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Numerical study of fractional quadratic Riccati differential equation using Padé approximation

Veyis Turut

Department of Mathematics, Faculty of Arts and Sciences, Batman University, Batman,Turkey

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Abstract: In this paper, univariate Padé approximation is applied to fractional power series solutions of fractional quadratic Riccati differential equation.. As it is seen from the tables, univariate Padé approximation gives reliable solutions and numerical results.

Keywords: Fractional Riccati differential equation, Univariate Padé approximation, Caputo sense.

1 Introduction

It is well konown that Fractional differential equations(FDEs) have a three-hundred years history, but its applications return to the last two decades. Applications of Fractional differential equations (FDEs) can be seen in many fields such as wave propagation, economics, electricity, control, viscoelasticity, porous media, telecommunication lines, electromagnetic, biology, chemistry, etc.

Riccati differential equation is important in the optimal control systems. Sub-equation method [1], the Legendre wavelet method [2], the modified Homotopy perturbation method [3], the Homotopy analysis method [4, 5], the new Homotopy perturbation method [6, 7], the Haar wavelet operational matrix method [8], Enhanced Homotopy perturbation method [9], the Adomian decomposition method [10], the variational iteration method [11], Adams-Bashforth-Moulton method [12], Homotopy perturbation method [13], are some of the numerical methods which has been used to solve Fractional Riccati differential equations (FDEs).

The univariate multivariate Padé approximation have been proved by many authors to be a powerful mathematical tool for addressing various kinds of linear and nonlinear problems [14-20]. The reliability of the method and the reduction in the burden of computational work gives this method wider application. More details about definitions and theorems about Padé approximations can be found in [21,22].

In this paper univariate Padé approximation was applied on the frractional power series solutions of fractional quadratic Riccati differential equation of the form [3]

$$\frac{d^{\alpha}y}{dx^{\alpha}} = A(t) + B(t)y + C(t)y^2, \quad t > 0 \; m-1 < \alpha \leq m \tag{1}$$

subject to the initial conditions

$$y^{(j)}(0) = c_j, \quad j = 0,1,\ldots,m-1, \tag{2}$$

* Corresponding author e-mail: veyisturut@gmail.com

where where $A(t)$, $B(t)$ and $C(t)$ are given functions, $c_j, j = 0, 1, \ldots, m$, are arbitrary constants and α is a parameter describing the order of the fractional derivative [3].

The paper is organized as follows. We began by introducing some part of modified homotopy perturbation method (MHPM) that constructed and used by Odibat and Momani [3] in section 2. Then the univariate Padé approximation is presented which are required for establishing our results in section 3. In Section 4 the applications of the Padé approximation are presented to construct approximate solutions to fractional quadratic Riccati differential equations with initial conditions. In Section 4 two examples present to demonstrate the efficiency of the method. Concluding remarks are given in the last section.

2 The modified homotopy perturbation method (MHPM)

In this section, the algorithm of the new modification of the homotopy perturbation method that constructed by Odibat and Momani in [3] is presented . To illustrate the basic ideas of the new modification, Odibat and Momani in [3] considered the following nonlinear differential equation of fractional order:

$$D_*^{\alpha} u(t) + L(u(t)) + N(u(t)) = f(t), \quad t > 0, \; m-1 < \alpha \leq m, \tag{3}$$

where L is a linear operator which might include other fractional derivatives of order less than α, N is a nonlinear operatör which also might include other fractional derivatives of order less than α, f is a known analytic function and D_*^{α} is the Caputo fractional derivative of order a, subject to the initial conditions

$$u^k(0) = c_k, \; k = 0, 1, 2, \ldots, m-1. \tag{4}$$

Odibat and Momani in [3] constructed the following equations by using the definitions and theorems of the homotopy perturbation method:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots \tag{5}$$

$$L(u_0 + pu_1 + p^2 u_2 + \cdots) = L_0(u_0) + pL_1(u_0, u_1) + p^2 L_2(u_0, u_1, u_2) + \cdots \tag{6}$$

$$N(u_0 + pu_1 + p^2 u_2 + \cdots) = N_0(u_0) + pN_1(u_0, u_1) + p^2 N_2(u_0, u_1, u_2) + \cdots \tag{7}$$

More details about the construction of above equations can be seen in [3].

3 Univariate Padé approximation

Consider a formal power series

$$f(x) = c_0 + c_1 c + c_2 x^2 + \cdots \tag{8}$$

with $(c_0 \neq 0)$ [21]. The Padé approximation problem of order (m, n) or $[m, n]$ for f consists in finding polynomials

$$p(x) = \sum_{i=0}^{i} a_2 x^i, \; q(x) = \sum_{i=0}^{a} b_2 x^{\frac{z}{x}} \tag{9}$$

such that in the power series $(f-p)(x)$ [21]. To find the coefficients we get following linear systems of equations

$$\begin{cases} c_0 b_0 = 0 \\ c_1 b_0 + c_0 b_1 = a_1 \\ \vdots \\ c_m b_0 + c_{m-1} b_1 + \cdots + c_{m-n} b_n = a_m \end{cases} \tag{10}$$

$$\begin{cases} c_{m+1} b_0 + c_m b_1 + \ldots + c_{m-n+1} b_n = a_m \\ \vdots \\ c_{m+n} b_0 + c_{m-n+1} b_1 + \ldots + c_m b_n = 0 \end{cases} \tag{11}$$

with $c_i = 0$ for $i < 0$ [21].

In general a solution for the coefficients a_i is known after substitution of a solution for the b_i in the left hand side of (10). So the crucial point is to solve the homogeneous system of n equations (11) in the $n+1$ unknowns b_i. This system has at least one nontrivial solution because one of the unknowns can be chosen freely [21].

In short, by solving the equations (10) and (11) the coefficients a_i and b_i are found. Then the Padé equations (9) are found . After finding these polynomials we get The Padé approximation of order (m,n) or $[m,n]$ for f.

4 Applications and results

In this section univariate Padé series solutions of fractional Riccati differential equations with initial conditions shall be illustrated by two examples. The full modified homotopy perturbation method solutions of examples can be seen in [3].

Example 1. Consider the following fractional Riccati equation [3]:

$$\frac{d^\alpha u}{dt^\alpha} = -u^2(t) + 1, \;\; t > 0, \tag{12}$$

where $0 < \alpha \le 1$ subject to the initial condition

$$u(0) = 0. \tag{13}$$

The exact solution for equation (12) is given as $u_1(t) = \frac{e^{2t}-1}{e^{2t}+1}$ in [3]. Odibat and Momani obtained following solution in [3] by applying modified homotopy perturbation method on (9) and (10)

$$\begin{aligned} u_1(t) =& 4t - \frac{10t^3}{3} + \frac{4t^5}{5} - \frac{17t^7}{315} - 6\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \left[\frac{6}{\Gamma(3-\alpha)} + \frac{6}{\Gamma(4-\alpha)} + \frac{4\Gamma(5-\alpha)}{\Gamma(4-\alpha)^2}\right]\frac{\Gamma(4-\alpha)}{\Gamma(5-\alpha)} t^{4-\alpha} \\ & - \left[\frac{2}{3\Gamma(3-\alpha)} + \frac{4}{\Gamma(4-\alpha)} + \frac{16}{\Gamma(6-\alpha)}\right]\frac{\Gamma(6-\alpha)}{\Gamma(7-\alpha)} t^{6-\alpha} \\ & - \left[\frac{1}{\Gamma(3-\alpha)^2} + \frac{2}{\Gamma(4-2\alpha)} + \frac{2\Gamma(5-\alpha)}{\Gamma(4-\alpha)\Gamma(5-2\alpha)}\right]\frac{\Gamma(5-2\alpha)}{\Gamma(6-2\alpha)} t^{5-2\alpha} + 4\frac{t^{3-2\alpha}}{\Gamma(4-2\alpha)} - \frac{t^{4-3\alpha}}{\Gamma(5-3\alpha)} \end{aligned} \tag{14}$$

Table 1: Numerical Values for exact solution $u_1(t)$ and padé approximations of $u_1(t)$ for $\alpha = 1$.

t	$r_{3,3}(t)$	$r_{2,2}(t)$	$r_{2,4}(t)$	$u_1(t)$
0.001	0.009995666679	0.009995668547	0.009995666689	0.009999666667
0.002	0.01996533376	0.01996539332	0.01996533387	0.01999733367
0.003	0.02988300325	0.02988345453	0.02988300499	0.02999100345
0.004	0.03972268031	0.03972457628	0.03972269338	0.03997868046
0.005	0.04945837500	0.04946413852	0.04945843699	0.04995837492
0.006	0.05906410368	0.05907837731	0.05906432470	0.05992810372
0.007	0.06851389074	0.06854457031	0.06851453728	0.06988589038
0.008	0.07778177016	0.07784120397	0.07778340559	0.07778176911
0.009	0.08684178710	0.08694812095	0.08684548857	0.08975778470
0.1	0.09566799962	0.09584664540	0.09567567230	0.09966799456

Table 2: Numerical Values for exact solution $u_1(t)$ and padé approximations of $u_1(t)$ for $\alpha = 1$.

t	$r_{5,2}(t)$	$r_{4,2}(t)$	$r_{3,4}(t)$	$u_1(t)$
0.001	0.009995666682	0.009995666676	0.009995666683	0.009999666667
0.002	0.01996533375	0.01996533377	0.01996533376	0.01999733367
0.003	0.02988300323	0.02988300324	0.02988300324	0.02999100345
0.004	0.03972268032	0.03972268034	0.03972268035	0.03997868046
0.005	0.04945837494	0.04945837498	0.04945837492	0.04995837492
0.006	0.05906410352	0.05906410366	0.05906410351	0.05992810372
0.007	0.06851389034	0.06851389072	0.06851389029	0.06988589038
0.008	0.07778176912	0.07778177014	0.07778176913	0.07778176911
0.009	0.08684178478	0.08684178716	0.08684178470	0.08975778470
0.1	0.09566799463	0.09566799959	0.09566799458	0.09966799456

By applying equations (10) and (11) to put equation (14) into Padé series, following Padé equations respectively $r_{3,3}(t)$, $r_{2,2}(t)$, $r_{2,4}(t)$, $r_{5,2}(t)$,$r_{4,2}(t)$ and $r_{3,4}(t)$ were obtained for $\alpha = 1$and different values of m and n;

$$r_{3,3}(t) = \frac{t - 4.3025641042t^3}{1.000000000 + 0.03076923075t^2} \tag{15}$$

$$r_{2,2}(t) = \frac{1.000000000t}{1.000000000 + 4.333333332t^2} \tag{16}$$

$$r_{2,4}(t) = \frac{1.000000000t}{1 + 4.333333333t^2 + 18.64444444t^4} \tag{17}$$

$$r_{5,2}(t) = \frac{t - 3.928571428t^3 - 1.620634923t^5}{1.000000000 + 0.4047619052t^2} \tag{18}$$

$$r_{4,2}(t) = \frac{0.9999999997t - 4.302564101t^3}{0.9999999997 + 0.03076923073t^2} \tag{19}$$

$$r_{3,4}(t) = \frac{t - 4.305238660t^3}{1 + 0.02809467050t^2 - 0.01158976104t^4} \tag{20}$$

Example 2. Consider the following fractional Riccati equation [3]:

$$\frac{d^\alpha u}{dt^\alpha} = 2u(t) - u^2(t) + 1, \ \ t > 0, \tag{21}$$

where $0 < \alpha \leq 1$ subject to the initial condition

$$u(0) = 0. \tag{22}$$

Table 3: Numerical Values for exact solution $u_2(t)$ and padé approximations of $u_2(t)$ for $\alpha = 1$.

t	$r_{3,2}(t)$	$r_{2,2}(t)$	$r_{4,1}(t)$	$u_2(t)$
0.001	0.01010032327	0.01010032307	0.01010032327	0.0101003299
0.002	0.02040250481	0.02040249827	0.02040250473	0.0204026117
0.003	0.03090817612	0.03090812532	0.03090817515	0.0309087185
0.004	0.04161871586	0.04161849710	0.04161871038	0.0416204315
0.005	0.05253524444	0.05253456220	0.05253522337	0.0525394351
0.006	0.06365861941	0.06365688488	0.06365855602	0.0636673102
0.007	0.07498943171	0.07498560185	0.07498927070	0.0750055287
0.008	0.08652800277	0.08652037616	0.08652764146	0.0865554472
0.009	0.09827438236	0.09826034792	0.09827364482	0.0983183006
0.1	0.1102283475	0.1102040816	0.1102269503	0.1102951967

The exact solution for equation (21) is given as $u_2(t) = 1+\sqrt{2}\tanh\left(\sqrt{2}t+\frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$ in [3]. Odibat and Momani obtained following solution in [3] by applying modified homotopy perturbation method on (21) and (22)

$$\begin{aligned} u_2(t) =& 4t+6t^2-\frac{2t^3}{3}-3t^4+\frac{t^5}{15}+\frac{34t^6}{90}-\frac{17t^7}{315}-\frac{6t^{2-\alpha}}{\Gamma(3-\alpha)}-\frac{16t^{3-\alpha}}{\Gamma(4-\alpha)}+\left[\frac{10}{\Gamma(3-\alpha)}+\frac{2}{\Gamma(4-\alpha)}\right]\frac{\Gamma(4-\alpha)}{\Gamma(5-\alpha)}t^{4-\alpha} \\ &+\left[\frac{2}{\Gamma(3-\alpha)}+\frac{8}{\Gamma(4-\alpha)}+\frac{20}{\Gamma(5-\alpha)}+\frac{4\Gamma(4-\alpha)}{\Gamma(3-\alpha)\Gamma(5-\alpha)}\right]\frac{\Gamma(5-\alpha)}{\Gamma(6-\alpha)}t^{5-\alpha} \\ &-\left[\frac{2}{3\Gamma(3-\alpha)}+\frac{4}{\Gamma(5-\alpha)}+\frac{16}{\Gamma(6-\alpha)}+\frac{4\Gamma(4-\alpha)}{\Gamma(3-\alpha)\Gamma(5-\alpha)}\right]\frac{\Gamma(6-\alpha)}{\Gamma(7-\alpha)}t^{6-\alpha}+\frac{4t^{3-2\alpha}}{\Gamma(4-2\alpha)}+\frac{6t^{4-2\alpha}}{\Gamma(5-2\alpha)} \\ &-\left[\frac{1}{\Gamma(3-\alpha)^2}+\frac{2}{\Gamma(4-2\alpha)}+\frac{2}{\Gamma(5-2\alpha)}+\frac{2\Gamma(4-\alpha)}{\Gamma(3-\alpha)\Gamma(5-2\alpha)}\right]\frac{\Gamma(5-2\alpha)}{\Gamma(6-2\alpha)}t^{5-2\alpha}-\frac{t^{4-3\alpha}}{\Gamma(5-3\alpha)} \end{aligned} \tag{23}$$

By applying equations (10) and (11) to put equation (23) into Padé series, following Padé equations respectively $r_{3,2}(t)$,$r_{2,2}(t)$, $r_{5,2}(t)$, $r_{4,1}(t)$, $r_{4,3}(t)$and $r_{6,1}(t)$ were obtained for $\alpha = 1$and different values of m and n;

$$r_{3,2}(t) = \frac{t+0.7599999996t^2+1.173333333t^3}{1-0.2400000004t+1.080000000t^2} \tag{24}$$

$$r_{2,2}(t) = \frac{0.9999999998t-0.9999999983t^2}{0.9999999998-1.999999998t+1.666666665t^2} \tag{25}$$

$$r_{4,1}(t) = \frac{t+0.3999999999t^2-0.2666666671t^3-1.200000000t^4}{1.-0.6000000001t} \tag{26}$$

$$r_{5,2}(t) = \frac{t+1.380378659t^2+0.8632625740t^3-0.7236565312t^4-0.9305284636t^5}{1+0.3803786576t+0.1495505833t^2} \tag{27}$$

$$r_{4,3}(t) = \frac{0.9999999999t-0.4199272075t^2-0.2258113443t^3-1.719381256t^4}{0.9999999999-1.419927207t+0.8607825301t^2-1.106854717t^3} \tag{28}$$

$$\begin{aligned} r_{6,1}(t) =& (0.9999999999t+1.142857143t^2+0.4761904758t^3-0.9523809522t^4-0.7428571429t^5 \\ &+0.2920634920t^6)/(0.9999999999+0.1428571428t) \end{aligned} \tag{29}$$

5 Conclusion

As it is seen from the tables in two examples, it can be said that the obtained numerical results by using univariate padé approximation are very powerful and efficient. It provides us with a simple way to adjust and control the convergence region of solution series. This numerical study illustrates the validity and great potential of the Padé approximation for

Table 4: Numerical Values for exact solution $u_2(t)$ and padé approximations of $u_2(t)$ for $\alpha = 1$.

t	$r_{5,2}(t)$	$r_{4,3}(t)$	$r_{6,1}(t)$	$u_2(t)$
0.001	0.01010032327	0.01010032327	0.01010032328	0.0101003299
0.002	0.02040250477	0.02040250478	0.02040250477	0.0204026117
0.003	0.03090817567	0.03090817569	0.03090817570	0.0309087185
0.004	0.04161871343	0.04161871342	0.04161871342	0.0416204315
0.005	0.05253523505	0.05253523498	0.05253523504	0.0525394351
0.006	0.06365859095	0.06365859071	0.06365859093	0.0636673102
0.007	0.07498935889	0.07498935825	0.07498935890	0.0750055287
0.008	0.08652783838	0.08652783654	0.08652783852	0.0865554472
0.009	0.09827404503	0.09827404022	0.09827404526	0.0983183006
0.1	0.1102277053	0.1102276939	0.1102277058	0.1102951967

fractional Riccati differential equations. The basic ideas of this approach can be further employed to solve other strongly problems in fractional calculus.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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