y} \in \Im_0^1(M_n)$. Therefore $\widetilde{S}$ has the components $\widetilde{S}_{IK}^J$ such that

$$\widetilde{S}_{\alpha\gamma}^{\overline{\beta}} = -y^{\varepsilon} R_{\varepsilon\alpha\gamma}^{\beta} \tag{15}$$

all others being zero, with respect to the induced coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ in $t(B_m)$.

Since the components ${}^{cc}\Gamma_I{}^J{}_K$ of ${}^{cc}\nabla$ are given by (11), it follows from (13) and (15) that the horizontal lift ${}^{HH}\nabla$ of a projectable linear connection ∇ has the components ${}^{HH}\Gamma_I{}^J{}_K$ such that

$$\begin{cases} {}^{HH}\Gamma_a{}^b{}_c = {}^{HH}\Gamma_a{}^b{}_\gamma = {}^{HH}\Gamma_a{}^b{}_{\overline{\gamma}} = {}^{HH}\Gamma_\alpha{}^b{}_c = {}^{HH}\Gamma_\alpha{}^b{}_{\overline{\gamma}} = {}^{HH}\Gamma_{\overline{\alpha}}{}^b{}_c = {}^{HH}\Gamma_{\overline{\alpha}}{}^b{}_\gamma = {}^{HH}\Gamma_{\overline{\alpha}}{}^b{}_{\overline{\gamma}} = 0, \\ {}^{HH}\Gamma_\alpha{}^b{}_\gamma = \Gamma_\alpha{}^b{}_\gamma, \\ {}^{HH}\Gamma_a{}^\beta{}_c = {}^{HH}\Gamma_a{}^\beta{}_\gamma = {}^{HH}\Gamma_a{}^\beta{}_{\overline{\gamma}} = {}^{HH}\Gamma_\alpha{}^\beta{}_c = {}^{HH}\Gamma_\alpha{}^\beta{}_{\overline{\gamma}} = {}^{HH}\Gamma_{\overline{\alpha}}{}^\beta{}_c = {}^{HH}\Gamma_{\overline{\alpha}}{}^\beta{}_\gamma = {}^{HH}\Gamma_{\overline{\alpha}}{}^\beta{}_{\overline{\gamma}} = 0, \\ {}^{HH}\Gamma_\alpha{}^\beta{}_\gamma = \Gamma_\alpha{}^\beta{}_\gamma, \\ {}^{HH}\Gamma_a{}^{\overline{\beta}}{}_c = {}^{HH}\Gamma_a{}^{\overline{\beta}}{}_\gamma = {}^{HH}\Gamma_a{}^{\overline{\beta}}{}_{\overline{\gamma}} = {}^{HH}\Gamma_\alpha{}^{\overline{\beta}}{}_c = {}^{HH}\Gamma_{\overline{\alpha}}{}^{\overline{\beta}}{}_c = {}^{HH}\Gamma_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\gamma}} = 0, \\ {}^{HH}\Gamma_\alpha{}^{\overline{\beta}}{}_{\overline{\gamma}} = \Gamma_\alpha{}^\beta{}_\gamma, \\ {}^{HH}\Gamma_\alpha{}^{\overline{\beta}}{}_\gamma = y^\varepsilon \partial_\varepsilon \Gamma_\alpha{}^\beta{}_\gamma - y^\varepsilon R_{\varepsilon\alpha\gamma}^\beta, \\ {}^{HH}\Gamma_{\overline{\alpha}}{}^{\overline{\beta}}{}_\gamma = \Gamma_\alpha{}^\beta{}_\gamma. \end{cases} \tag{16}$$

with respect to the induced coordinates in $t(B_m)$. Where ${}^{HH}\Gamma_I{}^J{}_K$ are the components of ${}^{HH}\nabla$ in $t(B_m)$.

Proof. For convenience sake we only consider ${}^{HH}\Gamma_\alpha{}^{\overline{\beta}}{}_\gamma$. According to (11), (13) and (15), in fact:

$$\widetilde{S}_{\alpha\gamma}^{\overline{\beta}} = {}^{HH}\Gamma_\alpha{}^{\overline{\beta}}{}_\gamma - {}^{cc}\Gamma_\alpha{}^{\overline{\beta}}{}_\gamma$$
$$-y^\varepsilon R_{\varepsilon\alpha\gamma} = {}^{HH}\Gamma_\alpha{}^{\overline{\beta}}{}_\gamma - y^\varepsilon \partial_\varepsilon \Gamma_\alpha{}^\beta{}_\gamma$$
$${}^{HH}\Gamma_\alpha{}^{\overline{\beta}}{}_\gamma = y^\varepsilon \partial_\varepsilon \Gamma_\alpha{}^\beta{}_\gamma - y^\varepsilon R_{\varepsilon\alpha\gamma}^\beta.$$

Thus, we have ${}^{HH}\Gamma_\alpha{}^{\overline{\beta}}{}_\gamma = y^\varepsilon \partial_\varepsilon \Gamma_\alpha{}^\beta{}_\gamma - y^\varepsilon R_{\varepsilon\alpha\gamma}^\beta$. Similarly, we can easily find other components of ${}^{HH}\Gamma_I{}^J{}_K$.

Theorem 2. *Let $X,Y \in \Im_0^1(B_m)$. Then we obtain*

$${}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y) = 0.$$

Proof. If $X,Y \in \Im_0^1(B_m)$ and

$$\begin{pmatrix} \left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)\right)^b \\ \left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)\right)^\beta \\ \left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)\right)^{\overline{\beta}} \end{pmatrix}$$

are the components of $\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)\right)^J$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t(B_m)$, then we have

$$\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)\right)^J = {}^{vv}X^{a\,HH}\nabla_a({}^{vv}Y)^J + {}^{vv}X^{\alpha\,HH}\nabla_\alpha({}^{vv}Y)^J + {}^{vv}X^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^J.$$

Firstly, if $J=b$, we have

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)\right)^{b} &= {}^{vv}X^{a\,HH}\nabla_{a}\underbrace{{}^{vv}Y^{b}}_{0}+{}^{vv}X^{\alpha\,HH}\nabla_{\alpha}\underbrace{{}^{vv}Y^{b}}_{0}+{}^{vv}X^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}\underbrace{({}^{vv}Y^{b})}_{0}\\ &=0\end{aligned}$$

by virtue of (5) and (16). Secondly, if $J=\beta$, we have

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)\right)^{\beta} &= {}^{vv}X^{a\,HH}\nabla_{a}\underbrace{{}^{vv}Y^{\beta}}_{0}+{}^{vv}X^{\alpha\,HH}\nabla_{\alpha}\underbrace{{}^{vv}Y^{\beta}}_{0}+{}^{vv}X^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}\underbrace{({}^{vv}Y^{\beta})}_{0}\\ &=0\end{aligned}$$

by virtue of (5) and (16). Thirdly, if $J=\overline{\beta}$, then we have

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)\right)^{\overline{\beta}} &= \underbrace{{}^{vv}X^{a}}_{0}\,{}^{HH}\nabla_{a}({}^{vv}Y)^{\overline{\beta}}+\underbrace{{}^{vv}X^{\alpha}}_{0}\,{}^{HH}\nabla_{\alpha}({}^{vv}Y)^{\overline{\beta}}+{}^{vv}X^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^{\overline{\beta}}\\ &= X^{\alpha}(\underbrace{\partial_{\overline{\alpha}}Y^{\beta}}_{0}+{}^{HH}\Gamma_{\overline{\alpha}\,c}^{\beta}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma_{\overline{\alpha}\,\gamma}^{\beta}\underbrace{({}^{vv}Y)^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}\,\overline{\gamma}}^{\beta}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\ &=0\end{aligned}$$

by virtue of (5) and (16). Thus Theorem 2 is proved.

Theorem 3. *Let $\widetilde{Y}$ be a projectable vector field on M_n with projections Y on B_m. If $X\in\Im_0^1(B_m)$, then*

$$^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})=0.$$

Proof. If $\widetilde{Y}\in\Im_0^1(M_n)$, $X\in\Im_0^1(B_m)$ and

$$\begin{pmatrix}\left({}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{b}\\ \left({}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{\beta}\\ \left({}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{\overline{\beta}}\end{pmatrix}$$

are the components of $\left({}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{J}$ with respect to the coordinates $(x^b,x^\beta,x^{\overline{\beta}})$ on $t(B_m)$, then we have

$$\left({}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{J}={}^{vv}X^{a\,HH}\nabla_{a}({}^{HH}\widetilde{Y})^{J}+{}^{vv}X^{\alpha\,HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{J}+{}^{vv}X^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{J}.$$

Firstly, if $J=b$, we have

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{b} &= \underbrace{{}^{vv}X^{a}}_{0}\,{}^{HH}\nabla_{a}({}^{HH}\widetilde{Y})^{b}+\underbrace{{}^{vv}X^{\alpha}}_{0}\,{}^{HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{b}+\underbrace{{}^{vv}X^{\overline{\alpha}}}_{X^{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{b}\\ &= X^{\alpha}(\underbrace{\partial_{\overline{\alpha}}Y^{b}}_{0}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}\,c}^{b}}_{0}({}^{HH}\widetilde{Y})^{c}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}\,\gamma}^{b}}_{0}({}^{HH}\widetilde{Y})^{\gamma}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}\,\overline{\gamma}}^{b}}_{0}({}^{HH}\widetilde{Y})^{\overline{\gamma}})\\ &=0\end{aligned}$$

by virtue of (5), (9) and (16). Secondly, if $J=\beta$, we have

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{\beta} &= \underbrace{{}^{vv}X^{a}}_{0}\,{}^{HH}\nabla_{a}({}^{HH}\widetilde{Y})^{\beta}+\underbrace{{}^{vv}X^{\alpha}}_{0}\,{}^{HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{\beta}+\underbrace{{}^{vv}X^{\overline{\alpha}}}_{X^{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{\beta}\\ &= X^{\alpha}(\underbrace{\partial_{\overline{\alpha}}Y^{\beta}}_{0}+\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\,c}}_{0}({}^{HH}\widetilde{Y})^{c}+\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\,\gamma}}_{0}({}^{HH}\widetilde{Y})^{\gamma}+\underbrace{{}^{HH}\Gamma^{\beta}_{\overline{\alpha}\,\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y})^{\overline{\gamma}})\\ &=0\end{aligned}$$

by virtue of (5), (9) and (16). Thirdly, if $J=\overline{\beta}$, then we have

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})\right)^{\overline{\beta}} &= \underbrace{{}^{vv}X^{a}}_{0}\,{}^{HH}\nabla_{a}({}^{HH}\widetilde{Y})^{\overline{\beta}}+\underbrace{{}^{vv}X^{\alpha}}_{0}\,{}^{HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{\overline{\beta}}+\underbrace{{}^{vv}X^{\overline{\alpha}}}_{X^{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{\overline{\beta}}\\ &= X^{\alpha}(-\underbrace{\partial_{\overline{\alpha}}y^{\varepsilon}}_{\delta^{\varepsilon}_{\alpha}}\Gamma^{\beta}_{\varepsilon\,\gamma}Y^{\gamma}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\,c}}_{0}({}^{HH}\widetilde{Y})^{c}+{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\,\gamma}({}^{HH}\widetilde{Y})^{\gamma}+\underbrace{{}^{HH}\Gamma^{\overline{\beta}}_{\overline{\alpha}\,\overline{\gamma}}}_{0}({}^{HH}\widetilde{Y})^{\overline{\gamma}})\\ &= -X^{\alpha}\Gamma^{\beta}_{\alpha\,\gamma}Y^{\gamma}+X^{\alpha}\Gamma^{\beta}_{\alpha\,\gamma}Y^{\gamma}\\ &=0\end{aligned}$$

by virtue of (5), (9) and (16). The proof is completed.

Theorem 4. *Let $\widetilde{X}$ and $\widetilde{Y}$ be projectable vector fields on M_n with projection $X\in\Im^1_0(B_m)$ and $Y\in\Im^1_0(B_m)$, respectively. We have:*

$$ {}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})={}^{HH}(\nabla_{X}Y)\,. $$

Proof. (i) If $\widetilde{X},\widetilde{Y}\in\Im^1_0(M_n)$ and

$$\begin{pmatrix}\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{b}\\ \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{\beta}\\ \left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{\overline{\beta}}\end{pmatrix}$$

are the components of $\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{J}$ with respect to the coordinates $(x^{b},x^{\beta},x^{\overline{\beta}})$ on $t(B_m)$, then we have

$$\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{J}={}^{cc}\widetilde{X}^{a}\,{}^{HH}\nabla_{a}({}^{HH}\widetilde{Y})^{J}+{}^{cc}\widetilde{X}^{\alpha}\,{}^{HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{J}+{}^{cc}\widetilde{X}^{\overline{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{J}.$$

Firstly, if $J=b$, we have

$$\begin{aligned}\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{b} &= {}^{cc}\widetilde{X}^{a}\,{}^{HH}\nabla_{a}({}^{HH}\widetilde{Y})^{b}+{}^{cc}\widetilde{X}^{\alpha}\,{}^{HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{b}+{}^{cc}\widetilde{X}^{\overline{\alpha}}\,{}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{b}\\ &= X^{a}\,{}^{HH}\nabla_{a}({}^{HH}\widetilde{Y})^{b}+X^{\alpha}\,{}^{HH}\nabla_{a}({}^{HH}\widetilde{Y})^{b}+(y^{\varepsilon}\partial_{\varepsilon}X^{\alpha})\,{}^{HH}\nabla_{a}({}^{HH}\widetilde{Y})^{b}\\ &= X^{a}(\partial_{a}Y^{b}+\underbrace{{}^{HH}\Gamma^{b}_{a\,c}}_{0}Y^{c}+\underbrace{{}^{HH}\Gamma^{b}_{a\,\gamma}}_{0}Y^{\gamma}+\underbrace{{}^{HH}\Gamma^{b}_{a\,\overline{\gamma}}}_{0}Y^{\overline{\gamma}})+X^{\alpha}(\partial_{\alpha}Y^{b}+\underbrace{{}^{HH}\Gamma^{b}_{\alpha\,c}}_{0}Y^{c}+\underbrace{{}^{HH}\Gamma^{b}_{\alpha\,\gamma}}_{\Gamma^{b}_{\alpha\,\gamma}}Y^{\gamma}+\underbrace{{}^{HH}\Gamma^{b}_{\alpha\,\overline{\gamma}}}_{0}Y^{\overline{\gamma}})\\ &\quad+(y^{\varepsilon}\partial_{\varepsilon}X^{\alpha})\,(\underbrace{\partial_{\overline{\alpha}}Y^{b}}_{0}+\underbrace{{}^{HH}\Gamma^{b}_{\overline{\alpha}\,c}}_{0}Y^{c}+\underbrace{{}^{HH}\Gamma^{b}_{\overline{\alpha}\,\gamma}}_{0}Y^{\gamma}+\underbrace{{}^{HH}\Gamma^{b}_{\overline{\alpha}\,\overline{\gamma}}}_{0}Y^{\overline{\gamma}})\\ &= X^{\alpha}\partial_{\alpha}Y^{b}+X^{\alpha}\Gamma^{b}_{\alpha\,\gamma}Y^{\gamma}=X^{\alpha}\left(\partial_{\alpha}Y^{b}+\Gamma^{b}_{\alpha\,\gamma}Y^{\gamma}\right)\end{aligned}$$

by virtue of (7), (9) and (16). Secondly, if $J=\beta$, we have

$$\begin{aligned}
\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})\right)^{\beta} &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_{a}({}^{HH}\widetilde{Y})^{\beta}+{}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{\beta}+{}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{\beta}\\
&= X^{a\,HH}\nabla_{a}({}^{HH}\widetilde{Y})^{\beta}+X^{\alpha\,HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{\beta}+(y^{\varepsilon}\partial_{\varepsilon}X^{\alpha})\,{}^{HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{\beta}\\
&= X^{a}(\underbrace{\partial_{a}Y^{\beta}}_{0}+\underbrace{{}^{HH}\Gamma_{a\,c}^{\beta}}_{0}Y^{c}+\underbrace{{}^{HH}\Gamma_{a\,\gamma}^{\beta}}_{0}Y^{\gamma}+\underbrace{{}^{HH}\Gamma_{a\,\overline{\gamma}}^{\beta}}_{0}Y^{\overline{\gamma}})+X^{\alpha}(\partial_{\alpha}Y^{\beta}+\underbrace{{}^{HH}\Gamma_{\alpha\,c}^{\beta}}_{0}Y^{c}+\underbrace{{}^{HH}\Gamma_{\alpha\,\gamma}^{\beta}}_{\Gamma_{\alpha\,\gamma}^{\beta}}Y^{\gamma}+\underbrace{{}^{HH}\Gamma_{\alpha\,\overline{\gamma}}^{\beta}}_{0}Y^{\overline{\gamma}})\\
&\quad+(y^{\varepsilon}\partial_{\varepsilon}X^{\alpha})\,(\underbrace{\partial_{\overline{\alpha}}Y^{\beta}}_{0}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}\,c}^{\beta}}_{0}Y^{c}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}\,\gamma}^{\beta}}_{0}Y^{\gamma}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}\,\overline{\gamma}}^{\beta}}_{0}Y^{\overline{\gamma}})\\
&= X^{\alpha}\partial_{\alpha}Y^{\beta}+X^{\alpha}\Gamma_{\alpha\,\gamma}^{\beta}Y^{\gamma}=X^{\alpha}\left(\partial_{\alpha}Y^{\beta}+\Gamma_{\alpha\,\gamma}^{\beta}Y^{\gamma}\right)
\end{aligned}$$

by virtue of (7), (9) and (16). Thirdly, if $J=\overline{\beta}$, then we have

$$\begin{aligned}
\left({}^{HH}\nabla_{{}^{cc}\widetilde{X}}^{HH}\widetilde{Y}\right)^{\overline{\beta}} &= {}^{cc}\widetilde{X}^{a\,HH}\nabla_{a}({}^{HH}\widetilde{Y})^{\overline{\beta}}+{}^{cc}\widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{HH}\widetilde{Y})^{\overline{\beta}}+{}^{cc}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{HH}\widetilde{Y})^{\overline{\beta}}\\
&= X^{a\,HH}\nabla_{a}\left(-y^{\varepsilon}\Gamma_{\varepsilon\,\sigma}^{\beta}Y^{\sigma}\right)+X^{\alpha\,HH}\nabla_{\alpha}\left(-y^{\varepsilon}\Gamma_{\varepsilon\,\sigma}^{\beta}Y^{\sigma}\right)+(y^{\varepsilon}\partial_{\varepsilon}X^{\alpha})\,{}^{HH}\nabla_{\overline{\alpha}}\left(-y^{\varepsilon}\Gamma_{\varepsilon\,\sigma}^{\beta}Y^{\sigma}\right)\\
&= -X^{a}\underbrace{\partial_{a}\Gamma_{\varepsilon\,\sigma}^{\beta}}_{0}y^{\varepsilon}Y^{\sigma}-X^{a}\underbrace{\partial_{a}y^{\varepsilon}}_{0}\Gamma_{\varepsilon\,\sigma}^{\beta}Y^{\sigma}-X^{a}\Gamma_{\varepsilon\,\sigma}^{\beta}y^{\varepsilon}\underbrace{\partial_{a}Y^{\sigma}}_{0}-X^{\alpha}\partial_{\alpha}\Gamma_{\varepsilon\,\sigma}^{\beta}y^{\varepsilon}Y^{\sigma}\\
&\quad-X^{\alpha}\underbrace{\partial_{\alpha}y^{\varepsilon}}_{0}\Gamma_{\varepsilon\,\sigma}^{\beta}Y^{\sigma}-X^{\alpha}\Gamma_{\varepsilon\,\sigma}^{\beta}y^{\varepsilon}\partial_{\alpha}Y^{\sigma}+X^{\alpha}y^{\varepsilon}\partial_{\varepsilon}\Gamma_{\alpha\,\sigma}^{\beta}Y^{\sigma}-X^{\alpha}y^{\varphi}\partial_{\varphi}\Gamma_{\alpha\,\sigma}^{\beta}Y^{\sigma}+X^{\alpha}y^{\varphi}\partial_{\alpha}\Gamma_{\varphi\,\sigma}^{\beta}Y^{\sigma}\\
&\quad-X^{\alpha}y^{\varphi}\Gamma_{\varphi\,\phi}^{\beta}\Gamma_{\alpha\,\sigma}^{\phi}Y^{\sigma}+X^{\alpha}y^{\varphi}\Gamma_{\alpha\,\phi}^{\beta}\Gamma_{\varphi\,\sigma}^{\phi}Y^{\sigma}-X^{\alpha}\Gamma_{\alpha\,\sigma}^{\beta}\Gamma_{\varepsilon\,\phi}^{\sigma}y^{\varepsilon}Y^{\phi}-\Gamma_{\varepsilon\,\sigma}^{\beta}y^{\varepsilon}X^{\alpha}\partial_{\alpha}Y^{\sigma}+\Gamma_{\varepsilon\,\sigma}^{\beta}y^{\varepsilon}X^{\alpha}\partial_{\alpha}Y^{\sigma}\\
&= -\Gamma_{\varepsilon\,\sigma}^{\beta}y^{\varepsilon}X^{\alpha}\partial_{\alpha}Y^{\sigma}+\Gamma_{\varphi\,\phi}^{\beta}\Gamma_{\alpha\,\sigma}^{\phi}X^{\alpha}y^{\varphi}Y^{\sigma}
\end{aligned}$$

by virtue of (7), (9) and (16). Thus, we have ${}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})={}^{HH}(\nabla_{X}Y)$.

Theorem 5. *Let $\widetilde{X}$ be a projectable vector field on M_n with projections X on B_m. If $Y\in\Im_0^1(B_m)$, then*

$$^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)={}^{vv}(\nabla_{X}Y).$$

Proof. If $\widetilde{X}\in\Im_0^1(M_n)$, $Y\in\Im_0^1(B_m)$ and $\begin{pmatrix}\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{b}\\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{\beta}\\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{\overline{\beta}}\end{pmatrix}$ are the components of $\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{J}$ with respect to the coordinates $(x^{b},x^{\beta},x^{\overline{\beta}})$ on $t(B_m)$, then we have

$$\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{J}={}^{HH}\widetilde{X}^{a\,HH}\nabla_{a}({}^{vv}Y)^{J}+{}^{HH}\widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{vv}Y)^{J}+{}^{HH}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^{J}.$$

Firstly, if $J=b$, we have

$$\begin{aligned}
\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{b} &= {}^{HH}\widetilde{X}^{a\,HH}\nabla_{a}({}^{vv}Y)^{b}+{}^{HH}\widetilde{X}^{\alpha\,HH}\nabla_{\alpha}({}^{vv}Y)^{b}+{}^{HH}\widetilde{X}^{\overline{\alpha}\,HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^{b}\\
&= X^{a}(\partial_{a}\underbrace{{}^{vv}Y^{b}}_{0}+{}^{HH}\Gamma_{a\,c}^{b}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma_{a\,\gamma}^{b}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{a\,\overline{\gamma}}^{b}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\
&\quad+X^{\alpha}(\partial_{\alpha}\underbrace{{}^{vv}Y^{b}}_{0}+{}^{HH}\Gamma_{\alpha\,c}^{b}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma_{\alpha\,\gamma}^{b}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{\alpha\,\overline{\gamma}}^{b}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\
&\quad+{}^{HH}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}\underbrace{{}^{vv}Y^{b}}_{0}+{}^{HH}\Gamma_{\overline{\alpha}\,c}^{b}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma_{\overline{\alpha}\,\gamma}^{b}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}\,\overline{\gamma}}^{b}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\
&= 0
\end{aligned}$$

by virtue of (5), (9) and (16). Secondly, if $J=\beta$, we have

$$\begin{aligned}
\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{\beta} &= {}^{HH}\widetilde{X}^{a}\underbrace{{}^{HH}\nabla_{a}({}^{vv}Y)^{\beta}}_{0}+{}^{HH}\widetilde{X}^{\alpha}\underbrace{{}^{HH}\nabla_{\alpha}({}^{vv}Y)^{\beta}}_{0}+{}^{HH}\widetilde{X}^{\overline{\alpha}}\underbrace{{}^{HH}\nabla_{\overline{\alpha}}({}^{vv}Y)^{\beta}}_{0}\\
&= X^{a}(\partial_{a}\underbrace{{}^{vv}Y^{\beta}}_{0}+{}^{HH}\Gamma_{a}{}^{\beta}{}_{c}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma_{a}{}^{\beta}{}_{\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{a}{}^{\beta}{}_{\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\
&\quad +X^{\alpha}(\partial_{\alpha}\underbrace{{}^{vv}Y^{\beta}}_{0}+{}^{HH}\Gamma_{\alpha}{}^{\beta}{}_{c}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma_{\alpha}{}^{\beta}{}_{\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{\alpha}{}^{\beta}{}_{\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\
&\quad +{}^{HH}\widetilde{X}^{\overline{\alpha}}(\partial_{\overline{\alpha}}\underbrace{{}^{vv}Y^{\beta}}_{0}+{}^{HH}\Gamma_{\overline{\alpha}}{}^{\beta}{}_{c}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma_{\overline{\alpha}}{}^{\beta}{}_{\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}}{}^{\beta}{}_{\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\
&= 0
\end{aligned}$$

by virtue of (5), (9) and (16). Thirdly, if $J=\overline{\beta}$, then we have

$$\begin{aligned}
\left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)\right)^{\overline{\beta}} &= {}^{HH}\widetilde{X}^{a HH}\nabla_{a}\underbrace{{}^{vv}Y^{\overline{\beta}}}_{0}+{}^{HH}\widetilde{X}^{\alpha HH}\nabla_{\alpha}\underbrace{{}^{vv}Y^{\overline{\beta}}}_{0}+{}^{HH}\widetilde{X}^{\overline{\alpha} HH}\nabla_{\overline{\alpha}}\underbrace{{}^{vv}Y^{\overline{\beta}}}_{0}\\
&= X^{a}(\underbrace{\partial_{a}({}^{vv}Y)^{\overline{\beta}}}_{0}+{}^{HH}\Gamma_{a}{}^{\overline{\beta}}{}_{c}({}^{vv}Y)^{c}+\underbrace{{}^{HH}\Gamma_{a}{}^{\overline{\beta}}{}_{\gamma}}_{0}({}^{vv}Y)^{\gamma}+\underbrace{{}^{HH}\Gamma_{a}{}^{\overline{\beta}}{}_{\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\
&\quad +X^{\alpha}(\partial_{\alpha}({}^{vv}Y)^{\overline{\beta}}+{}^{HH}\Gamma_{\alpha}{}^{\overline{\beta}}{}_{c}({}^{vv}Y)^{c}+{}^{HH}\Gamma_{\alpha}{}^{\overline{\beta}}{}_{\gamma}\underbrace{({}^{vv}Y)^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{\alpha}{}^{\overline{\beta}}{}_{\overline{\gamma}}}_{\Gamma_{\alpha}{}^{\beta}{}_{\gamma}}({}^{vv}Y)^{\overline{\gamma}})\\
&\quad +\left(-y^{\varepsilon}\Gamma_{\varepsilon}{}^{\alpha}{}_{\beta}Y^{\beta}\right)(\underbrace{\partial_{\overline{\alpha}}Y^{\beta}}_{0}+{}^{HH}\Gamma_{\overline{\alpha}}{}^{\overline{\beta}}{}_{c}\underbrace{{}^{vv}Y^{c}}_{0}+{}^{HH}\Gamma_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\gamma}\underbrace{{}^{vv}Y^{\gamma}}_{0}+\underbrace{{}^{HH}\Gamma_{\overline{\alpha}}{}^{\overline{\beta}}{}_{\overline{\gamma}}}_{0}({}^{vv}Y)^{\overline{\gamma}})\\
&= X^{\alpha}\partial_{\alpha}Y^{\beta}+X^{\alpha}\Gamma_{\alpha}{}^{\beta}{}_{\gamma}Y^{\gamma}=X^{\alpha}\left(\partial_{\alpha}Y^{\beta}+\Gamma_{\alpha}{}^{\beta}{}_{\gamma}Y^{\gamma}\right)\\
&= (\nabla_{X}Y)^{\beta}
\end{aligned}$$

by virtue of (5), (9) and (16). On the other hand, we know that ${}^{vv}(\nabla_{X}Y)$ have the components

$$ {}^{vv}(\nabla_{X}Y)=\begin{pmatrix}0\\0\\(\nabla_{X}Y)^{\beta}\end{pmatrix}$$

with respect to the coordinates $(x^{b},x^{\beta},x^{\overline{\beta}})$ on $t(B_m)$. Thus, we have ${}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)={}^{vv}(\nabla_{X}Y)$ in $t(B_m)$.

5 Conclusion

In this paper, we consider horizontal lifting problem of projectable linear connection on M to the semi-tangent bundle tM. In this context, the following equations have been obtained:

(i) ${}^{HH}\nabla_{{}^{vv}X}({}^{vv}Y)=0,$
(ii) ${}^{HH}\nabla_{{}^{vv}X}({}^{HH}\widetilde{Y})=0,$
(iii) ${}^{HH}\nabla_{{}^{cc}\widetilde{X}}({}^{HH}\widetilde{Y})={}^{HH}(\nabla_{X}Y),$
(iv) ${}^{HH}\nabla_{{}^{HH}\widetilde{X}}({}^{vv}Y)={}^{vv}(\nabla_{X}Y).$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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